

CHERN CLASSES AND COHOMOLOGY FOR RANK 2 REFLEXIVE SHEAVES ON \mathbf{P}^3

MARGHERITA ROGGERO AND PAOLO VALABREGA

The paper shows that, if F is a nonsplit rank 2 reflexive sheaf on \mathbf{P}^3 , then the knowledge of the numbers $d_n = h^2(F(n)) - h^1(F(n))$ gives an explicit algorithm to compute the Chern classes c_1, c_2, c_3 and the dimensions $h^0(F(n))$, for all n (in particular the first integer a such that the sheaf $F(a)$ has some nonzero section). If the sheaf is a vector bundle it is also proved that the knowledge of the numerical sequence $\{h^1(F(n))\}$ together with the first Chern class gives all the information as above. In some special cases, i.e. when $h^1(F(n)) \neq 0$ for at most three values of n , an algorithm is also produced to compute the first Chern class from the sequence $\{h^1(F(n))\}$. Vector bundles with natural cohomology are also discussed.

It must be remarked that, if one knows not only the dimensions $h^1(F(n))$, for all n , but also the whole structure of the Rao-module $\bigoplus H^1(F(n))$, then the first Chern class c_1 is uniquely determined (as it is shown in a paper by P. Rao).

n1. F is a rank 2 nonsplit reflexive sheaf on $\mathbf{P}^3 = \mathbf{P}$. Its Chern classes are c_1, c_2, c_3 ; if it is normalized, then $c_1 = 0$ or -1 . Once and for all $h^i(F(n)) = \dim F(n)$, $i = 0, 1, 2, 3$.

Now we give a list of well-known properties useful throughout the paper.

1. If $c_1(F) = c_1$, the associated normalized reflexive sheaf is defined as $F^n = F(\varepsilon)$ where

$$\varepsilon = \begin{cases} \frac{c_1}{2} & \text{for } c_1 \text{ even,} \\ -\frac{c_1 + 1}{2} & \text{for } c_1 \text{ odd.} \end{cases}$$

2. With every reflexive sheaf F there are two associated numbers:

$$a = a(F) = \text{smallest integer } n \text{ such that } h^0(F(n)) \neq 0,$$

$$a_1 = a_1(F) = \text{smallest integer } n \geq a \text{ such that}$$

$$h^0(F(n)) > h^0(\mathcal{O}_P(n - a)).$$

Since F is not split, then every general nonzero section of $F(a)$ gives rise to a zero locus which is necessarily a curve in \mathbf{P} (see [H1], n.1 and [H2], n.4); this is false for a split sheaf. The same is true for

$F(n)$, $n \geq a_1$ (see [GRV], Th. 0.1: it is stated for bundles, but it is easy to check it holds for any reflexive sheaf, exactly with the same proof: actually we shall use it only for bundles).

3. If a section of $F(n)$ gives rise to a curve, then F and the ideal sheaf \mathbf{I} of the curve are connected by the following exact sequence (see [H1], n.1 and [H2], n.4):

$$(*) \quad 0 \rightarrow \mathcal{O}_P \rightarrow F(n) \rightarrow \mathbf{I}(2n + c_1) \rightarrow 0.$$

It follows that, if in addition F is normalized, then $h^0(F(n)) > 1$ implies that $n > 0$, because $n \leq 0$ means that the curve lies on a surface of degree $2n + c_1 \leq 0$, which is absurd.

In particular $a_1 > 0$ and $h^0(F(a)) = 1$ if $a \leq 0$, because in both cases there is a curve (see 2 above).

4. The Euler-Poincaré function of F is

$$\begin{aligned} \chi(F(n)) &= (n + 3)(n + 2)(n + 1)/3 + c_1(n + 2)(n + 1)/2 \\ &\quad - (n + 2)c_2 + (c_3 - c_1c_2)/2. \end{aligned}$$

5. The normalized sheaf F is stable if $a(F) = a > 0$, properly semistable if $a = c_1 = 0$, unstable otherwise; a general sheaf is stable, properly semistable, unstable according to the corresponding property for the normalized sheaf F^n .

6. If F is a nonsplit vector bundle, then $h^1(F(n)) \neq 0$ for some n (Horrocks' theorem: see [BH], n.5, Lemma 1). Actually it is true: if c_1 is even, then $h^1(F(\varepsilon - 1)) \neq 0$; if c_1 is odd, then $h^1(F(n)) \neq 0$ for $n = \varepsilon - 1, \varepsilon, \varepsilon + 1$. This depends upon [CV1], Theorems 1.3 and 1.6; in fact it is enough to consider, for $n \gg 0$, a smooth irreducible curve C zero locus of a section of $F(n)$; then, by (*) above $h^1(F(m)) = h^1(\mathbf{I}(n + m + c_1))$, for all m .

Moreover $h^1(F(n)) \neq 0$ for all n such that $a - 2 \leq n \leq a_1 - 2$ ([CV2], Cor. 8).

7. For a reflexive sheaf F the condition $h^1(F(n)) = 0$ for every n is far from forcing the sheaf to split; in fact every locally complete intersection curve C is zero locus of a section of some reflexive sheaf F ([H2], Th. 4.1), but C may as well be arithmetically normal, which implies (by (*) above): $h^1(F(n)) = h^1(\mathbf{I}(n)) = 0$ for every n . If C is not a complete intersection, then F cannot split ([H1], Cor. 1.2).

8. A condition that forces a reflexive sheaf to be a bundle is given in [R], Theorem 2.3: the normalized sheaf F is a vector bundle if and only if $h^2(F(p)) = 0$ for some $p \leq -2$; for a general sheaf this condition becomes: $h^2(F(p)) = 0$ for some $p \leq -2 + \varepsilon$.

If

$$h^2(F(p)) = 0 \quad \text{for} \begin{cases} p = \varepsilon - 3 & \text{if } c_1 \text{ is even,} \\ p = \varepsilon - 4 \text{ or } \varepsilon - 3 \text{ or } \varepsilon - 2 & \text{if } c_1 \text{ is odd} \end{cases}$$

then the sheaf is a split bundle ([R], Cor. 3.2).

9. The dual F^\vee of F is the reflexive sheaf $F(-c_1)$ ([H1], #2 and [H2], Prop. 1.10: recall that the first Chern class can be defined through the determinant itself). Moreover, if F is a vector bundle, Serre's duality says that $h^i(F(n)) = h^{3-i}(F(-n - c_1 - 4))$ for every n and $i = 0, 1$. If F is an arbitrary reflexive sheaf, then the above equality holds only for $i = 0$ ([H2], Prop. 2.5), while for $i = 1$ we have the following exact sequence

$$(**) \quad 0 \rightarrow H^1(F(-n - c_1 - 4)) \rightarrow H^2(F(n)) \rightarrow H^0 \text{Ext}^1(F(n), \mathcal{O}_{\mathbf{P}}(-4)) \\ \rightarrow H^2(F(-n - c_1 - 4)) \rightarrow H^1(F(n)) \rightarrow 0$$

together with the equality: $h^0 \widetilde{\text{Ext}}^1(F(n), \mathcal{O}_{\mathbf{P}}(-4)) = c_3$ (see [H2], Prop. 1.10, Th. 2.5, claim and proof of Prop. 2.6, Cor. 2.2: the sequence (**)) is given with duals but $F^\vee = F(-c_1)$).

For a reflexive sheaf, we introduce, for every integer n in \mathbf{Z} , the number $d_n = d_n(F) = h^2(F(n)) - h^1(F(n))$ (see also [BH], n.3). For $n \ll 0$, $d_n(F) = h^2(F(n))$, because $h^1(F(n)) = 0$. If F is a vector bundle, then $d_n = h^1(F(-n - c_1 - 4)) - h^1(F(n))$, because of Serre's duality (not true in general for a reflexive sheaf); therefore, $d_n(F) = 0$ both for $n \ll 0$ and for $n \gg 0$.

We start with the following key lemma:

LEMMA 1. *Let F be a nonsplit reflexive sheaf.*

- (i) *if F is normalized and $n \leq 0$, then $h^0(F(n)) = h^0(\mathcal{O}_{\mathbf{P}}(n - a))$;*
- (ii) *if F is general and $n + a + c_1 \leq 0$, $h^0(F(n)) = h^0(\mathcal{O}_{\mathbf{P}}(n - a))$;*
- (iii) *if F is general and $n + a + c_1 > 0$, then $h^0(F(n)) = \chi(F(n)) - d_n$.*

Proof. (i) if $a > 0$, then $n < a$, so both members are 0. If $a \leq 0$, take a curve Y corresponding to the sheaf $F(a)$, getting the exact sequence (see above)

$$0 \rightarrow \mathcal{O}_{\mathbf{P}} \rightarrow F(a) \rightarrow \mathbf{I}_Y(2a + c_1) \rightarrow 0$$

and also (twisting by $n - a$)

$$0 \mapsto \mathcal{O}_{\mathbb{P}}(n - a) \mapsto F(n) \mapsto \mathbf{I}_Y(n + a + c_1) \mapsto 0.$$

It is now enough to remark (because $c_1 \leq 0$ for a normalized sheaf that $n + a + c_1 \leq 0$, which implies that no surface of degree $n + a + c_1$ contains Y).

(ii) The same exact sequence as above works: now c_1 is arbitrary but $n + a + c_1 \leq 0$ by hypothesis.

(iii) We have: $h^0(F(n)) = \chi(F(n)) - d_n + h^0(F(-n - c_1 - 4))$, but the last term is 0 because $-n - c_1 - 4 < a$ by hypothesis.

We want to show that the knowledge of the numbers d_n gives some information about F , i.e. about the Chern classes, the number $a(F)$, the property of being stable or unstable,

First of all we give a criterion to decide whether F is stable or not, based on the knowledge of the numbers d_n 's and of c_1 , hence of the normalized sheaf $F^n = F(\varepsilon)$ (see above). We recall that, for every n ,

$$d_n(F) = h^2(F(n)) - h^1(F(n)) = h^2(F(\varepsilon)(n - \varepsilon)) - h^1(F(\varepsilon)(n - \varepsilon)) = d_{n-\varepsilon}(F(\varepsilon)) = d_{n-\varepsilon}(F^n).$$

PROPOSITION 1. *Let F be a nonsplit reflexive sheaf and n, m two integers. Then*

$$\begin{aligned} &nd_{\varepsilon} + md_{-1+\varepsilon} - (2m + 3n)d_{-2+\varepsilon} + (m + 2n)d_{-3+\varepsilon} \\ &= \begin{cases} 2n + (m + 3n)(c_1 + 2\varepsilon) & (\text{stable sheaf}), \\ n & (\text{properly semistable sheaf}), \\ 0 & (\text{unstable sheaf}). \end{cases} \end{aligned}$$

Proof. First we prove the claim when F is normalized, i.e. when $\varepsilon = 0$. By Lemma 1 we have, for $h \leq 0$:

$$h^0(F(h)) = h^0\mathcal{O}_{\mathbb{P}}(h - a).$$

Moreover, if $h \geq -3$, then $h^3(F(h)) = h^0(F(-h - c_1 - 4)) = h^0\mathcal{O}_{\mathbb{P}}(-h - c_1 - 4 - a)$ because $-h - c_1 - 4 \leq 0$. Substitute now d_h with $\chi F(h) - h^0F(h) + h^3F(h)$ in $nd_0 + md_{-1} - (2m + 3n)d_{-2} + (m + 2n)d_{-3}$ and use Lemma 1 and duality; if we put: $\varphi(n) = n + 1$ for $n \geq 0$, $\varphi(n) = 0$ for $n < 0$, we get:

$$\begin{aligned} &2n + (3n + m)c_1 + n(-\varphi(-a) - 2\varphi(-a - 1) + 2\varphi(-a - 1 - c_1) \\ & \qquad \qquad \qquad + \varphi(-a - 2 - c_1)) \\ & + m(\varphi(-a - c_1 - 1) - \varphi(-a - 1)). \end{aligned}$$

Therefore, if F is stable ($a > 0$) the result is $2n + (3n + m)c_1$; if it is property semistable ($a = 0$ and $c_1 = 0$) the result is n ; if it is unstable the result is 0.

Assume now that F is general; then it is enough to remark that $d_{n+\varepsilon}(F) = d_n(F^n)$ for all n and moreover $c_1 + 2\varepsilon = c_1(F(\varepsilon))$ ([H2], Cor. 2.2).

PROPOSITION 2. *Let F be a nonsplit rank 2 reflexive sheaf on \mathbf{P} . If it is stable then $a - \varepsilon$ is the smallest positive integer n such that*

$$d_{n+\varepsilon} - (n + 2)d_{-1+\varepsilon} + (n + 1)d_{-2+\varepsilon} \neq 2\binom{n + 3}{3} + (c_1 + 2\varepsilon)\binom{n + 2}{3}.$$

If it is unstable then $-a + \varepsilon - c_1$ is the smallest positive integer n such that $d_{n+\varepsilon} - (n + 2)d_{-1+\varepsilon} + (n + 1)d_{-2+\varepsilon} \neq 0$.

Proof. First assume that F is normalized ($\varepsilon = 0$), that $a > 0$ and $c_1 = 0$. Then we have (for any $n \geq 0$): $d_n = \chi(F(n)) - h^0(F(n))$, by Lemma 1, (iii); therefore

$$\begin{aligned} d_n - (n + 2)d_{-1} + (n + 1)d_{-2} &= \chi(F(n)) - h^0(F(n)) - (n + 2)(\chi(F(-1))) \\ &= n^3/3 + 2n^2 + 11n/3 + 2 - (n + 2)c_2 - h^0(F(n)) - (n + 2)(-c_2) \\ &= 2\binom{n + 3}{3} - h^0(F(n)) \end{aligned}$$

which gives the result because the sheaf has no nonzero section up to $n = a - 1$ and at least one nonzero section for $n = a$.

If $c_1 = 0$ and $a < 0$, then, for all n such that $0 \leq n \leq -a$, we have:

$$\begin{aligned} d_n - (n + 2)d_{-1} + (n + 1)d_{-2} &= \chi(F(n)) - h^0(F(n)) + h^0(F(-n - 4)) \\ &\quad - (n + 2)(\chi(F(-1)) - h^0(F(-1)) + h^0(F(-3))). \end{aligned}$$

Since $0 \leq n \leq -a$, then we are able to use Lemma 1, (i) and (ii), to see that the second member is simply

$$\begin{aligned} 2\binom{n + 3}{3} - \binom{n - a + 3}{3} + h^0\mathcal{O}_{\mathbf{P}}(-n - a - 4) \\ + (n + 2)(h^0\mathcal{O}_{\mathbf{P}}(-3 - a) - h^0\mathcal{O}_{\mathbf{P}}(-1 - a)) \end{aligned}$$

which is easily seen to be 0 if $n < -a$ and 1 if $n = -a$.

The case $c_1 = -1$ can be dealt with using very similar arguments.

Now pass to F arbitrary: it is enough to recall that $d_n(F) = d_{n-\varepsilon}(F(\varepsilon))$ and $c_1(F(\varepsilon)) = c_1 + 2\varepsilon$ ([H2], Cor. 2.2).

REMARK 1. It is not evident at all that the same sequence $\{d_n\}$ for two different reflexive sheaves gives rise to the same a ; actually this is true, because the d_n 's determine also c_1 , as it will be seen in Theorem 1.

COROLLARY 1. *A reflexive sheaf F splits if and only if $d_n = 0$ for all n .*

Proof. If F is split, then it is well known that the intermediate cohomology vanishes.

Conversely, assume that all the d_n 's vanish and F does not split; then, by Prop. 1, F must be an unstable sheaf; now use Prop. 2 to see that $d_n \neq 0$ either for $n = -a + 2\varepsilon - c_1$ or for $n = -2 + \varepsilon$ or for $n = -1 + \varepsilon$. This is a contradiction.

REMARK 2. The same result could be proved using the nonsplitting criteria of [R] (see 8 above).

PROPOSITION 3. *Let F be a nonsplit rank 2 reflexive sheaf on \mathbf{P} . Then we have:*

(i) *if F is stable or properly semistable, then $c_2 = -\varepsilon^2 - c_1\varepsilon + d_{-2+\varepsilon} - d_{-1+\varepsilon}$;*

(ii) *if F is unstable, then $c_2 = -\varepsilon^2 - c_1\varepsilon + d_{-2+\varepsilon} - d_{-1+\varepsilon} - (a - \varepsilon)^2 - (a - \varepsilon)(c_1 + 2\varepsilon) = -a^2 - ac_1 + d_{-2+\varepsilon} - d_{-1-\varepsilon}$.*

Proof. First we assume that F is normalized ($\varepsilon = 0$). If $a \geq 0$, then $c_2 = \chi(F(-2)) - \chi(F(-1)) = d_{-2} - d_{-1}$. If $a < 0$, then we have:

$$\begin{aligned} d_{-2} - d_{-1} &= \chi(F(-2)) - \chi(F(-1)) - h^0(F(-2)) + h^0(F(-2 - c_1)) \\ &\quad + h^0(F(-1)) - h^0(F(-3 - c_1)) \\ &= c_2 - h^0(\mathcal{O}_{\mathbf{P}}(-2 - a)) + h^0(\mathcal{O}_{\mathbf{P}}(-2 - a - c_1)) \\ &\quad + h^0(\mathcal{O}_{\mathbf{P}}(-a - 1)) - h^0(\mathcal{O}_{\mathbf{P}}(-a - 3 - c_1)) \\ &= c_2 + a^2 + ac_1 + (c_1^2 + c_1)/2 = c_2 + a^2 + ac_1. \end{aligned}$$

If the sheaf is arbitrary we pass to $F(\varepsilon)$ and recall that $c_2(F(\varepsilon)) = c_2(F) + c_1(F)\varepsilon + \varepsilon^2$ ([H2], Cor. 2.2).

REMARK 3. The previous results show that, if we have the sequence $\{d_n\}$, for all n , and c_1 (hence ε), then there is an algorithm to compute explicitly both a and c_2 .

LEMMA 2. *Let F be a nonsplit rank 2 reflexive sheaf; then*

- (i) $d_n = 0$ for all $n \gg 0$;
- (ii) $d_n = c_3$ for all $n \ll 0$.

Proof. The intermediate cohomology for any coherent sheaf is 0 for $n \gg 0$; therefore (i) is obvious. For (ii) we remark that the first cohomology module is finite, hence 0 for $n \ll 0$ and then use the exact sequence (**) of n.1, which for $n \ll 0$ has only two nonzero terms ($h^2(F(n))$ and c_3).

By the previous lemma and Corollary 1 every nonsplit reflexive sheaf F has two associated numbers (depending upon $\{d_n\}$: if all the d_n 's are 0, then they are not defined):

$$p = p(F) = \min\{n/d_m = 0 \ \forall m \geq n\};$$

$$q = q(F) = \max\{n/d_m = c_3 \ \forall m \leq n\}.$$

Now we show some relations between the d_n 's, the Chern classes, $a(F)$ and $h^0(F(n))$, for all n . Actually we see that, if the sequence $\{d_n\}$ is known for all n , then all the numerical characters are explicitly computable from it.

THEOREM 1. *Let F be a nonsplit rank 2 reflexive sheaf. Then we have:*

- (i) $c_3 = d_q$;
- (ii) $c_1 = -p - q - 4$;
- (iii) c_2 is uniquely determined by c_1 and $\{d_n\}$ as in Proposition 3;
- (iv) a is uniquely determined by c_1 and $\{d_n\}$ as in Proposition 2;
- (v) $h^0(F(n))$ is uniquely determined by c_1 and $\{d_n\}$ as in Lemma 1, (ii) and (iii).

Proof. (i) follows from Lemma 2.

For (ii) we observe that, by the above exact sequence (**) of n.1, we have:

$$c_3 = d_n + d_{-n-c_1-4};$$

therefore

$$d_{-q-c_1-4+r} = c_3 - d_{q-r} = \begin{cases} 0 & \text{for } r \leq 0, \\ t \neq 0 & \text{for } r > 0, \end{cases}$$

which means (by the definition itself) that $p = -q - c_1 - 4$.

For (iii), (iv), (v), we remark that, by (ii), p and q determine c_1 , hence also ε .

REMARK 4. The equality

$$c_3 = d_n + d_{-n-c_1-4}$$

shows that the pairs (n, d_n) are symmetric with respect to $(-(c_1 + 4)/2, c_3/2)$.

For a rank 2 nonsplit vector bundle we know that, by Serre's duality, $h^2(F(n)) = h^1(F(-n - c_1 - 4))$; moreover $c_3 = 0$ ([H2], Prop. 2.6). So the previous result becomes as follows:

COROLLARY 2. *Let F be a nonsplit rank 2 vector bundle over \mathbf{P} . Then the sequence $\{h^1(F(n))\}$ and the Chern class c_1 determine uniquely the sequence $\{d_n\}$, hence c_2 , a and $h^0(F(n))$, for all n .*

REMARK 5. In [GRV] these results were already obtained, but only for a bundle G having the same sequence $\{h^1(G(n))\}$ as a bundle associated to a skew union of complete intersection.

n.2. Corollary 2 in n.1 leads us to raise the following question: is it enough to know the sequence $\{h^1(F(n))\}$, all n , in order to get also c_1 ? We do not know in general, but we discuss some examples and partial results.

EXAMPLE 1. If there is m such that $h^1(F(n)) = 0$ for $n \neq m$, $h^1(F(m)) \neq 0$, then the sheaf $F(m+1)$ is a nullcorrelation bundle. In fact, if the normalized bundle $F(\varepsilon) = E$ is not stable, that is, $a(E) \leq 0$, then $h^1(E(n)) \neq 0$ for $a-2 \leq n \leq a_1-2$ (see n.1,6; see also [S], Th. 3.1 and Prop. 3.2), which includes at least 2 values.

Therefore the bundle is stable. In this case either the 1-cohomology vanishes at 2 values of n or it is a nullcorrelation bundle (see [CV1], Prop. 4).

PROPOSITION 4. *Let F be a rank 2 vector bundle such that $h^1(F(n))$ does not vanish exactly at 2 values $n = m$ and $n = p > m$; then we have:*

the bundle is stable;

$$p = m + 1;$$

$$c_1 = -2m - 2;$$

$$c_2 = 2 + (m + 1)^2;$$

$$h^1(F(n)) = \begin{cases} 2 & \text{for } n = m, \\ 2 & \text{for } n = m + 1 \end{cases}$$

(the same dimensions as a twist of the bundle associated to 3 skew lines).

Proof. by [CV1], Th. 1.3, 1.5 and 1.6 and [CV2], Cor. 8 (see above, n.1, 6), the hypothesis implies that the bundle has even c_1 , i.e. the normalized bundle $F^n = E$ has $c_1 = 0$, the 1-cohomology cannot vanish exactly at -1 and at 0 and $a(E) - 2 \geq -1$, which means that the bundle is stable. Now we are able to compute explicitly such a cohomology at both levels where it does not vanish. Working on the normalized bundle E , with $c_1 = 0$, $c_2 > 1$ (by [H1], Lemma 3.2), we have:

$$\begin{aligned} \chi(E) &= -h^1(E) = 2 - 2c_2 < 0; \\ \chi(E(-1)) &= -h^1(E(-1)) = -c_2; \\ \chi(E(1)) &= h^0(E(1)) = 8 - 3c_2 > 0. \end{aligned}$$

Therefore c_2 must be either 1 or 2, and hence $c_2 = 2$.

Now we go back to the original bundle and find that the first non-vanishing level must be $m = -c_1/2 - 1$, whence the expected value $c_1 = 2(-m - 1)$. The claim about c_2 follows from the general formula connecting the second Chern class of a bundle and of a twist.

PROPOSITION 5. *Let F be a nonsplit rank 2 vector bundle such that $c_1 = 0$ and moreover $h^1(F(n))$ does not vanish exactly for three values of n . Then:*

- (i) $h^1(F(n)) \neq 0$ for $n = -1, 0, 1$;
- (ii) F is stable;
- (iii) $3 \leq c_2 \leq 5$ and moreover

$$h^1(F(n)) = \begin{cases} c_2 & \text{for } n = -1, \\ 2c_2 - 2 & \text{for } n = 0, \\ t & \text{for } n = 1, \end{cases}$$

where $t = 3c_2 - 8 + h^0(F(1))$ and $h^0(F(1)) \leq 5 - c_2$.

Proof. By [CV1], Prop. 15 (see n.1, 6), the cohomology is not 0 at $n = -1$ and at $n = 0$ (to avoid a nullcorrelation bundle, with only one non vanishing group). Moreover, the cohomology does not vanish between $a - 2$ and $a_1 - 2$; since $a_1 > 0$ (Remark 1), then $a \geq 0$. Now put: $c_2(F) = c$. If $a = 0$ the three vanishing groups correspond to $n = -2, -1, 0$. Then we have: $\chi(F(1)) = h^0(F(1)) = 8 - 3c \geq 4$, which means either $c = 0$ or $c = 1$; but both cases are absurd, because F has a section which, being the first one, gives rise to a curve of degree c (see [H1], Lemma 1.3).

At last $a > 0$ and the bundle is stable. Now we want to show that the third nonvanishing group corresponds exactly to $n = 1$. In fact, assume that $h^1(F(1)) = 0$; then we have: $\chi(F(1)) = 8 - 3c = h^0(F(1)) + h^1(F(-5)) \geq 0$, which implies $c \leq 2$: but $c \leq 1$ is excluded because the bundle is stable ([H1], Lemma 3.2) and $c = 2$ is excluded by [H1], Lemma 9.4 (because there should be only two nonvanishing groups). Hence the three nonvanishing groups correspond to $n = -1, 0, 1$ and the dimensions can be computed by $\chi(F(-1)) = -c$, $\chi(F) = 2 - 2c$, $\chi(F(1)) = 8 - 3c = h^0(F(1)) - h^1(F(1))$. By $\chi(F(2)) = h^0(F(2)) = 20 - 4c$ we see that $20 - 4c \geq 0$, hence $c \leq 5$; moreover $c \geq 3$, because 2 and 1 are excluded by [H1], Lemma 3.2 and Lemma 9.4)

For the value of t it is enough to observe that $h^0(F(2))$ is 0 for $c = 5$ and so $h^0(F(1)) = 0$; if $c = 4$, then $h^0(F(2)) = 4$. Hence $F(1)$ has at most one nonzero section; if $c = 3$, then $h^0(F(2)) = 8$ and $F(1)$ has at most two independent sections.

PROPOSITION 6. *Let F be a nonsplit rank 2 vector bundle such that $c_1 = -1$ and moreover $h^1(F(n))$ does not vanish exactly at three values of n . Then*

- (i) $h^1(F(n)) \neq 0$ for $n = -1, 0, 1$;
- (ii) $c_2 = 2$;
- (iii) F is stable;
- (iv)

$$h^1(F(n)) = \begin{cases} 1 & \text{for } n = -1, \\ 2 & \text{for } n = 0, \\ 1 & \text{for } n = 1. \end{cases}$$

Proof. First of all put: $c = c_2$. By [CV1], Th. 1.6 see n.1, 6), the three nonvanishing groups must correspond to $n = -1, 0, 1$; therefore $a - 2 \geq -2$ and $a \geq 0$ (see n.1, 6). If $a = 0$, then, because $\chi(F(2)) = h^0(F(2)) = 14 - 7c/2 \geq 10$, we see that $c < 8/7$. Hence $c = 0$ (because c must be odd: [H1], Cor. 2.2); but this is absurd because it is the degree of a curve, zero locus of a section of the bundle. Hence $a > 0$ and the bundle is stable. If $a = 1$ and $h^0(F(1)) > 1$, then there should exist a curve, zero locus of a section of $F(1)$, lying on a plane, which is absurd because it cannot be complete intersection as the bundle is not split ([H1], Cor. 1.2). Therefore either $a \geq 2$ or $a = 1$ and there is only one independent section for $n = a = 1$. Now we observe that $\chi(F(1)) = 5 - 5c/2$, $\chi(F(-2)) = c/2 = h^1(F(-1))$, $\chi(F) = 1 - 3c/2 = -h^1(F)$, $\chi(F(2)) = h^0(F(2)) = 14 - 7c/2$; hence

we see that $c \leq 4$ and therefore either $c = 2$ or $c = 4$ (c being even for an odd first Chern class). If $c = 2$, then we have the following cohomology: $h^1(F(-1)) = 1$, $h^1(F) = 2$, $h^1(F(1)) = 1$ ($F(2)$ gives rise to two skew conics). The case $c = 4$ is not allowed because it implies $a \geq 3$ and hence natural cohomology, which is excluded by [HH], Ex. 1.6.3.

THEOREM 2. *Let F be a nonsplit rank 2 vector bundle such that $h^1(F(n))$ does not vanish exactly at 3 values of n ; then we have:*

- (i) *the 1-cohomology does not vanish at 3 consecutive values m , $m + 1$, $m + 2$;*
- (ii) *F is stable;*
- (iii) *setting $(a, b, c) = (h^1(F(m)), h^1(F(m+1)), h^1(F(m+2)))$: either $(a, b, c) = (1, 2, 1)$ and then $c_1 = -2m - 3$ or $(a, b) = (h, 2h - 2)$, with $3 \leq h \leq 5$, and then $c_1 = -2m - 2$, $3h - 8 \leq 2h - 3$. Furthermore, each of these seven cases actually occurs.*

Proof. (i) The three values are consecutive for a normalized bundle, by Propositions 5 and 6; hence for every bundle.

(ii) The property is true for a normalized bundle (Propositions 5 and 6), hence for all bundles.

(iii) The case $(a, b, c) = (1, 2, 1)$ is considered in Proposition 6, and corresponds to an odd c_1 , while the case $(a, b) = (h, 2h - 2)$ is considered in Proposition 5 and corresponds to an even c_1 .

If c_1 is even, then the normalized bundle $F(-c_1/2)$ has nonvanishing cohomology at $n = -1, 0, 1$ by Proposition 5; therefore we see that $m = -c_1/2 - 1$. If c_1 is odd, then the normalized bundle $F(-(c_1 + 1)/2)$ has nonvanishing cohomology at $n = -1, 0, 1$ by Proposition 6; therefore $m = -(c_1 + 1)/2 - 1$.

Now we see that all the seven cases occur.

For $c_1 = 0$, the three values for c_2 are 3, 4, 5; then

—if $c_2 = 3$, the sequences (3,4,2) and (3,4,3) for the 1-cohomology correspond to 4 skew lines lying either on one or no quadric surface;

—if $c_2 = 3$, the sequence (3,4,1) corresponds to an instanton bundle with $a \leq 2$ ([H1], Th. 8.2 and [HH], examples of n.1);

—if $c_2 = 4$, 5 skew lines give rise to the sequence (4,6,5), but also the sequence (4,6,4) is allowed ([HH], n.1);

—if $c_2 = 5$, the sequence (5,8,7) exists and corresponds to a bundle with natural cohomology (again [HH], n.1).

As long as c_1 is odd, such a bundle corresponds to a skew union of two conics ([HH], Ex. 1.6.2).

REMARK 6. If the bundle has exactly 4 nonvanishing groups, then it is still possible to describe everything (the four values are consecutive, c_1 even and odd are not allowed together and can be explicitly computed by the position of the nonvanishing cohomology, ...), like in the previous theorem, but it is required, beside all the previous machinery, also the classification of stable bundles with $c_1 = -1$ and $c_2 = 2$ (see [HS]).

For five nonvanishing groups there is a case that cannot be easily excluded; hence for $n \geq 5$ the difficulty increases, because of the lack of classification for normalized bundles. Therefore our technique cannot work for arbitrary n (even if it is not excluded that some extra n can be dealt with).

EXAMPLE 2. Let F be a vector bundle with natural cohomology ([HH]) and $c_1(F) = -1$. Then $h^1(F(n)) \neq 0$ for $n = -1, 0, 1$ by [CV1], Th. 1.6 (n.1, 6). Therefore $h^2(F(-1)) = h^0(F(-1)) = h^2(F) = h^0(F) = 0$, because of the natural cohomology; hence F is stable ($a \geq 1$). Moreover $h^2(F(-1)) = 0 = h^1(F(-2))$ implies that F is an instanton bundle, which means that $h^1(F(n)) = 0$ for all $n \leq -2$ ([HH], Prop. 1.4). We have also:

$$\begin{aligned} h^1(F(-1)) &= -\chi(F(-1)) = c_2(F)/2, \\ h^1(F) &= -\chi(F) = 3c_2(F)/2 - 1. \end{aligned}$$

Now let G be any bundle such that $c_1(G) = 0$ and $h^1(G(n)) = h^1(F(n))$ for all n . By Proposition 2, with $n = 1, m = 0$, we have:

$$\begin{aligned} 2h^1(G(-1)) - h^1(G) &= 2h^1(F(-1)) - h^1(F) = 1 - c_2(F)/2 \\ &= \begin{cases} 2 & (G \text{ stable}), \\ 1 & (G \text{ properly semistable}), \\ 0 & (G \text{ unstable}). \end{cases} \end{aligned}$$

So $c_2(F) = 2$, because F is stable (hence $c_2 > 0$ by [H1], Lemma 3.4). This implies that $h^1(F(-1)) = 1, h^1(F) = 2$, while $h^1(F(1)) = -\chi(F(1)) = -5 + 5c_2(F)/2 = 0$, which is absurd (n.1, 6).

This means that no bundle G with $c_1(G) = 0$ can have $h^1(G(n)) \stackrel{\cong}{=} h^1(F(n))$, for all n .

Using the same techniques as above one can show that, if F is a bundle with natural cohomology and $c_1(F) = 0$, then no bundle G with $c_1(G) = -1$ can have $h^1(G(n)) = h^1(F(n))$, for all n .

EXAMPLE 3. It is possible that two normalized vector bundles have the same positive spectrum (see [H2], n.7), hence the same dimensions of the first cohomology groups for every $n \leq 0$, but different first Chern classes. For instance ([H2], n.8) take the skew union Y of a plane cubic and a complete intersection of two quadrics, which gives rise to a bundle F ; then $E = F(-2)$ has first Chern class 0 and spectrum $(-1, 0, 1)$. If we take the skew union X of a plane quartic and the complete intersection of a cubic and a quadric, then X gives rise to a vector bundle which, twisted by -3 , has first Chern class -1 and has spectrum $(-2, -1, 0, 1)$. It is easy to see that the first cohomology modules have different dimensions when $n > 0$.

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UNIVERSITÀ DI TORINO
VIA CARLO ALBERTO, 10
10121 TORINO, ITALY

AND

POLITECNICO DI TORINO
CORSO DUCA DEGLI ABRUZZI, 24
10129 TORINO, ITALY

