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EMBEDDING A 2-COMPLEX K IN $\mathbb{R}^4$ WHEN $H^2(K)$ IS A CYCLIC GROUP

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We prove that every finite 2-dimensional cell complex with cyclic second cohomology embeds in $\mathbb{R}^4$ tamely.

1. Introduction. It has long been known that every compact PL (piecewise-linear) manifold embeds in euclidean space of double dimension. The analogous result, however, is not true for arbitrary simplicial complexes (see [**2**]). In [**6**] an obstruction to embedding n-complexes in $\mathbb{R}^{2n}$ was found. Since that obstruction is not homotopy invariant and is in general difficult to calculate, it is natural to ask if a certain class of n-complexes which can be easily described embeds in $\mathbb{R}^{2n}$. It has been known that every n-complex with cyclic nth cohomology embeds in $\mathbb{R}^{2n}$ if $n \neq 2$ (see [**5**]). If $n > 2$ one can use the techniques of [7] to prove it. The same techniques are much harder to apply when $n = 2$ and if they are successful they yield embeddings which are not smooth but only tame on each 2-cell (recall that an embedding $D^2 \to \mathbb{R}^4$ is tame if it can be extended to an embedding $D^2 \times D^2 \to \mathbb{R}^4$). At present the author does not even know whether every contractible 2-complex embeds in $\mathbb{R}^4$ piecewise smoothly.

In [**4**] it was shown that the case $n = 2$ really is different from other dimensions (§3). Here we establish a result analogous to other dimensions.

THEOREM. *If K is a finite 2-complex such that $H^2(K)$ is cyclic then K can be embedded in $\mathbb{R}^4$.*

Note. All homology and cohomology groups will be with integer coefficients; Z denotes the ring of integers.

The case $H^2(K) = 0$ was proved in [**4**]. The general case can be reduced to the case when $H^2(K)$ is infinite cyclic. This case is basically in two steps. First it is proved for the case when $H_2(K)$ is generated by an embedded orientable surface. For arbitrary K with $H^2(K) = Z$ the situation is reduced to the previous case by constructing a tower of maps and 2-complexes

$$K_r \xrightarrow{p_r} K_{r-1} \xrightarrow{p_{r-1}} \cdots \xrightarrow{p_1} K_1 \xrightarrow{p_0} K_0 = K$$

such that K_{j-1} can be embedded in $\mathbb{R}^4$ if K_j can and such that K_r embeds in $\mathbb{R}^4$.

In what follows all embeddings of K in $\mathbb{R}^4$ will be smooth in the interior of each cell except for a finite number of points in the interiors of 2-cells where they will still be tame. Thus if we construct such an embedding of a subdivided K it will still be tame on the original K. Therefore we can assume without loss of generality whenever it is convenient that K is either a simplicial complex or that all the attaching maps are homeomorphisms.

2. A special case. In what follows K will be a finite connected 2-complex.

LEMMA 1. *Suppose $H^2(K) = Z$ and suppose that $H_2(K)$ is generated by an embedded orientable surface $F \subset K$. Then K can be embedded in $\mathbb{R}^4$.*

Proof. Let e_0 be a 2-cell of F. Then the inclusion $(K - \text{int}(e_0), F - \text{int}(e_0)) \subset (K, F)$ gives rise to the following commutative diagram

$$\begin{array}{ccccccc} H^2(K, F) & \longrightarrow & H^2(K) & \longrightarrow & H^2(F) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ H^2(K - \text{int}(e_0), F - \text{int}(e_0)) & \longrightarrow & H^2(K - \text{int}(e_0)) & \longrightarrow & 0 & & \end{array}$$

in which both rows are exact. Since $H^2(K) \to H^2(F)$ is an isomorphism the first homomorphism in the top row is trivial. The first vertical map is an isomorphism (by excision); therefore the first homomorphism in the bottom row is also trivial. This implies that $H^2(K - \text{int}(e_0))$ is 0.

By attaching 2-cells to $K - \text{int}(e_0)$ we can obtain an acyclic 2-complex L. Denote $L \cup e_0$ again by K. Clearly if this K can be embedded in $\mathbb{R}^4$ so can the original 2-complex.

Choose an embedding of $F \cup K^{(1)}$ in $\mathbb{R}^3 \times 0 \subset \mathbb{R}^4$ which is smooth on F and on each edge of K. Identify $F \cup K^{(1)}$ with its image under this embedding. Then $F \cup K^{(1)} \subset \mathbb{R}^3 \times 0$. Let $H \times 0$ be a regular neighborhood of $K^{(1)}$ in $\mathbb{R}^3 \times 0$. $H \times 0$ is a handlebody with spine $K^{(1)}$. There is a natural projection $p: \partial(H \times 0) \to K^{(1)}$ such that $H \times 0$ is the mapping cylinder of p. Thus every point in $H \times 0$ can be thought of as a class $[x, t]$ where $x \in \partial H$, $t \in I$ $(= [0, 1])$, and $[x, 1] = p(x)$. let $\hat{p} : H \to K^{(1)}$ be defined by $\hat{p}([x, t]) = p(x)$.

Let U be a regular neighborhood of $K^{(1)}$ in K. ∂U is a union of circles $C_0, \dots, C_g$ where C_i corresponds to the 2-cell e_i of K and where g is the genus of H (because L is acyclic). Suppose $\partial U \cap F = C_0 \cup \cdots \cup C_k$. Orient F and assume that $C_0, \dots, C_k$ have the induced orientation. Also choose orientations for the curves $C_{k+1}, \dots, C_g$. U and H can be chosen in such a way that $(H \times 0) \cap F = U \cap F$ and so that $U \cap F = \hat{p}^{-1}(p(\partial U \cap F)) = \{[x, t] \in H \times 0;\ x \in \partial U \cap F,\ t \in I\}$. Embed $C_{k+1} \cup \cdots \cup C_g$ smoothly in $H \times 1$ in such a way that $p|C_j : C_j \to K^{(1)}$ is the attaching map for e_j. Let $U_j = \{([x, t], 1 - t) \in H \times [-1, 1] | x \in C_j,\ t \in I\}$. U_j is an embedding of the collar of e_j into $H \times [0, 1]$. $(\bigcup_{j=k+1}^{g} U_j) \cup (F \cap H \times 0)$ is an embedding of U into $H \times [-1, 1]$ which we can assume to be piecewise smooth.

Since L is acyclic, $C_1, \dots, C_g$ form a basis for $H_1(\partial(H \times [-1, 1]))$. Let T be a maximal tree of $K^{(1)}$ and let $s_1, \dots, s_g$ be the edges of $K^{(1)} - T$. If m_i is the midpoint of s_i let

$$S_i = (\hat{p}^{-1}(m_k) \times \{-1, 1\}) \cup p^{-1}(m_i) \times [-1, 1] \subset \partial(H \times [-1, 1]).$$

Then S_i is an embedded 2-sphere. Choose an orientation for S_i. For each $i = 1, \dots, g$ choose an oriented simple closed curve a_i in $\partial(H \times [-1, 1])$ such that $a_i \cdot S_j = \delta_{ij}$. Then $\{a_1, \dots, a_g\}$ is a basis for $H_1(\partial(H \times [-1, 1]))$. Suppose $C_i \sim \sum p_{ij} a_j$, $i = 1, \dots, g$, in $\partial(H \times [-1, 1])$ ($\sim$ stands for homologous). Then $\det(p_{ij}) = \pm 1$. Let Σ_i' be a union of suitably oriented disjoint copies of spheres $S_1, \dots, S_g$ representing the class $\sum_{j=1}^{g} q_{ij}[S_j]$ in $H_2(\partial(H \times [-1, 1]))$ where $(q_{ij}) = (p_{ji})^{-1}$. Then

$$C_i \cdot \Sigma_j' = \sum_{k,l} p_{ik} q_{jl} a_k \cdot S_l = \sum_{k=1}^{g} p_{ik} q_{jk} = \delta_{ij}.$$

The intersection number $\Sigma' \cdot F$ is zero (it is the intersection of closed orientable surfaces in $\mathbb{R}^4$). Since $\Sigma_i' \cap F = \Sigma_i' \cap (C_0 \cup \cdots \cup C_k)$, the intersection number $\Sigma_j' \cdot (C_0 \cup \cdots \cup C_k)$ in $\partial(H \times [-1, 1])$ is also zero. Since $\Sigma_i' \cdot (C_1 \cup \cdots \cup C_k) = 0$, for $i > k$, it follows that $\Sigma_i' \cdot C_0 = 0$, for $i > k$. Therefore we can pipe together the intersections of Σ_i' with C_j, $j = 0, \dots, g$, along $C_0 \cup \cdots \cup C_g$ to obtain for each $i > k$ a surface $\Sigma_i'' \subset \partial(H \times [-1, 1])$ such that $\Sigma_i'' \cap F = \varnothing = \Sigma_i'' \cap C_j$, $i \neq j$, and such that $\Sigma_i'' \cap C_i$ is a point. Since all the "pipes" lie either in $H \times 1$ or in a neighborhood of $\partial H \times 0$ in $\partial(H \times [-1, 1])$, one can choose half of a symplectic basis for each $H_1(\Sigma_i'')$, $i > k$, represented by smooth simple closed curves in $\partial H \times (0, 1] \cup H \times 1$. Since $M' = \mathbb{R}^3 \times [0, \infty) - \text{int}(H \times [-1, 1])$ is simply connected, we can cap off these curves by regularly immersed discs in M'. By performing surgeries along these discs change each Σ_i'', $i > k$, into a singular 2-sphere Σ_i. All the singularities lie in M'. Furthermore, $\Sigma_i \cap (U \cup F) = \Sigma_i' \cap C_i$ is a point. Note also that $\Sigma_i \cap \Sigma_j \cap \text{int}(H \times [-1, 1]) = \varnothing$, and that $\Sigma_i \cdot \Sigma_j = 0$, for $i \neq j$, $i, j > k$.

Cap off the curves $C_{k+1}, \dots, C_g$ by regularly immersed discs $D'_{k+1}, \dots, D'_g$, respectively, lying in $\mathbb{R}^3 \times [1, \infty)$. This extends the embedding of $F \cup U$ to a regular immersion of K into $\mathbb{R}^4$. Since $D_i' \cdot \Sigma_j = \delta_{ij}$ for all $i, j > k$, we can use the spheres Σ_j to pipe off the intersections between the discs $D'_{k+1}, \dots, D'_g$, in order to get immersed discs $D_{k+1}, \dots, D_g$, respectively, such that $D_i \cdot D_j = 0$, for $i \neq j$. Again $\Sigma_i \cdot D_j = \delta_{ij}$, for $i, j > k$.

Let M be the union of M' and a regular neighborhood of $\Sigma_{k+1} \cup \dots \cup \Sigma_g$ which misses F. Since $\Sigma_j - M'$ is a union of embedded discs, for $j = k+1, \dots, g$, M is simply connected. The discs $D_{k+1}, \dots, D_g$ and the classes $x_i = [\Sigma_i] \in H_2(M)$, $i > k$, satisfy the conditions of Theorem 3.1 of [**3**]. Applying Theorem 1.1 of [**3**] we get $g - k$ tamely embedded discs $B^2_{k+1}, \dots, B^2_g$ in M such that $B^2_j \cap \partial M = C_j$. This, in turn, defines an embedding of K in $\mathbb{R}^4$.

3. The case $H^2(K) = Z$. Let B be a ball of radius r and let $F \colon B \times I \to B$ have the following properties: $F_0 = \text{id}$, $F_t | \partial B = \text{id}$, for $t \in [0, 1]$, and F_t is a homeomorphism of B for $t \in [0, 1)$. Then the homotopy $H : B \times B^k \times I \to B \times B^k$ given by

$$H((x, y), t) = (F(x, (1 - |y|)t), y)$$

is the identity on $\partial(B \times B^k)$. Furthermore, H_t is one-to-one on $B \times B^k - B \times 0$, for all $t \in I$, and $H_t | B \times 0 = F_t \times 0$.

LEMMA 2. *Let K be a finite 2-dimensional cell complex, such that all the 2-cells are attached via homeomorphisms. Let g be an embedding of K into $\mathbb{R}^4$. Then there exists a homotopy with compact support $H \colon \mathbb{R}^4 \times I \to \mathbb{R}^4$, such that $H_0 = \text{id}$, and such that H_t is homeomorphism for $t \in [0, 1)$, which does one of the following three types of deformations*:

(i) *for an edge s of K, H_1 maps $g(s)$ to a point and is* 1-1 *elsewhere*;

(ii) *for a* 2*-cell* e *with boundary a union of two edges* s_1, s_2 *having pairs of common endpoints,* H *is a deformation retraction of* $g(e)$ *onto* $g(s_1)$, *which is fixed on* $g(s_1)$.

(iii) *for two* 2*-cells* e_1, e_2 *with* $e_1 \cap e_2$ *being an arc* A, H_1 *maps* $g(e_1)$ *homeomorphically onto* $g(e_2)$, *and is* 1-1 *on* $g(K) - g(e_1 \cup e_2)$. *Furthermore,* H *is fixed on* $g(e_2)$.

If K_1 *is the* 2*-complex obtained from* K *by the identifications defined by* H_1 *then* $H_1 g\colon K \to \mathbb{R}^4$ *factors through* K_1. *The factoring map* $K_1 \to \mathbb{R}^4$ *is an embedding.*

Proof. Define a homotopy $F\colon 2B^k \times I \to 2B^k$ as follows:
For type (i) let $k = 1$, and let

$$F(x, t) = \begin{cases} (1-t)x & \text{for } |x| \le 1, \\ (1+t)x - 2tx/|x| & \text{for } 1 \le |x| \le 2. \end{cases}$$

F squeezes $[-1, 1]$ to 0 and linearly stretches the rest of $[-2, 2]$.

For type (ii) let $k = 2$, and let

$$F((x, y), t) = \begin{cases} (x, y(1-t)) & \text{for } |x| \le 1,\ 0 \le y \le A(x), \\ (x, (1/(A(x) - B(x)))((A(x)(1-t) - B(x))y + tA(x)B(x))) & \\ \qquad\qquad \text{for } |x| \le 1,\ A(x) \le y \le B(x), & \\ (x, y) & \text{elsewhere}, \end{cases}$$

where $A(x) = \sqrt{1-x^2}$, $B(x) = \sqrt{4-x^2}$. F shrinks $D^2 \cap \mathbb{R}^2_+$ to $[-1, 1] \times 0$.

For type (iii) let $k = 3$ and define F as follows:
Let $\delta : [0, 2\pi] \times I \to [0, 2\pi]$ be the homotopy

$$\delta(\alpha, t) = \begin{cases} (1-t)\alpha & \text{for } 0 \le \alpha \le \pi/2, \\ (1+t/3)\alpha - 2\pi t/3 & \text{for } \alpha \ge \pi/2. \end{cases}$$

δ shrinks $[0, \pi/2]$ to 0 and stretches $[\pi/2, 2\pi]$ over $[0, 2\pi]$. A point in $\mathbb{R}^3$ can be represented as a pair of a real and a complex number. Let

$$F((x, r \cdot \exp(i\alpha)), t) = \begin{cases} (x, r \cdot \exp(i\delta(\alpha, t))) & \text{for } \rho \le 1, \\ (x, r \cdot \exp(i[(2-\rho)\delta(\alpha, t) + (\rho - 1)\alpha])) & \text{for } \rho \in [1, 2], \end{cases}$$

where $\rho = \sqrt{x^2 + r^2}$.

In each case $F_t|\partial(2B^k)$ is identity for all $t \in I$.

For type (i) $g(s)$ has a regular neighborhood N homeomorphic to $[-2, 2] \times B^3$. Let $\varphi : [-2, 2] \times B^3 \to N$ be a homeomorphism such that $\varphi([-1, 1] \times 0) = g(s)$.

For type (ii) $g(e)$ has a regular neighborhood N homeomorphic to $2D^2 \times B^2$. Let $\varphi : 2D^2 \times B^2 \to N$ be a homeomorphism such that $\varphi((D^2 \cap \mathbb{R}^2_+) \times 0) = g(e)$, and such that $\varphi([-1, 1] \times 0) = g(s_1)$.

For type (iii), since $D = g(e_1 \cup e_2)$ is a tame disc such that its interior doesn't intersect $g(K) - D$, there exists a homeomorphism φ from $2B^3 \times [-1, 1]$ onto a regular neighborhood N of D, satisfying the following two properties: $\varphi(B^3 \times 0) \cap (g(K) - D) = \varnothing$, and φ maps $\{(x, y, z, 0) \in B^3 \times 0 | y \geq 0, z \geq 0, yz = 0\}$ onto D so that $g(A) = \varphi(\{(x, 0, 0, 0) \in B^3 \times 0\})$.

Given φ and F for each type we define the desired homotopy H by

$$H(x, t) = \begin{cases} x & \text{for } x \in N, \\ \varphi(F(u, (1 - |v|t), v) & \text{for } (u, v) \in 2B^k \times B^{4-k}, \\ & x = \varphi(u, v). \end{cases}$$

Suppose $f \colon F \to K$ represents a generator of $H_2(K)$. We can assume (by subdividing F and K appropriately) that f is simplicial and non-degenerate on each simplex (compare with **[1]**, p. 11). We dealt with the case when f is an embedding in Lemma 1. Assume now that the singular set S of f (S is the closure of the set $\{x \in F | f^{-1}(f(x))$ contains more than one point$\}$) is non-empty. We will successively replace K by "nicer" complexes and finally reduce the problem of embeddability of K in $\mathbb{R}^4$ to the situation of Lemma 1.

Case 1. S is 0-dimensional.

If $\Sigma = f(S) = \{y_1, \dots, y_r\}$ then $F_0 = f(F)$ is obtained from F by identifying the points of each set $f^{-1}(y_j)$, $j = 1, \dots, t$. Suppose $f^{-1}(y_1) = \{v_1, v_2, \dots, v_l\}$. Construct F_1 from F by identifying the points of each set $f^{-1}(y_1) - \{v_1\}, f^{-1}(y_2), \dots, f^{-1}(y_r)$. Note that F_1 is not a surface. Clearly there exists a map f_1 making the following diagram commutative:

$$\begin{array}{ccc} F & \xrightarrow{f_1} & F_1 \\ & \searrow^{f} & \downarrow p_1 \\ & & F_0 \end{array}$$

where $p_1 \colon F_1 \to F_0$ denotes the natural projection. The singular set S_1 of f_1 is equal to $S - \{v_1\}$.

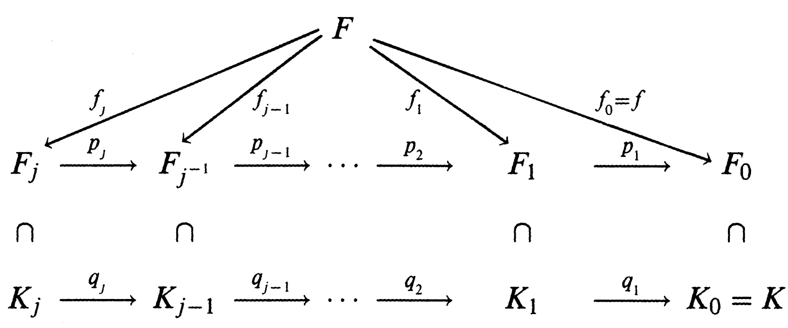
Attach the endpoints of an arc A to F_1 to w_1 and w_2, where $w_i = f_1(v_i)$. The resulting space $\widehat{F}_1$ is homotopy equivalent to F_0. For example, the map $\hat{p}_1 : \widehat{F}_1 \to F_0$ defined to be p_1 on F_1 and sending A to y_1 is a homotopy equivalence. It is easy to find a homotopy inverse $q \colon F_0 \to \widehat{F}_1$. Suppose $\alpha \colon I \to A$ is a parametrization of A such that $\alpha(0) = w_1$. If σ is a simplex of dimension greater than zero in F_1, with vertex w_1, then σ is a cone over a simplex τ. Define

$$q(x) = \begin{cases} x & \text{for } x \notin p_1(\mathrm{st}(w_1)), \\ [u, 2t-1] & \text{for } [u, t] \in \sigma = C(\tau),\ x = p_1([u, t]), \\ & t \in [1/2, 1], \\ \alpha(1-2t) & \text{for } t \in [0, 1/2]. \end{cases}$$

Here $\mathrm{st}(w_1)$ denotes the star of w_1, and $C(\tau)$ is the cone over τ with the vertex w_1 corresponding to the value $t = 0$.

Clearly q is 1-1 on each 1-simplex of F_0. If $L = \overline{K - F_0}$ then K is obtained from F_0 by attaching L along a graph G in $F_0^{(1)}$. If σ is a cell attached to G via an attaching map ψ then attach σ to $\widehat{F}_1$ via $q\psi$. This gives us a new complex K_1 homotopy equivalent to K by an obvious extension $q_1 \colon K_1 \to K$ of $\hat{p}_1$. By subdividing $\mathrm{st}(y_1)$ we can always assume that K_1 is again a simplicial complex with A one of its 1-simplices. $H_2(K_1)$ is generated by the mapping $f_1 \colon F \to K_1$ which has one less point in its singular set than f. Using Lemma 2 successively (one deformation of type (i) along A followed by a sequence of deformations of type (ii)) we see that if K_1 can be embedded in $\mathbb{R}^4$ then so can K.

Repeating the same construction we get the following commutative diagram



where the maps in the bottom row are homotopy equivalences, $H_2(K_i)$ is generated by $f_i \colon F \to F_i \subset K_i$, $i = 0, \ldots, j$, and f_j is an embedding. Furthermore, if K_i can be embedded in $\mathbb{R}^4$ so can K_{i-1}, $i = 1, \ldots, j$. Also K_j embeds in $\mathbb{R}^4$ by Lemma 1. This proves

PROPOSITION 1. *Suppose K is a finite simplicial complex. Suppose that $H^2(K) = Z$ and that $H_2(K)$ is represented by a non-degenerate simplicial map $f\colon F \to K$ of an orientable surface F into K. If the singular set of f is 0-dimensional then K can be embedded in $\mathbb{R}^4$.*

Case 2. S is 1-dimensional.

Then $\Sigma = f(S)$ is also at most 1-dimensional. F_0 is obtained from F by identifying the points of each $f^{-1}(y)$, $y \in \Sigma^{(0)}$, and by identifying the components of each $f^{-1}(\sigma)$ (by simplicial isomorphisms) where σ runs over the interiors of the edges of Σ. Let $f^{-1}(\sigma_0)$ be a union of open edges $s_1, \ldots, s_r$, for some open edge $\sigma_0 \in \Sigma$. Construct F_1 from F by identifying the points of each set $f^{-1}(y)$, $y \in \Sigma^{(0)}$, and by identifying the components of $s_2 \cup \cdots \cup s_r$ and of the sets $f^{-1}(\sigma)$ where σ runs over open 1-simplices of $\Sigma - \sigma_0$ (again via simplicial isomorphisms). As in Case 1 there exists a map f_1 making the diagram

$$\begin{array}{ccc} F & \xrightarrow{f_1} & F_1 \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p_1} \\ & & F_0 \end{array}$$

commute where $p_1\colon F_1 \to F_0$ is the natural projection. The singular set S_1 of f_1 has one less edge than $S : S_1 = S - s_1$.

Attach a 2-cell D to $z_1 \cup z_2 \subset F_1$ via a homeomorphism where $z_j = f_1(s_j)$. The resulting space $\widehat{F}_1$ is homotopy equivalent to F_0. The extension $\hat{p}_1\colon \widehat{F}_1 \to F_0$ of $p_1\colon F_1 \to F_0$ which squeezes D to z_1 is a homotopy equivalence. Suppose, as before, that $L = \overline{K - F_0}$ is attached to F_0 along a graph G. Then $\widehat{G} = p^{-1}(G) - z_1$ is homeomorphic to G and L can be attached to $\widehat{F}_1$ along $\widehat{G}$ in the obvious way to construct a 2-complex K_1 which is homotopy equivalent to K. Let $q_1\colon K_1 \to K$ be the obvious extension of $\hat{p}_1\colon \widehat{F}_1 \to F_0$. $H_2(K_1)$ is generated by $f_1\colon F \to K_1$ which has one less edge in its singular set than f. Also, by using one deformation of type (ii) from Lemma 2 we see that if K_1 embeds in $\mathbb{R}^4$ then so does K. As in Case 1 we repeat the above procedure to get a commutative diagram

$$
\begin{array}{ccccccccc}
 & & & & F & & & & \\
 & \swarrow^{f_l} & & \swarrow^{f_{l-1}} & & \searrow^{f_1} & & \searrow^{f_0=f} & \\
F_l & \xrightarrow{p_l} & F_{l-1} & \xrightarrow{p_{l-1}} & \cdots & \xrightarrow{p_2} & F_1 & \xrightarrow{p_1} & F_0 \\
\cap & & \cap & & & & \cap & & \cap \\
K_l & \xrightarrow{q_l} & K_{l-1} & \xrightarrow{q_{l-1}} & \cdots & \xrightarrow{q_2} & K_1 & \xrightarrow{q_1} & K_0 = K
\end{array}
$$

where the bottom maps are homotopy equivalences, the singular set of f_l is 0-dimensional, and K_{i-1} embeds in $\mathbb{R}^4$ if K_i does, for $i = 1, \dots, l$. Combining this with Proposition 1 we get

PROPOSITION 2. *Suppose $H^2(K) = Z$, and suppose that a generator of $H_2(K)$ is represented by a non-degenerate simplicial map $f: F \to K$ where F is an orientable surface. If the singular set of f is 1-dimensional then K embeds in $\mathbb{R}^4$.*

Case 3. *S is 2-dimensional.*

Choose a point b_σ in the interior of each 2-cell σ of F. Let S_k be the collection of all open 2-cells σ such that $f^{-1}(f(b_\sigma))$ contains k points. Denote by Z_k the union of 2-cells σ such that $\mathrm{int}(\sigma) \in S_k$. Represent the homology class of $f: F \to K$ by a linear combination $\sum x_e e$ where e runs over the 2-cells of K. By choosing appropriate orientations for the 2-cells of $f(F)$ we can assume that all the coefficients x_e are non-negative. Furthermore, F can be chosen so that $S_k = \{f^{-1}(\mathrm{int}(e)) | x_e = k\}$, for all k (see **[2]**, p. 11). Let $M = \max\{k | S_k \neq \varnothing\}$. Since S is 2-dimensional, M is greater than 1. S_M does not contain all the open 2-cells of F because the coefficients x_e have no common factor. Therefore there exists a 2-cell σ_1 such that $\mathrm{int}(\sigma_1) \in S_M$ and such that the intersection of σ_1 with $\overline{F - Z_M}$ contains an open edge s_1. Let $\Sigma = f(S)$. Construct F_1 from F

(1) by identifying the points of each $f^{-1}(y)$, $y \in \Sigma^{(0)}$,

(2) by identifying the components of $f^{-1}(\tau)$ where τ runs over the open edges of $\Sigma - f(s_1)$,

(3) by identifying the components of $f^{-1}(e)$ where e runs over all closed 2-cells of $\Sigma - f(\sigma_1)$,

(4) by gluing together $s_2, \ldots, s_m$ where $s_1, \ldots, s_m$ are the components of $f^{-1}(f(s_1))$, and

(5) by gluing together $\sigma_2, \ldots, \sigma_m$, where $\sigma_1, \ldots, \sigma_m$ are closed 2-cells whose union is $f^{-1}(f(\sigma_1))$.

As before, let all the identifications be via simplicial isomorphisms. f can again be factored as $p_1 f_1$ where $p_1: F_1 \to F_0$ is the natural projection. p_1 is a homotopy equivalence. If, as before, K is obtained from F_0 by attaching L $(= \overline{K - F_0})$ along a graph $G \subset F_0$, construct K_1 by attaching L to F_1 along $p_1^{-1}(G) - f_1(s_1) \approx G$ in the obvious way. K_1 is homotopy equivalent to K. Let $q_1: K_1 \to K$ be the natural extension of p_1. $H_2(K)$ is generated by $f_1: F \to K_1$. The singular set of f_1 has one less 2-simplex than S. Also, by Lemma 2 (using type (iii) deformation) K embeds in $\mathbb{R}^4$ if K_1 does.

As in the previous two cases we can repeat the above procedure to get a commutative diagram

$$\begin{array}{ccccccccc} & & & & F & & & & \\ & \swarrow^{f_d} & & \swarrow^{f_{d-1}} & & \searrow^{f_1} & & \searrow^{f_0 = f} & \\ F_d & \xrightarrow{p_d} & F_{d-1} & \xrightarrow{p_{d-1}} & \cdots & \xrightarrow{p_2} & F_1 & \xrightarrow{p_1} & F_0 \\ \cap & & \cap & & & & \cap & & \cap \\ K_d & \xrightarrow{q_d} & K_{d-1} & \xrightarrow{q_{d-1}} & \cdots & \xrightarrow{q_2} & K_1 & \xrightarrow{q_1} & K_0 = K \end{array}$$

where $f_i: F \to K_i$ represents a generator of $H_2(K_i)$, $i = 0, \ldots, d$, where the singular set of f_d is 1-dimensional, and where K_{i-1} embeds in $\mathbb{R}^4$ if K_i does, for $i = 1, \ldots, d$. Since, by Proposition 2, K_d embeds in $\mathbb{R}^4$ this proves the following result.

LEMMA 3. *If K is a finite 2-complex such that $H^2(K)$ is infinite cyclic then K embeds in $\mathbb{R}^4$.*

4. Proof of the theorem. Suppose $H^2(K) = Z/mZ$. Then $H_1(K)$ is isomorphic to the direct sum of Z/mZ and a free abelian group F. Let $x \in H_1(K)$ correspond to a generator of Z/mZ. Since the second cohomology does not change if 1-cells are attached to K, we can assume that $K^{(1)}$ is connected. Therefore x can be represented by a closed curve $C: S^1 \to K^{(1)}$. Denote by L the 2-complex obtained from K by attaching an additional 2-cell e using C as the attaching map. Let p be a point of $\text{int}(e)$ and let y be a generator of $H_1(\text{int}(e) - p)$. Since $H_2(K) = 0$ the Meyer-Vietoris sequence of the pair $\{L - p, \operatorname{int}(e)\}$ gives rise to the following exact sequence:

$$0 \to H_2(L) \to H_1(\operatorname{int}(e) - p) \to H_1(K) \to H_1(L) \to 0.$$

Because y gets mapped to x, $H_1(L)$ is free and $H_2(L)$ is isomorphic to Z. Therefore $H^2(L) = Z$. By Lemma 3 L embeds in $\mathbb{R}^4$. Since $K \subset L$ we also get an embedding of K into $\mathbb{R}^4$. This finishes the proof of the theorem.

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