PACIFIC JOURNAL OF MATHEMATICS
Vol. 148, No. 2, 1991

OPERATORS PRESERVING DISJOINTNESS ON REARRANGEMENT INVARIANT SPACES

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Let X and Y be two rearrangement invariant spaces on a measure space (Ω, Σ, μ) with a finite, nonatomic measure μ. We show that if there exists a non-zero order continuous disjointness preserving operator $T: X \to Y$, then $X \subseteq Y$. This result has many consequences. For example, if $T: L_p(\Omega, \Sigma, \mu) \to L_q(\Omega, \Sigma, \mu)$ $(0 < p < q \le \infty)$ preserves disjointness, then $T \equiv 0$.

1. Notation and preliminary facts. Recall that a (linear) operator $T: X \to Y$ between vector lattices is said to be a *disjointness preserving operator* if $|x_1| \wedge |x_2| = 0$ in X implies $|Tx_1| \wedge |Tx_2| = 0$ in Y. All vector lattices are assumed to be Archimedean, and all operators on normed or linear metric spaces are assumed to be continuous.

Let (Ω, Σ, μ) be a measure space with a finite σ-additive nonatomic measure and $S(\Omega, \Sigma, \mu)$ be the space of all (equivalence classes of) measurable real valued functions. Throughout the work we will use the representation of the space S as the space $C_\infty(Q)$ of all continuous extended functions on the Stone space Q of S. (See **[10]** for details.) We retain the same notation μ for the corresponding measure on Q, which is defined on the σ-algebra Σ_Q consisting of all subsets of the form $(E \backslash N) \cup (N \backslash E)$, where E is a *clopen* (closed and open) subset of Q and N is a first category subset of Q. It is well known that $\mu(D) = 0$ if and only if D is a nowhere dense subset of Q. (Any extremally disconnected space Q with such a measure is sometimes called a hyperstonian space.) A subspace X of $S(\Omega, \Sigma, \mu)$ is called a rearrangement invariant (r.i.) ideal if

(i) X is an order ideal in S, and

(ii) If $x \in X$, $y \in S$, and x and y are equimeasurable, in symbols $x \sim y$, then $y \in X$.

If, in addition, X is equipped with a Banach norm $\|\cdot\|$ such that

(iii) $x_1, x_2 \in X$ and $|x_1| \le |x_2| \Rightarrow \|x_1\| \le \|x_2\|$, and

(iv) $x_1, x_2 \in X$ and $x_1 \sim x_2 \Rightarrow \|x_1\| = \|x_2\|$,

then X is called a r.i. Banach function space. We refer to [7] for the basic facts concerning r.i. ideals and Banach spaces. (Let us mention incidentally that up to an equivalent renorming (i), (ii), and (iii) imply (iv). See **[1]** or [7, p. 115].) All necessary information about Banach and vector lattices can be found in [**4**, **10**].

2. The following theorem is the main result of this article.

THEOREM 1. *If* X *and* Y *are r.i. ideals and* $X \not\subseteq Y$, *then every order continuous disjointness preserving operator* $T: X \to Y$ *is identically equal to zero, i.e.,* $T \equiv 0$.

We precede the proof of this theorem with several immediate corollaries.

COROLLARY 2. *Let* X *and* Y *be two r.i. Banach function spaces and* X *have order continuous norm. If* $T: X \to Y$ *is a nonzero disjointness preserving operator, then* $X \subseteq Y$.

An alternative proof of this corollary can be obtained using Lemma 5.2 in [**6**].

COROLLARY 3. *There is no nontrivial disjointness preserving operator from* $L_p(\Omega, \Sigma, \mu)$ *into* $L_q(\Omega, \Sigma, \mu)$ *for* $0 < p < q \leq \infty$.

REMARK. In a special case of L_p-spaces $(1 \leq p \leq \infty)$, when Ω is an open subset of R^n and μ is Lebesgue measure, this result was earlier obtained by a quite different method by M. Drahklin [**5**].

COROLLARY 4 (*L. Potepun* [**9**]). *Order isomorphic r.i. ideals coincide. That is, if* X *and* Y *are order isomorphic r.i. ideals, then* $X = Y$.

Proof. Let T be an order isomorphism of X onto Y. Obviously, T and T^{-1} are order continuous and, hence, by Theorem 1, $X \subseteq Y$ and $Y \subseteq X$, i.e., $X = Y$. The original proof in [**9**] was much more difficult. □

3. Three auxiliary lemmas. The space Q and measure μ below are as defined above.

LEMMA 5. *Let* A *be a nonvoid clopen subset of* Q *and let* φ *be a continuous open mapping from* A *into* Q. *Put* $B = \varphi(A)$. *Then there exists a nonvoid clopen subset* B_1 *of* B *and a constant* $K > 0$ *such that for any measurable* $D \subset B_1$

$$K^{-1}\mu(D) \leq \mu(\varphi^{-1}(D)) \leq K\mu(D).$$

Proof. The set $B = \varphi(A)$ is evidently a clopen subset of Q. We introduce a new measure γ on the σ-algebra Σ_Q by letting $\gamma(D) := \mu(\varphi^{-1}(D \cap B))$, $(D \in \Sigma_Q)$. Obviously, B is the support set of the measure γ. Let us verify that γ is absolutely continuous with respect to μ. Take an arbitrary measurable set D with $\mu(D) = 0$. Hence D is nowhere dense in Q. Since φ is open the set $\varphi^{-1}(D)$ $(= \varphi^{-1}(D \cap B))$ is also nowhere dense and thus $\mu(\varphi^{-1}(D \cap B)) = 0$. This proves that γ is absolutely continuous with respect to μ and, consequently, by the Radon-Nikodym theorem there exists a nonnegative function $h \in L_1(\Omega, \Sigma, \mu)$ such that $\gamma(D) = \int_D h\, d\mu$ for each measurable set D. Take a nonvoid clopen subset $B_1 \subset B$ and a constant $K > 0$ so that $K^{-1} \leq h(q) \leq K$ for each $q \in B_1$. Clearly B_1 and K satisfy the desired properties. □

LEMMA 6. *Let X and Y be two r.i. ideals on a (finite nonatomic measure) space (Ω, Σ, μ). If $X \not\subset Y$, then for each set $D \in \Sigma$ with $\mu(D) > 0$ there is a function $x \in X$ such that its support* $\operatorname{supp}(x) \subset D$ *and $x \notin Y$. Moreover, x can be chosen to be a step function.*

The proof is straightforward and is omitted. We only mention that for infinite measures this lemma is false and it is the only place where the finiteness of the measure μ is essential (see 5.4 below).

LEMMA 7. *Let Y be a r.i. ideal and $\tilde{y} = \sum_{n=1}^{\infty} d_n \chi_{E_n} \in Y$ be a step function, where $\{E_n\}$ $(n = 1, 2, \ldots)$ is a sequence of pairwise disjoint measurable sets. Also, let $\{D_n\}$ be a second sequence of pairwise disjoint measurable sets such that $K^{-1} \leq \mu(D_n)/\mu(E_n) \leq K$ for some $K > 0$. Then the step function $x = \sum_{n=1}^{\infty} d_n \chi_{D_n}$ likewise belongs to Y.*

4. Proof of Theorem 1. Let $T: X \to Y$ be an order continuous disjointess preserving operator from X into Y and let $X \not\subset Y$. We must show that $T \equiv 0$. The gist of the proof lies in an application of the multiplicative representation of disjointness preserving operators obtained in [**2**].

By Theorem A in [**2**], the operator T admits a global multiplicative representation, i.e., there exists a clopen set $E \subset Q$, a function $e \in C_\infty(Q)$ and a continuous mapping φ from E into Q, such that for each $x \in X$ and each $q \in Q$

$$(Tx)(q) = e(q)x(\varphi(q)), \quad \text{if } q \in E, \quad \text{and} \quad (Tx)(q) = 0 \text{ otherwise.}$$

The order continuity of T implies that the mapping φ is open (see [**2**, Lemma 4.1] or [**8**, Prop. 8]). Without loss of generality we may assume that $T \geq 0$. If $T \not\equiv 0$, then the set E is nonvoid and $E_0 := \{q \in E : 0 < e(q) < \infty\}$ is a dense open subset of E. (It is possible that $E_0 = E$.) Let us fix some constant $M > 0$ such that the clopen set $A = \mathrm{cl}\{q \in E_0 : M^{-1} < e(q) < M\}$ is nonvoid.

If we restrict the mapping φ to A and let $B = \varphi(A)$, then the continuous open mapping $\varphi: A \to B$ satisfies the conditions of Lemma 5. Therefore there exists a nonvoid clopen set $B_1 \subset B$ and a constant $K > 0$ such that $K^{-1} \leq \mu(D)/\mu(\varphi^{-1}(D) \cap A) \leq K$ for each measurable $D \subset B_1$. The condition $X \not\subseteq Y$ implies by Lemma 6 that there exists a step function $x = \sum_{n=1}^{\infty} d_n \chi_{D_n}$ such that $x \in X$, $x \notin Y$, $D_n \subset B_1$, and $D_n \cap D_m = 0$ $(n \neq m)$. Since $x \in X$, the function $y = Tx \in Y$. Now let us express y in terms of the multiplicative representation of T. We have

$$y = Tx = e(x \circ \varphi) = e(\cdot)x(\varphi(\cdot)) = e(\cdot)\left(\sum_{n=1}^{\infty} d_n \chi_{D_n}\right)(\varphi(\cdot))$$

$$= e(\cdot)\sum_{n=1}^{\infty} d_n \chi_{D_n}(\varphi(\cdot)) = e(\cdot)\sum_{n=1}^{\infty} d_n \chi_{\varphi^{-1}(D_n)}(\cdot).$$

Since $y \in Y$, we see that $y\chi_A \in Y$ and hence

$$y\chi_A = e\sum_{n=1}^{\infty} d_n \chi_{\varphi^{-1}(D_n) \cap A}.$$

As we know $e(q) \in [M^{-1}, M]$ for each $q \in A$ and therefore the function $\tilde{y} = \sum_{n=1}^{\infty} d_n \chi_{\varphi^{-1}(D_n) \cap A}$ belongs to Y if and only if $y\chi_A \in Y$. Letting $E_n = \varphi^{-1}(D_n) \cap A$, we see that $\tilde{y} = \sum_{n=1}^{\infty} d_n \chi_{E_n} \in Y$ and $K^{-1} \leq \mu(D_n)/\mu(E_n) \leq K$. By Lemma 7 this implies that $x \in Y$, a contradiction, and the proof is finished. □

5. Examples and comments. First, we show that the hypotheses of Theorem 1 cannot be weakened.

5.1. The condition $X \not\subseteq Y$ is essential, since if $X \subseteq Y$, then the identity imbedding id: $X \to Y$ is a nonzero order continuous disjointness preserving operator.

5.2. Here we show that the assumption of order continuity of $T: X \to Y$ cannot be dropped. Indeed, let a r.i. space X have a nonzero discrete functional f. Then for each Y we can easily construct a nonzero disjointness preserving operator $T: X \to Y$. To this end take an arbitrary $y \in Y$, $y \neq 0$ and define $Tx = f(x)y$. It is evident that $T \not\equiv 0$ and T preserves disjointness. (A similar argument explains why we do not consider the case of atomic measure spaces. This case is of no interest since each discrete r.i. space always has a nonzero order continuous discrete functional.)

5.3. Recall that a norm $\|\cdot\|$ on a normed lattice Z is said to be *strictly monotone* if $0 \le z_1 < z_2$ implies $\|z_1\| < \|z_2\|$.

PROPOSITION 8. *If X and Y are r.i. Banach function spaces with strictly monotone norms and T is a positive isometry from X into Y, then $X \subseteq Y$ (and $X = Y$ if T is also onto).*

Proof. It is easy to see (and this observation is due to A. S. Veksler) that each positive isometry preserves disjointness provided the norm in Y is strictly monotone. Thus, Theorem 1 is applicable and hence $X \subseteq Y$. If T is also onto, then, as is shown in [**3**, Thm. 1], T is necessarily an order isomorphism, and now Corollary 4 yields the desired equality $X = Y$. □

5.4. *The case of infinite measure.* Let us assume that $\mu(\Omega) = \infty$. It is a little bit surprising that Theorem 1 does not hold in this case. A simple example is as follows. Take $X = L^2(\mathbf{R})$ and $Y = L^2(\mathbf{R}) \cap L^1(\mathbf{R})$. Clearly X and Y are r.i. Banach function spaces with order continuous norms, $X \not\subseteq Y$ but, nevertheless, there exist nonzero order continuous disjointness preserving operators from X into Y. For example, $T_1 x := x\chi_{[a,b]}$ (where $a < b$ are arbitrary real numbers), or $T_2 x(t) := x(t)/(t^2+1)$ are such operators. Nevertheless, the following version of Theorem 1 still holds.

COROLLARY 9. *Let $\mu(\Omega) = \infty$. If there exists a nonzero order continuous disjointness preserving operator $T: X \to Y$, where X and Y are r.i. ideals, then for each set D of finite measure the subspace $X_D = \{x \in X : \operatorname{supp}(x) \subset D\}$ belongs to Y.*

Proof. Since $T \not\equiv 0$ and T is order continuous there is $x_1 \in X$ such that $y_1 = Tx_1 \neq 0$ and $\mu(E_1) < \infty$ where $E_1 = \operatorname{supp}(x)$. Choose a set E_2 of finite measure for which $y_1\chi_{E_2} \neq 0$. Now put $E = E_1 \cup E_2 \cup D$ and define T_E by $T_E x = \chi_E T(x\chi_E)$. Obviously, T_E

is a nonzero order continuous disjointness preserving operator from the r.i. ideal X_E into the r.i. ideal Y_E. By Theorem 1, $X_E \subseteq Y_E$. In particular, $X_D \subset Y$. □

We have treated the case of real spaces only, but the results remain true for complex spaces as well.

In conclusion the author would like to thank Drs. E. Arenson, A. Kitover and A. Mekler for many stimulating discussions, and the referee for his help and valuable suggestions.

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Received July 18, 1988 and in revised form January 19, 1990.

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