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SPANNED AND AMPLE VECTOR BUNDLES WITH LOW CHERN NUMBERS

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Here we classify paris (V, E) with V projective variety, $\dim(V) = n$, E ample and spanned rank-2 vector bundle and, if $n = 2k$, $c_2(E)^k = 1$, if $n = 2k+1$, $c_1(E)c_2(E)^k = 2$. In both cases $V = P^n$ and E is the direct sum of two line bundles of degree 1.

Introduction. In the last few years a few papers appeared (e.g. **[LP]**, **[LS]**, **[W1]**, **[W2]**) giving classifications, under suitable assumptions, of pairs (V, E) with V projective variety and E an ample, spanned vector bundle with low Chern classes. It is natural to arise the following conjecture (which is proved in 2.2 in a stronger form if the bundle is the direct sum of r line bundles):

Conjecture. Fix integers n, r s, $i_1, \ldots, i_s$, with $s > 0$, $0 < r$, $0 < i_t \leq \min(r, n)$, $i_1 + \cdots + i_s = n$. Fix an irreducible, complete variety V, $\dim(V) = n$, and an ample vector bundle E, E spanned by global sections. Then

$$c_{i-1}(E) \cdots c_{i_s}(E) \geq \binom{r}{i_1} \cdots \binom{r}{i_s}$$

and if we have equality, then $V \cong \mathbf{P}^n$ and $E \cong r\mathcal{O}_V(1)$.

Here we work over an algebraically closed field $\mathbf{K}$ and prove the following results.

THEOREM 1. *Fix an even integer $n = 2k > 0$. Let V be an integral complete variety and E a rank-2 ample vector bundle on V, E spanned by its global sections and with $c_2(E)^k = 1$. Assume either V Cohen-Macaulay or* $\mathrm{char}(\mathbf{K}) = 0$. *Then $V \cong \mathbf{P}^n$ and $E \cong 2\mathcal{O}_V(1)$.*

THEOREM 2. *Fix an odd integer $n = 2k+1 > 0$. Let V be an integral complete variety and E a rank-2 ample vector bundle on V, E spanned by its global sections and with $c_1(E)c_2(E)^k = 2$. Assume either V Cohen-Macaulay or* $\mathrm{char}(\mathbf{K}) = 0$. *Then $V \cong \mathbf{P}^n$ and $E \cong 2\mathcal{O}_V(1)$.*

For a fixed variety V, Theorem 1 follows from the conjecture of **[LS]**; hence Theorem 1 was known in several cases proved in **[LP]**, **[W1]**, 3.4, **[W2]**.

This paper is dedicated to Alessandra.

NOTATIONS. For a projective space X, we write $\mathcal{O}(1)$ instead of $\mathcal{O}_X(1)$ when there is no danger of misunderstanding. A vector bundle is called spanned if it is spanned by its global sections. We use $|L|$, $L \in \mathrm{Pic}(Y)$, for the linear system associated to the sections of L.

1. Proof of Theorem 1.

LEMMA 1.1. *Let V be an integral complete variety,* $\dim(V) = n$, *and E an ample vector bundle on V,* $\mathrm{rk}(E) = r$, *E spanned by a linear subspace W of its sections. Then* $\dim(W) \geq n + r$.

Proof. Set $L := \mathcal{O}_{P(E)}(1)$. Using $(\mathbf{P}(E), L)$ instead of (V, E) we reduce to the case $r = 1$. Now use that the map induced by $|L|$ is finite by the ampleness of E. □

We omit the proof of the following general result suggested by the referee, since it will be either used in cases (e.g. under assumption ($) in the proofs of Theorems 1 and 2) in which the existence of a suitable section with zero-locus of the right codimension is trivial or proved directly (claim in the proof of 1.3).

LEMMA 1.2. *Let E be a rank-v ample vector bundle on an n-fold X. Assume that E is spanned by its sections. Let $x_1, \ldots, x_k$ be k points in X. If $kv \leq$* $\dim(X)$, *then there is $s \in H^0(E)$ such that $\{x_1, \ldots, x_k\} \subseteq (s)_0$ and* $\mathrm{codim}(s)_0 = v$. *Furthermore we may assume that x_2 is a tangent vector at x_1.*

LEMMA 1.3. *Let S be an integral complete surface and E a rank-2 spanned ample vector bundle with $c_2(E) = 1$. Then $(S, E) \cong (\mathbf{P}^2, 2\mathcal{O}(1))$.*

Proof. If S is normal, the result is well known (see e.g. **[B]** if $\mathrm{char}(\mathbf{K}) > 0$). Assume that S is not normal. Let $p \colon S' \to S$ be the normalization. Let $E' := p^*(E)$. We have $(S', E') \cong (\mathbf{P}^2, \mathcal{O}(1))$. Fix a nonnormal point $x \in S$, hence with $\mathrm{length}(p^{-1}(x)) > 1$.

Claim. There is a section s of E with $x \in (s)_0$ and $\mathrm{codim}((s)_0) = 2$.

Assume the claim. Then $p^*(s)$ is a section of E' vanishing in codimension 2. Since $c_2(E') < \mathrm{length}(p^{-1}(x))$, we get a contradiction.

Proof of the claim. Let F be the fiber of the projection $t\colon \mathbf{P}(E) \to S$ over x and $L := \mathcal{O}_{\mathbf{P}(E)}(1)$ the tautological line bundle. Let $h\colon \mathbf{P}(E) \to |L|$ be the map induced by $|L|$. Since L is spanned, $h(F)$ is a line. Since L is ample, $h^{-1}(h(F))$ is a curve. Set $A := t(h^{-1}(h(F))$, and let $A(1), \dots, A(s)$ be the irreducible components of A of dimension 1 (if any). Let $P(i)$ be a general point of $A(i)$. Since $h^{-1}(h(F)) \cap t^{-1}(P(i))$ is finite, a general section of E vanishing at x does not vanish at $P(i)$. By Bertini's theorem ([**K**]) applied to $S \backslash A$, we get that a general section s of E with $x \in (s)_0$ vanishes only in codimension 2. □

The proof of Theorem 1 will be divided into several steps ((a), ..., (m)). It will give also Theorem 2 and most of the results stated in the next section.

Proof of Theorem 1. (a) Take $s \in H^0(E)$ with $X := (s)_0$ of codimension 2. We want to prove that $(X, E|X) \cong (\mathbf{P}^{n-2}, 2\mathcal{O}(1))$. Since $c_2(E|X)^{k-1} = 1$, X is generically reduced and $Y := X_{\text{red}}$ is irreducible. By the inductive assumption, $(Y, E|Y) \cong (\mathbf{P}^{n-2}, 2\mathcal{O}(1))$. If V is Cohen-Macaulay, then X is Cohen-Macaulay; since X is generically reduced, it is reduced. Now assume char$(\mathbf{K}) = 0$. Then a general section s' of E has $W := (s')_0$ reduced, hence $(W, E|W) \cong (\mathbf{P}^{n-2}, 2\mathcal{O}(1))$. Set $H := \det(E)$. For every closed subscheme Z of V, let p_Z be the Hilbert polynomial of $\mathcal{O}_Z$ with respect to H. Using either flatness or the short exact sequences determined by s and s', we see that $p_X = p_W$. Assume that X is not reduced and let $k \geq 0$ be the dimension of the support of the nilradical of $\mathcal{O}_X$. For a fixed Z and a general $D \in |L|$, we have $p_{Z \cap D}(n) = p_Z(n) - p_Z(n-1)$. Hence $p_{X \cap D} = p_{W \cap D}$. Taking k general divisors of L, we get a contradiction.

(b) Set $T := \{(s)_0\colon s \in H^0(E), \operatorname{codim}((s)_0) = 2\}$. By the proof of the claim in the proof of 1.3 (or by 1.2), for every $x \in V$, there is $S \in T$ with $x \in S$.

(c) Now we prove that V is smooth. Indeed by (b) and (a) for every $x \in V$ there is a smooth subvariety S of codimension 2 and locally complete intersection in V, with $x \in S$.

(d) Now we give a few definitions. A curve $C \subset V$ is called a line if $C \cong \mathbf{P}^1$ and $E|C \cong 2\mathcal{O}\ (1)$. A line C is called of type T if it is contained in some $S \in T$. Fix any $S \in T$. For any smooth codimension 2 subvariety Y of V which is an embedded deformation of S, we have $(Y, E|Y) \cong (\mathbf{P}^{n-2}, 2\mathcal{O}(1))$ by the invariance of Chern numbers under deformations. Call G the set of such Y. A line is called of type G if it is contained in some $Y \in G$.

(e) Now we prove that for all x, $y \in V$, $x \neq y$, there is $S \in T$ with $\{x,y\} \subset S$.

Proof. For any $z \in V$, let $F(z)$ be the fiber over z of the projection $\mathbf{P}(E) \to V$. Let $L := \mathcal{O}_{\mathbf{P}(E)}(1)$ be the tautological line bundle and $h\colon \mathbf{P}(E) \to |L|$ be the induced map. Since L is spanned, $h(F(z))$ is a line for all z, hence $h(F(x)) \cup h(F(y))$ spans a linear space O of dimension at most 3. We conclude as in the proof of the claim in the proof of 1.3.

(f) By (e) for all x, $y \in V$ with $x \neq y$ there is a line of type T containing x and y. Now we want to prove that there is a unique line containing x and y. Take $S \in T$ with $\{x,y\} \subset S$ and let C be a line containing x and y. Since $E|C \cong 2\mathcal{O}(1)$, any section of E vanishing on x and y vanishes on C. Thus $C \subset S$, hence the uniqueness of the line containing x and y. Hence every line is of type T. Since $E|S$ is the normal bundle to S in V, every line is a smooth point of the Hilbert scheme of V. Write (x,y) for the line containing x and y, $x \neq y$.

(g) Now assume the existence of a divisor $D > 0$ with $h^0(E(-D)) \neq 0$. Fix x, y, z in V with $x \in D$, $y \notin D$. Take $X \in T$, X containing x, uy. Since $D \cap S$ is a positive divisor, we have $\mathcal{O}(D)|S \cong \mathcal{O}_S(1)$. Fix a section s of $E(-D)$, $s \neq 0$. We have seen that s does not vanish on S or vanishes identically on S. By (e) we get easily that s vanishes nowhere on S, hence it does not vanish at x or at y. Thus s does not vanish at all and shows that $E(-D)$ is the extension of a line bundle M by the trivial line bundle. Set $R := M(D)$. We have seen that $R|S \cong \mathcal{O}_S(1)$ for every $S \in T$. Since R is a quotient of E, R is ample and spanned. Fix $A \in |R|$, $u \in A$, $z \in A$, $v \notin A$, $S \in T$, $Q \in T$ with $u \in S \cap Q$, $v \in S$, $z \in Q$. Since $R|S \cong \mathcal{O}_S(1)$, one sees that A is smooth and irreducible, and that for any two points of A, the line containing them is contained in A. Fix $W \in T$. Since $h^0(R|W) = n - 1 \leq h^0(R) - 2$, there are A, $B \in R$ with $A \neq B$, $W \subseteq (A \cap B)$. Since $A \cap B$ contains the line joining any two of its points, $(A \cap B)_{\text{red}} = W$. Fix $a \in W$ and $C \in |R|$ with $a \notin C$. Since V is the union of the lines (a,t), $t \in C$, there is a line (a,m) not tangent to A at a. Take B' in the pencil of R spanned by A and B, with $m \in B'$. Then $A \cap B'$ is smooth at a, hence everywhere i.e. $(A \cap B) = W$. Thus $R^n = 1$. Since R is ample and spanned, and V is smooth $V \cong \mathbf{P}^n$, $R \cong \mathcal{O}(1)$. By (e) for every line I of V, $E|I \cong 2\mathcal{O}_I(1)$, hence $E \cong 2\mathcal{O}(1)$ (e.g. use [E]).

(h) From now on in this section, we make the following assumption (\$):

$$(\$) \qquad \text{For divisors } D > 0,\ h^0(E(-D)) = 0.$$

By (g) to prove Theorem 1 it is sufficient to assume (\$) and find a contradiction.

(i) First assume $h^0(E) > 6$; by 1.1 this is satisfied if $k > 2$. Fix any 3 points x, y, z of V. By assumption there is $s \in H^0(E)$ with $s(x) = x(y) = s(z) = 0, s \neq 0$. By (\$) there is $S \in T$ with $\{x, y, z\} \subset S$. We want to check that if $h^0(E) = 6$ there is $S \in G$ with $\{x, y, z\} \subset S$ and that if $z \notin (x, y)$, such a surface S is unique. First the uniqueness. If S, S' are surfaces with this property, $S \cap S'$ contains the line joining any two of its points, hence $S = S'$. Counting dimensions, we see that for general x, y, z there is $S \in G$ containing them. Since by the proof of (a) G is a complete family, this is true for all x, y, z; alternatively one can use the union of the lines (z, t) with $t \in (x, y)$.

(j) For every $S \in G$ and every $P \notin S$, let $D(P, S)$ be the union of the lines (P, t) with $t \in S$. $D(P, S)$ is a divisor. First we check that for any x, $y \in D(P, S)$, $(x, y) \subset D(P, S)$ (hence in particular $D(P, S)$ is irreducible). Take u, $v \in S$ such that $x \in (u, P)$, $y \in (v, P)$. Fix $t \in (x, y)$. By (i) there is $W \in G$ with $\{u, v, P\} \subset W$, hence with $t \in W$ and with $(t, P) \in W$; hence $(t, P) \cap (u, v) \neq \emptyset$, i.e. $t \in D(P, S)$. Note that the divisors $D(P, S)$ and $D(P', S')$ are algebraically equivalent. Hence for any two points a, $b \in V$, there is a divisor D algebraically equivalent to $D(P, S)$ and with $a \in D$, $b \notin D$; by Nakai's ampleness criterion $D(P, S)$ is ample.

(k) By (j) any line of type T not contained in $D(P, S)$ intersects $D(P, S)$ at most at a point. Fix a point x of $D(P, S)$. By (k) there is a line F of type T intersecting $D(P, S)$ only at x and transversally. Thus $\mathscr{O}(D(P, S))|F \cong \mathscr{O}(1)$. Thus the same is true for all lines (by (f) they are of type T), hence $\mathscr{O}(D(P, S))|S = \mathscr{O}(1)$. Fix $y \in D(P, S)$ and any line of type T through y and not contained in $D(P, S)$; since $D(P, S) \neq V$, the existence of such a line follows from (i) and (j); we get that $D(P, S)$ is smooth at y for every y. Since for suitable P', we have $D(P, S) \cap D(P', S) = S$ (set-theoretically), S is ample in $D(P, S)$ by the last part of (i).

(l) Set $A := D(P, S)$.

Claim. $E|A \cong 2\mathscr{O}_A(A)$.

Proof of the claim. Let $\mathscr{I}$ be the ideal sheaf of S in A; let $S(k)$ be the kth infinitesimal neighborhood of S in A, with ideal sheaf $\mathscr{I}^{k+1}$. Set $F := \text{Hom}(2\mathscr{A}_A(A), E|A)$. Since E is ample in A, there is an integer $k > 0$ such that $h^1(A, F \otimes \mathscr{I}^{k+1}) = 0$, thus $H^0(A, F) \to H^0(S(k), F|S(k))$ is surjective. Since $\mathscr{I}^k/\mathscr{I}^{k+1} \cong \mathscr{O}_S(k)$, from the isomorphism $E|S \cong 2\mathscr{O}_S(1)$ and the exact sequence

$$0 \to \mathscr{I}^t/\mathscr{I}^{t+1} \otimes F \to F|S(t) \to F|S(t-1) \to 0$$

we find that the restriction map $H^0(S(k), F|S(k)) \to H^0(S, F|S)$ is surjective. Thus there is $c \in H^0(A, F)$ which induces the isomorphism between $2\mathscr{O}_S(1)$ and $E|S$. Since S is ample in A, every divisor of A intersects S. Thus c induces an isomorphism at every point of A (take the determinant!).

(m) The same proofs as in (l) give that $E = 2\mathscr{O}(A)$, containing ($). The proof of Theorem 1 is over. □

2. Proof of Theorem 2.

Proof of Theorem 2. First assume $n = 1$. Let h be the morphism from V to a suitable Grassmannian Grass induced by $H^0(E)$. By assumption (for the Plucker embedding) $\deg(h)\deg(h(V)) = 2$. Thus $h(V)$ is smooth and rational. If $\deg(h) = 2$, $h(V)$ is a line and the restriction of the universal quotient bundle of Grass to $h(V)$ is not ample (see e.g. **[P]**, p. 123), contradicting the ampleness of E. Thus h is an isomorphism and E must be the direct sum of two line bundles of degree 1. If $n > 3$, the inductive proof of Theorem 1 works. If $n = 3$, however that proof has to be modified (in particular point (e) and its consequences). Thus we assume $n = 3$ and use the terminology "line of type T or of type G" as in the previous section.

(1) As in (b) of §1, for every $P \in W$, there is a line C of type T with $P \in C$. As in (c) of §1 this implies the smoothness of V.

(2) First assume the existence of a divisor $D > 0$ with $h^0(E(-D)) \neq 0$. By (1) there is a line A of type T not contained in D. Since $E|D$ is ample, $c_2(E|D) \neq 0$, hence for every line C of type T, $C \cap D \neq \varnothing$. Thus $\mathscr{O}(D)|C$ has degree 1 for every C of type T. Fix $s \in H^0(E)$ with $(s)_0 = A$ and $t \in H^0(E(-D))$, $t \neq 0$. Then $s|D$ shows that $c_2(E|D) = 1$. By 1.3 $D \cong \mathbf{P}^2$, $E|D \cong 2\mathscr{O}(1)$. Note that t either vanishes identically on a line not in D or has no zero there. Fix a point $P \in V$. By 1.1 and Bertini's theorem (**[K]**), we see that there are infinitely many lines of type T through P. Thus we see that if t vanishes at P, it vanishes in codimension 1. Enlarging if necessary D, we get a contradiction.

Thus $t(\mathcal{O}_V)$ is a subline bundle of $E(-D)$; let $M := E(-D)/s(\mathcal{O}_V)$, $R := M(D)$, hence R ample and spanned. Fix $A \in |R|$. As before we see that $(A, E|A) \cong (\mathbf{P}^2, 2\mathcal{O}(1))$. Since $h^0(V, A) > 3$, A contains a line B of type T. Thus $\mathcal{O}(A)|A \cong \mathcal{O}(1)$, hence $A^3 = 1$, and we get the thesis.

(2) From now on, we assume the following assertion ($):

($) there is no divisor $D > 0$ with $h^0(E(-D)) \neq 0$.

By 1.1 for any length 2 subscheme X of V there is a non-zero section of E vanishing there. By ($) there is a line of type T containing X. Such a line is unique by ($) (even taking lines not of type T). The uniqueness implies that every line is of type T.

(3) Fix any line S and $P \notin S$. Let $A = D(P, S)$ be the union of the lines (P, t) with $t \in S$. Let Q be the image of $H^0(E)$ into $H^0(A, E|A)$ by the restriction map. Take a general $s \in Q$ with $s(P) = 0$. By Bertini's theorem we see that s vanishes only in codimension 2 on A. By the last part of (2) we see that $P = (s)_0$ as a scheme. Thus by 1.3 $(A, E|A) \cong (\mathbf{P}^2, 2\mathcal{O}(1))$. In particular every section of $E|A$ vanishing on a scheme of length 2 vanishes on a "line" of A. Thus by the last statement in (2) every line intersecting A at more than one point is contained in A. Taking $D(P', S)$ for general P', we get $A^3 = 1$, hence $V = \mathbf{P}^3$. By [E] E splits and Theorem 2 is proved. □

REMARK 2.1. Fix (V, E). A line in V is a smooth rational curve C such that $E|C$ is a direct sum of line bundles of degree 1. Here are some properties a pair (V, E) can have: (i) through a general point there is a line; (ii) for two general points there is a line; (iii) for every pair of points there is a line containing them. In (ii) and (iii) we can ask also the uniqueness of the line. The proofs of Theorems 1 and 2, show that (ii) is true if in the statement of the theorems we omit the Cohen-Macaulay assumption; furthermore no pair (V, E) exists if in the statement of Theorem 2 we take $c_1(E)c_2(E)^k = 1$. One gets similar results, for instance if $r = 3$, $n = 1 + 3k$, $c_1(E)c_3(E)^k = 3$ (no such pair exists if $c_1(E)c_3(E)^k < 3$) and in a few similar cases.

Now we show that the conjecture holds (in a stronger form) for vector bundles which are direct sum of ample, spanned line bundles.

PROPOSITION 2.2. *Fix integers r, n, s, $i_1, \ldots, i_s$ with $r > 0$, $n > 0$, $s > 0$, $0 < i_t \leq \min(r, n)$ for all t, $i_1 + \cdots + i_s = n$. Let V be a complete, integral variety and $L_1, \ldots, L_r$ be ample and spanned line bundles on*

V. Set $E := L_1 \oplus \cdots \oplus L_r$ and $c = c_{i_1}(E)c_{i_2}(E)\cdots c_{i_s}(E)$. Then

$$c \geq d := \binom{r}{i_1} \cdots \binom{r}{i_s}$$

and if $c = d$, then $V \cong \mathbf{P}^n$ and L_t has degree one for all t.

Proof. The intersection number of any n ample line bundles is > 0. The result follows immediately from the following claim.

Claim. Fix any n ample, spanned, line bundles $M_1, \ldots, M_n$ in V. If their intersection number is one, then $V \cong \mathbf{P}^n$ and each M_t has degree one.

Proof of the claim. By induction on n, the cases with $n = 1$ and $n = 2$ being left to the reader; for $n = 2$ use for instance Hodge index theorem. Assume $n \geq 3$. Take $A \in |M_1|$. By induction we get $A \cong \mathbf{P}^{n-1}$ and each $M_t|A$, $t > 1$, has degree one. Set $U := M_1$, $J := M_2$. We get $UJ^{n-1} = 1$. Taking $B \in |J|$, we get $B \cong \mathbf{P}^{n-1}$ and JB of degree one. Thus $J^n = 1$, and the claim is easy. □

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