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THE STRUCTURE OF TWISTED SU(3) GROUPS

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In order to study how the C^*-algebra $C(S_\mu U(3))$ of twisted SU(3) groups introduced by Woronowicz is related to the deformation quantization of the Lie-Poisson SU(3), we need to understand the algebraic structure of $C(S_\mu U(3))$ better. In this paper, we shall use Bragiel's result about the irreducible representations of $C(S_\mu U(3))$ and the theory of groupoid C^*-algebras to give an explicit description of the C^*-algebra structure of $C(S_\mu U(3))$, which indicates that $C(S_\mu U(3))$ is some kind of foliation C^*-algebra of the singular symplectic foliation of the Lie-Poisson group SU(3).

In recent years, there has been a rapid growth of interest in the theory of quantum groups [**D**]. In particular, S. L. Woronowicz has developed a C^*-algebraic theory of quantum groups, which has motivated a lot of research [**B**, **Po**, **Ro**, **S**, **Va-So**, **Wo1**, **Wo2**].

In [**S**], the explicit knowledge of the C^*-algebra structure of $C(S_\mu U(2))$ [**Wo1**, **S**] has helped us to find a deformation quantization [**BFFLS**, **Ri1**, **Ri2**, **Ri3**] of the Lie-Poisson SU(2) [**D**, **Lu-We**], which is in a sense compatible with the quantization of the group structure of SU(2) by the "twisted groups" $S_\mu U(2)$. On the other hand, although both $C(S_\mu U(2))$ and $C(S_\mu U(3))$ [**Wo1**, **Wo2**] are defined as universal C^*-algebras of certain generators and relations, the algebraic structure of the latter seems to be much more complicated than that of the former. In [**B**], Bragiel classified the irreducible representations of the C^*-algebra $C(S_\mu U(3))$ of the twisted SU(3) groups (with $0 < \mu < 1$) and showed that $C(S_\mu U(3))$ is a type-I C^*-algebra [**Pe**]. In this paper, enlightened by the ideas in [**M-Re**, **Cu-M**], we shall use Bragiel's result and the theory of groupoid C^*-algebras [**Re**] to give an explicit description of the C^*-algebra structure of $C(S_\mu U(3))$, which indicates that $C(S_\mu U(3))$ is some kind of foliation C^*-algebra of the singular symplectic foliation of the Lie-Poisson group SU(3) [**Co**, **We**, **Lu-We**].

We shall use freely the concepts and properties of the theory of groupoid C^*-algebras throughout this paper. A good reference for this is [**Re**]. First let us fix notations. Let $\mathbb{T}$ be the unit circle in $\mathbb{C}$ and $\mathbb{T}^2$ be the two-torus embedded in $\mathbb{C}^2$. We shall denote by ϕ and

ψ the two canonical coordinate functions of $\mathbb{T}^2$ with values in $\mathbb{T}$. For any groupoid $\mathfrak{G}$, we denote by $\mathfrak{G}|P$ the reduction of $\mathfrak{G}$ by the subset P of the unit space of $\mathfrak{G}$ **[Re]**. If a locally compact group G acts on a space X by an action τ, we shall denote by $X \times_\tau G$ the corresponding transformation group groupoid.

We define $\mathfrak{G} := \overline{\mathbb{Z}}^3 \times_\alpha \mathbb{Z}^5 | \overline{\mathbb{Z}}_{\geq}{}^3$, where $\overline{\mathbb{Z}} = \mathbb{Z} \cup \{+\infty\}$, the subscript $\geq$ denotes the nonnegative part, and $\mathbb{Z}^5$ acts on $\overline{\mathbb{Z}}^3$ by translation determined by the first three components, i.e. $\alpha(\mu)(\nu) = \nu - (\mu_1, \mu_2, \mu_3)$ for $\mu \in \mathbb{Z}^5$ and $\nu \in \mathbb{Z}^3$. Since the last two copies of $\mathbb{Z}$ act trivially on $\overline{\mathbb{Z}}^3$, we have $C^*(\mathfrak{G}) \cong C^*(\mathfrak{G}_0) \otimes C^*(\mathbb{Z}^2) \cong C^*(\mathfrak{G}_0) \otimes C(\mathbb{T}^2)$, where $\mathfrak{G}_0 := \overline{\mathbb{Z}}^3 \times_\tau \mathbb{Z}^3 | \overline{\mathbb{Z}}_{\geq}{}^3$ and τ is the action by translation. We assume that under the above isomorphism, the standard basis elements e_4 and e_5 of $\mathbb{Z}^5$ correspond to the conjugates $\overline{\phi}$ and $\overline{\psi}$ of the canonical coordinate functions on $\mathbb{T}^2$ (instead of ϕ and ψ in order to be more compatible with the notations used in **[B]** for the later discussion). Recall that the regular representation ρ_3 of $C^*(\mathfrak{G}_0)$ on the open dense invariant subset $\mathbb{Z}_{\geq}{}^3$ is faithful **[M-Re]**, and hence $C^*(\mathfrak{G})$ can be faithfully represented on the Hilbert space $l^2(\mathbb{Z}_{\geq}{}^3) \otimes L^2(\mathbb{T}^2)$ through $\tilde{\rho}_3 := \rho_3 \otimes m$ where m is the representation of $C(\mathbb{T}^2)$ by multiplication operators on $L^2(\mathbb{T}^2)$.

In **[B]**, the irreducible representations of $C(S_\mu U(3))$ are classified into six 2-parameter families (with parameters in $\mathbb{T}^2$) of irreducible representations π_3, π_{21}, π_{22}, π_{11}, π_{12} and π_0 (listed here in the same order as in **[B]**) on Hilbert spaces $l^2(\mathbb{Z}_{\geq}{}^3)$, $l^2(\mathbb{Z}_{\geq}{}^2)$, $l^2(\mathbb{Z}_{\geq}{}^2)$, $l^2(\mathbb{Z}_{\geq}{}^1)$, $l^2(\mathbb{Z}_{\geq}{}^1)$, and $l^2(\mathbb{Z}_{\geq}{}^0) = \mathbb{C}$, respectively. The 2-parameter family of irreducible representations π (on a Hilbert space $\mathscr{H}_\pi$) in the above list determine a representation $\tilde{\pi}$ of $C(S_\mu U(3))$ on $\mathscr{H}_\pi \otimes L^2(\mathbb{T}^2)$. Since $\pi_3(u_{ij})$'s and $\pi_3(u_{ij}{}^*)$'s are (finite) linear combinations of weighted (multivariable) shifts on $l^2(\mathbb{Z}_{\geq}{}^3)$ with weight functions extendable to $\overline{\mathbb{Z}}_{\geq}{}^3$ continuously, and since the weight functions involved in each $\pi_3(u_{ij})$ or $\pi_3(u_{ij}{}^*)$ are products of the canonical functions ϕ, ψ, $\overline{\phi}$ and $\overline{\psi}$ on $\mathbb{T}^2$ and functions on $\overline{\mathbb{Z}}_{\geq}{}^3$ independent of the parameters in $\mathbb{T}^2$, it is easy to identify the 2-parameter family $\tilde{\pi}_3(u_{ij})$ or $\tilde{\pi}_3(u_{ij}{}^*)$ with an element in $C_c(\mathfrak{G}) \subseteq C^*(\mathfrak{G})$ (which is faithfully represented on $l^2(\mathbb{Z}_{\geq}{}^3) \otimes L^2(\mathbb{T}^2)$) for each u_{ij}. For example, with $C_c(\overline{\mathbb{Z}}_{\geq}{}^3)$ and $\mathbb{Z}^5$ canonically embedded in $C_c(\mathfrak{G})$, we have

$$\begin{aligned} &\tilde{\pi}_3(u_{11}{}^*) = e_1 f_{11}, \quad \tilde{\pi}_3(u_{12}{}^*) = e_2 f_{12}, \\ &\tilde{\pi}_3(u_{13}{}^*) = e_5 f_{13}, \quad \tilde{\pi}_3(u_{21}{}^*) = e_3 f_{21}, \\ &\tilde{\pi}_3(u_{31}{}^*) = e_4 f_{31}, \end{aligned}$$

where, for $(N, M, L) \in \overline{\mathbb{Z}}_{\geq}{}^3$,

$$
\begin{aligned}
f_{11}(N, M, L) &= (1-\mu^{2(N+1)})^{1/2}, \\
f_{12}(N, M, L) &= \mu^{N+1}(1-\mu^{2(M+1)})^{1/2}, \\
f_{13}(N, M, L) &= \mu^{2+N+M}, \\
f_{21}(N, M, L) &= \mu^{N}(1-\mu^{2(L+1)})^{1/2}, \\
f_{31}(N, M, L) &= \mu^{N+L}.
\end{aligned}
$$

Note that for $0 < \mu < 1$, the above expressions have canonical meaning even when N, M or L is ∞. Thus we can factor the homomorphism $\tilde{\pi}_3$ through $C^*(\mathfrak{G})$, i.e. there exists a homomorphism

$$\eta : C(S_\mu U(3)) \to C^*(\mathfrak{G})$$

such that $\tilde{\pi}_3 = \tilde{\rho}_3 \circ \eta$. We shall see later that η is in fact injective since all the representations $\tilde{\pi}$ of $C(S_\mu U(3))$ mentioned above can be factored through η.

Let us consider the following invariant subsets of the unit space of $\mathfrak{G}$,

$$
\begin{aligned}
X_3 &= \{(N, M, L) | N, M, L \in \mathbb{Z}_{\geq}\} = \mathbb{Z}_{\geq}{}^3, \\
X_{21} &= \{(N, M, L) | N, M \in \mathbb{Z}_{\geq} \text{ and } L = \infty\} \cong \mathbb{Z}_{\geq}{}^2, \\
X_{22} &= \{(N, M, L) | N, L \in \mathbb{Z}_{\geq} \text{ and } M = \infty\} \cong \mathbb{Z}_{\geq}{}^2, \\
X_{11} &= \{(N, M, L) | N \in \mathbb{Z}_{\geq} \text{ and } M = L = \infty\} \cong \mathbb{Z}_{\geq}, \\
X_{12} &= \{(N, M, L) | M \in \mathbb{Z}_{\geq} \text{ and } N = L = \infty\} \cong \mathbb{Z}_{\geq}
\end{aligned}
$$

and $X_0 = \{(\infty, \infty, \infty)\}$. We define $X_i = X_{i1} \cup X_{i2}$ for $i = 1, 2$, and σ_i (resp. σ_{in}) to be the quotient map from $C^*(\mathfrak{G}|\overline{X}_{i+1})$ to $C^*(\mathfrak{G}|\overline{X}_i)$ (resp. $C^*(\mathfrak{G}|\overline{X}_{in})$) for $i = 0, 1, 2$, (resp. $i = 1, 2$ and $n = 1, 2$) where $\overline{X}_i$ is the closure of X_i in the unit space of $\mathfrak{G}$. Since $\tilde{\pi}_3(u_{ij})\tilde{\pi}_3(u_{ij}{}^*) = f_{ij}{}^2$ for the u_{ij}'s listed above and they separate points in $\mathbb{Z}_{\geq} \times \overline{\mathbb{Z}}_{\geq}{}^2$, i.e. points (N, M, L) with $N < \infty$, it is easy to check that $C_c(X_3) = C_c(\mathbb{Z}_{\geq}{}^3) \subseteq \operatorname{Im}(\eta)$ (by considering the level sets of these f_{ij}'s). Now since those weights f_{ij} are non-vanishing on $\mathbb{Z}_{\geq}{}^3$ and $C_c(\mathbb{Z}_{\geq}{}^3) \subseteq \operatorname{Im}(\eta)$, the convolution algebra $C_c(\mathbb{Z}_{\geq}{}^3 \times_\alpha \mathbb{Z}^5)$ and hence $C^*(\mathbb{Z}_{\geq}{}^3 \times_\alpha \mathbb{Z}^5) \cong C(\mathbb{T}^2) \otimes \mathscr{K}$ are contained in the C^*-algebra generated by (the weighted shifts) $\eta(u_{ij}{}^*)$ of the $u_{ij}{}^*$'s listed above and hence in $\operatorname{Im}(\eta)$ where $\mathscr{K}$ is the algebra of compact operators (on $l^2(\mathbb{Z}_{\geq}{}^3)$ here).

Now we consider the diagonal homomorphism $(\sigma_{21}, \sigma_{22})$ from $C^*(\mathfrak{G})$ to $C^*(\mathfrak{G}|\overline{X}_{21}) \oplus C^*(\mathfrak{G}|\overline{X}_{22})$. It is easy to see that $\mathfrak{G}|\overline{X}_{2n} \cong \overline{\mathbb{Z}}^2 \times_{\alpha(2, n)} \mathbb{Z}^5 | \overline{\mathbb{Z}}_{\geq}{}^2$ where $\mathbb{Z}^5$ acts on $\overline{\mathbb{Z}}^2$ through the action $\alpha(2, n)$ in the way that 2 components (depending on n) of $\mathbb{Z}^5$ act on $\overline{\mathbb{Z}}^2$ by translation while the other 3 components act trivially. More precisely, $\alpha(2, 1)(\mu) \cdot \nu = \nu - (\mu_1, \mu_2)$ and $\alpha(2, 2)(\mu) \cdot \nu = \nu - (\mu_1, \mu_3)$ for $\mu \in \mathbb{Z}^5$ and $\nu \in \mathbb{Z}^2$. Thus

$$C^*(\mathfrak{G}|\overline{X}_{2n}) \cong C^*(\overline{\mathbb{Z}}^2 \times_\tau \mathbb{Z}^2|\overline{\mathbb{Z}}_{\geq}{}^2) \otimes C^*(\mathbb{Z}^3) \cong C^*(\overline{\mathbb{Z}}^2 \times_\tau \mathbb{Z}^2|\overline{\mathbb{Z}}_{\geq}{}^2) \otimes C(\mathbb{T}^3),$$

where the canonical generators of $\mathbb{Z}^3$ are e_3, e_4, e_5 when $n = 1$, and e_2, e_4, e_5 when $n = 2$. It is straightforward to check that $(\sigma_{21} \circ \eta)(u_{ij})$'s $(1 \leq i, j \leq 3)$ are supported in $\overline{\mathbb{Z}}^2 \times_{\alpha(2,1)} \mathbb{Z}^4|\overline{\mathbb{Z}}_{\geq}{}^2$ where $\mathbb{Z}^4$ is generated by e_1, e_2, e_3 and e_5 in $\mathbb{Z}^5$, while $(\sigma_{22} \circ \eta)(u_{ij})$'s are supported in $\overline{\mathbb{Z}}^2 \times_{\alpha(2,2)} \mathbb{Z}^4|\overline{\mathbb{Z}}_{\geq}{}^2$ with $\mathbb{Z}^4$ generated by e_1, e_2, e_3 and e_4 in $\mathbb{Z}^5$. Furthermore, from the weight functions f_{ij} listed above, it is easy to check that $C_c(X_2) \subseteq \operatorname{Im}(\sigma_2 \circ \eta)$ and hence

$$C^*(\mathbb{Z}^2 \times_{\alpha(2,1)} \mathbb{Z}^4|\mathbb{Z}_{\geq}{}^2) \oplus C^*(\mathbb{Z}^2 \times_{\alpha(2,2)} \mathbb{Z}^4|\mathbb{Z}_{\geq}{}^2) \cong 2\mathscr{K} \otimes C(\mathbb{T}^2)$$

is contained in the C^*-algebra generated by $(\sigma_{21}, \sigma_{22})(\eta(u_{ij}{}^*))$ and hence in $\operatorname{Im}((\sigma_{21}, \sigma_{22}) \circ \eta)$. Let ρ_2 be the faithful regular representation of $\overline{\mathbb{Z}}^2 \times_\tau \mathbb{Z}^2|\overline{\mathbb{Z}}_{\geq}{}^2$ on $l^2(\mathbb{Z}_{\geq}{}^2)$ and $\tilde{\rho}_{2n} = \rho_2 \otimes m$ be the corresponding faithful representation of

$$C^*(\overline{\mathbb{Z}}^2 \times_{\alpha(2,n)} \mathbb{Z}^4|\overline{\mathbb{Z}}_{\geq}{}^2) \cong C^*(\overline{\mathbb{Z}}^2 \times_\tau \mathbb{Z}^2|\overline{\mathbb{Z}}_{\geq}{}^2) \otimes C(\mathbb{T}^2)$$

on $l^2(\mathbb{Z}_{\geq}{}^2) \otimes L^2(\mathbb{T}^2)$, where the isomorphism identifies e_3, e_5 with $\overline{\phi}$, $\overline{\psi}$ if $n = 1$, and identifies e_4, e_2 with $\overline{\phi}$, $\overline{\psi}$ if $n = 2$. Then it can be easily checked that

$$\tilde{\rho}_{2n}(\sigma_{2n}(\eta(u_{ij}))) = \tilde{\pi}_{2n}(u_{ij})$$

(note that in the above identification, the symbols N and M used in **[B]** need be interchanged when $n = 2$) and hence $\tilde{\pi}_{2n}$ factors through η. Let $\eta_{2n} := \sigma_{2n} \circ \eta$.

Now we consider $\sigma_{12} \circ \sigma_2$ and $\sigma_{11} \circ \sigma_2$. Since clearly $\sigma_{12} \circ \sigma_2$ factors through σ_{21} and $\sigma_{11} \circ \sigma_2$ factors through σ_{21} and σ_{22}, we may talk about $\sigma_{12} \circ \sigma_{21}$ $(= \sigma_{12} \circ \sigma_2)$ and $\sigma_{11} \circ \sigma_{21} = \sigma_{11} \circ \sigma_{22}$ $(= \sigma_{11} \circ \sigma_2)$ by abuse of language. Note that

$$C^*(\mathbb{Z}^2 \times_{\alpha(2,1)} \mathbb{Z}^4|\mathbb{Z}_{\geq}{}^2) \oplus C^*(\mathbb{Z}^2 \times_{\alpha(2,2)} \mathbb{Z}^4|\mathbb{Z}_{\geq}{}^2) \subseteq C^*(\mathfrak{G}|X_2) \subseteq \ker(\sigma_{1n})$$

because $(\mathbb{Z}^2 \times_{\alpha(2,1)} \mathbb{Z}^4|\mathbb{Z}_{\geq}{}^2) \cup (\mathbb{Z}^2 \times_{\alpha(2,2)} \mathbb{Z}^4|\mathbb{Z}_{\geq}{}^2) \subseteq X_2$. It is again easy to see that $\mathfrak{G}|\overline{X}_{1n} \cong \overline{\mathbb{Z}} \times_{\alpha(1,n)} \mathbb{Z}^5|\overline{\mathbb{Z}}_{\geq}$ where $\mathbb{Z}^5$ acts on $\overline{\mathbb{Z}}$ through the action $\alpha(1, n)$ in the way that one component (depending on n) of $\mathbb{Z}^5$ act on $\overline{\mathbb{Z}}$ by translation while the other 4 components act trivially.

More precisely, $\alpha(1,1)(\mu)\cdot\nu = \nu - \mu_1$ and $\alpha(1,2)(\mu)\cdot\nu = \nu - \mu_2$ for $\mu\in\mathbb{Z}^5$ and $\nu\in\mathbb{Z}$. Thus

$$C^*(\mathfrak{G}|\overline{X}_{1n}) \cong C^*(\overline{\mathbb{Z}}\times_\tau \mathbb{Z}|\overline{\mathbb{Z}}_\geq)\otimes C^*(\mathbb{Z}^4) \cong C^*(\overline{\mathbb{Z}}\times_\tau \mathbb{Z}|\overline{\mathbb{Z}}_\geq)\otimes C(\mathbb{T}^4),$$

where the canonical generators of $\mathbb{Z}^4$ are e_2, e_3, e_4, e_5 when $n=1$, and e_1, e_3, e_4, e_5 when $n=2$. It is straightforward to check that $(\sigma_{11}\circ\sigma_2\circ\eta)(u_{ij})$'s $(1\le i, j\le 3)$ are supported in $\overline{\mathbb{Z}}\times_{\alpha(1,1)}\mathbb{Z}^3|\overline{\mathbb{Z}}_\geq$ where $\mathbb{Z}^3$ is generated by e_1, e_2 and e_3 in $\mathbb{Z}^5$, while the $(\sigma_{12}\circ\sigma_2\circ\eta)(u_{ij})$'s are supported in $\overline{\mathbb{Z}}\times_{\alpha(1,2)}\mathbb{Z}^4|\overline{\mathbb{Z}}_\geq$ with $\mathbb{Z}^4$ generated by e_1, e_2, e_3 and e_5 in $\mathbb{Z}^5$. Let ρ_1 be the faithful regular representation of $\overline{\mathbb{Z}}\times_\tau\mathbb{Z}|\overline{\mathbb{Z}}_\geq$ on $l^2(\mathbb{Z}_\geq)$ and $\tilde{\rho}_{11}=\rho_1\otimes m$ be the corresponding faithful representation of

$$C^*(\overline{\mathbb{Z}}\times_{\alpha(1,1)}\mathbb{Z}^3|\overline{\mathbb{Z}}_\geq)\cong C^*(\overline{\mathbb{Z}}\times_\tau\mathbb{Z}|\overline{\mathbb{Z}}_\geq)\otimes C(\mathbb{T}^2)$$

on $l^2(\mathbb{Z}_\geq{}^2)\otimes L^2(\mathbb{T}^2)$, where the isomorphism identifies e_3 and e_2 with $\overline{\phi}$ and $\overline{\psi}$ respectively. Then it can be easily checked that

$$\tilde{\rho}_{11}((\sigma_{11}\circ\sigma_2\circ\eta)(u_{ij})) = \tilde{\pi}_{11}(u_{ij})$$

and hence $\tilde{\pi}_{11}$ factors through η and $\eta_{11}:=\sigma_{11}\circ\sigma_2\circ\eta=\sigma_{11}\circ\eta_{21}=\sigma_{11}\circ\eta_{22}$. On the other hand, we have

$$C^*(\overline{\mathbb{Z}}\times_{\alpha(1,2)}\mathbb{Z}^4|\overline{\mathbb{Z}}_\geq)\cong C^*(\overline{\mathbb{Z}}\times_\tau\mathbb{Z}|\overline{\mathbb{Z}}_\geq)\otimes C(\mathbb{T}^3),$$

where the conjugates of the three canonical coordinate functions of $\mathbb{T}^3$ correspond to the generators e_1, e_3 and e_5 in $\mathbb{Z}^5$. Composing the above identification with $\mathrm{id}\otimes\kappa_{12}$, we get a homomorphism λ_{12} from $C^*(\overline{\mathbb{Z}}\times_{\alpha(1,2)}\mathbb{Z}^4|\overline{\mathbb{Z}}_\geq)$ to $C^*(\overline{\mathbb{Z}}\times_\tau\mathbb{Z}|\overline{\mathbb{Z}}_\geq)\otimes C(\mathbb{T}^2)$, where κ_{12} is the homomorphism from $C(\mathbb{T}^3)$ to $C(\mathbb{T}^2)$ induced by the map from $\mathbb{T}^2$ to $\mathbb{T}^3$ sending $z\in\mathbb{T}^2$ to $(z_1, -z_1, z_2)$. Let $\tilde{\rho}_{12}=\rho_1\otimes m$ be the faithful representation of $C^*(\overline{\mathbb{Z}}\times_\tau\mathbb{Z}|\overline{\mathbb{Z}}_\geq)\otimes C(\mathbb{T}^2)\supseteq \mathrm{Im}(\eta_{12})$, where $\eta_{12}=\lambda_{12}\circ(\sigma_{12}\circ\sigma_2\circ\eta)=\lambda_{12}\circ(\sigma_{12}\circ\sigma_{21}\circ\eta)$. (Here we use the convention that $f\circ g$ is meaningful whenever $\mathrm{Im}(g)\subseteq\mathrm{Dom}(f)$.) Then $\tilde{\rho}_{12}\circ\lambda_{12}$ defines a representation of $\mathrm{Im}(\sigma_{12}\circ\sigma_2\circ\eta)$ on $l^2(\mathbb{Z}_\geq)\otimes L^2(\mathbb{T}^2)$. It is straightforward to check that

$$(\tilde{\rho}_{12}\circ\lambda_{12})((\sigma_{12}\circ\sigma_2\circ\eta)(u_{ij})) = \tilde{\pi}_{12}(u_{ij})$$

(note that in **[B]**, M is replaced by N) for all i, j. From the weight functions f_{ij} listed above, it is easy to check that $C_c(X_1)\subseteq \mathrm{Im}(\sigma_1\circ\sigma_2\circ\eta)$. So by the formulas for $\pi_{1n}(u_{ij})$ in **[B]**, it is not hard to see that

$$\begin{aligned} &C^*(\mathbb{Z}\times_{\alpha(1,1)}\mathbb{Z}^3|\mathbb{Z}_\geq)\oplus\lambda_{12}(C^*(\mathbb{Z}\times_{\alpha(1,2)}\mathbb{Z}^4|\mathbb{Z}_\geq)) \\ &\qquad\cong 2C^*(\mathbb{Z}\times_\tau\mathbb{Z}|\mathbb{Z}_\geq)\otimes C(\mathbb{T}^2)\cong 2\mathscr{K}\otimes C(\mathbb{T}^2)\end{aligned}$$

is contained in the C^*-algebra generated by $(\eta_{11}, \eta_{12})(u_{ij}{}^*)$ and hence in $\operatorname{Im}((\eta_{11}, \eta_{12}))$. Notice that

$$C^*(\mathbb{Z} \times_{\alpha(1,1)} \mathbb{Z}^3|\mathbb{Z}_{\geq}) \oplus C^*(\mathbb{Z} \times_{\alpha(1,2)} \mathbb{Z}^4|\mathbb{Z}_{\geq}) \subseteq C^*(\mathfrak{G}|X_1)$$

is contained in the kernel of σ_0.

Now we consider $\sigma_0 \circ \sigma_1 \circ \sigma_2$. Since $\sigma_0 \circ \sigma_1 \circ \sigma_2$ clearly factors through $\sigma_{11} \circ \sigma_2$ and $\sigma_{12} \circ \sigma_2$, we may talk about $\sigma_0 \circ \sigma_{11} \circ \sigma_2 = \sigma_0 \circ \sigma_{12} \circ \sigma_2 = \sigma_0 \circ \sigma_1 \circ \sigma_2$ by abuse of language. Note that $C^*(\mathfrak{G}|X_0) = C^*(\mathbb{Z}^5) \cong C(\mathbb{T}^5)$ and that $(\sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \eta)(u_{ij})$'s $(1 \leq i, j \leq 3)$ are supported in $\mathbb{Z}^3$ generated by e_1, e_2 and e_3 in $\mathbb{Z}^5$. Composing the identification $C^*(\mathbb{Z}^3) \cong C(\mathbb{T}^3)$ with κ_0 (where the generators e_1, e_2, e_3 are identified with the conjugates of the corresponding coordinate functions of $\mathbb{T}^3$), we get a homomorphism λ_0 from $C^*(\mathbb{Z}^3)$ to $C(\mathbb{T}^2)$, where κ_0 is the homomorphism from $C(\mathbb{T}^3)$ to $C(\mathbb{T}^2)$ induced by the map from $\mathbb{T}^2$ to $\mathbb{T}^3$ sending $z \in \mathbb{T}^2$ to $(z_1, z_2, -z_1)$. Let $\tilde{\rho}_0 := m$. Then $\tilde{\rho} \circ \lambda_0$ is a representation of $C^*(\mathbb{Z}^3)$ on $L^2(\mathbb{T}^2)$. It is straightforward to check that

$$(\tilde{\rho}_0 \circ \eta_0)(u_{ij}) = \tilde{\pi}_0(u_{ij})$$

for all i, j, where $\eta_0 = \lambda_0 \circ \sigma_0 \circ \sigma_1 \circ \sigma_2 \circ \eta$ is a homomorphism from $C(S_\mu U(3))$ to $C^*(\mathbb{Z}^2) \cong C(\mathbb{T}^2)$. Comparing the definitions of κ_{12} and κ_0 and relating the generators of their domains $C^*(\mathbb{Z}^3)$ to those of $\mathbb{Z}^5$ as we specified above, it is easy to check that η_0 factors through η_{11} and η_{12}, say $\eta_0 = \tilde{\omega}_0 \circ (\eta_{11}, \eta_{12})$ for some $\tilde{\omega}_0$ defined on $\operatorname{Im}(\eta_{11}, \eta_{12})$. Note that $\ker(\tilde{\omega}_0)$ contains the subalgebra

$$C^*(\mathbb{Z} \times_{\alpha(1,1)} \mathbb{Z}^3|\mathbb{Z}_{\geq}) \oplus \lambda_{12}(C^*(\mathbb{Z} \times_{\alpha(1,2)} \mathbb{Z}^4|\mathbb{Z}_{\geq})) \cong 2\mathscr{K} \otimes C(\mathbb{T}^2).$$

Now we summarize what we have so far. There are homomorphisms $\eta_3 = \eta$, η_{21}, η_{22}, η_{11}, η_{12} and η_0 from $C(S_\mu U(3))$ to

$$C^*(\mathfrak{G}) = C^*(\overline{\mathbb{Z}}^3 \times_\alpha \mathbb{Z}^5|\overline{\mathbb{Z}}_{\geq}{}^3) = C(\overline{\mathbb{Z}}^3 \times_\tau \mathbb{Z}^3|\overline{\mathbb{Z}}_{\geq}{}^3) \otimes C(\mathbb{T}^2),$$
$$C^*(\overline{\mathbb{Z}}^2 \times_\tau \mathbb{Z}^2|\overline{\mathbb{Z}}_{\geq}{}^2) \otimes C(\mathbb{T}^2), \quad C^*(\overline{\mathbb{Z}}^2 \times_\tau \mathbb{Z}^2|\overline{\mathbb{Z}}_{\geq}{}^2) \otimes C(\mathbb{T}^2),$$
$$C^*(\overline{\mathbb{Z}} \times_\tau \mathbb{Z}|\overline{\mathbb{Z}}_{\geq}) \otimes C(\mathbb{T}^2), \quad C^*(\overline{\mathbb{Z}} \times_\tau \mathbb{Z}|\overline{\mathbb{Z}}_{\geq}) \otimes C(\mathbb{T}^2) \quad \text{and} \quad C(\mathbb{T}^2),$$

respectively, such that

(1) each η_i or η_{in} factors through η_j with $j > i$, where $\eta_i := (\eta_{i1}, \eta_{i2})$ if $i = 1, 2$. In fact, $\eta_{21} = \omega_{21} \circ \eta$, $\eta_{22} = \omega_{22} \circ \eta$, $\eta_{11} = \omega_{11} \circ \eta_{21}$, $\eta_{11} = \omega'_{11} \circ \eta_{22}$, $\eta_{12} = \omega_{12} \circ \eta_{21}$, $\eta_0 = \omega_0 \circ \eta_{11}$ and $\eta_0 = \omega'_0 \circ \eta_{12}$ for some ω's defined on the range of the corresponding η's.

(2) Let $\eta_i = \tilde{\omega}_i \circ \eta_{i+1}$ for a suitable homomorphism $\tilde{\omega}_i$ defined on $\mathrm{Im}(\eta_{i+1})$. Then $\ker(\tilde{\omega}_i)$ contains a copy of $C(\mathbb{T}^2) \otimes \mathscr{K}$ if $i = 2$, and contains two copies of $C(\mathbb{T}^2) \otimes \mathscr{K}$ if $i = 0$ or 1. Furthermore, $\mathrm{Im}(\eta_0) \cong C(\mathbb{T}^2)$. Note that $\mathrm{Ker}(\eta_i) = \eta_{i+1}^{-1}(\mathrm{Ker}(\tilde{\omega}_i))$.

(3) $\tilde{\pi}_i = \tilde{\rho}_i \circ \eta_i$ $(i = 0, 3)$ and $\tilde{\pi}_{in} = \tilde{\rho}_{in} \circ \eta_{in}$ $(i = 1, 2)$ for some faithful representations $\tilde{\rho}_i$ and $\tilde{\rho}_{in}$ on $\mathrm{Im}(\eta_i)$ and $\mathrm{Im}(\eta_{in})$ respectively. Since the irreducible representations of $C(S_\mu U(3))$ are classified by those 2-parameter families of π_0, π_{11}, π_{12}, π_{21}, π_{22}, and π_3, the spectrum of $C(S_\mu U(3))$ is a disjoint union of 6 copies of $\mathbb{T}^2$ as a set. On the other hand, by (1)–(3), all these representations π_i's (or π_{in}'s) factor through η_j (or η_{jn}) with $j > i$ and hence $\eta = \eta_3$ is faithful. Thus, the type I C^*-algebra $C(S_\mu U(3))$ has a composition sequence

$$0 \subseteq \mathscr{I}_3 = \mathrm{Ker}(\eta_2) \subseteq \mathscr{I}_2 = \mathrm{Ker}(\eta_1) \subseteq \mathscr{I}_1 = \mathrm{Ker}(\eta_0) \subseteq \mathscr{I}_0 = C(S_\mu U(3))$$

such that $\mathscr{I}_3 = \mathrm{Ker}(\tilde{\omega}_2)$, $\mathscr{I}_2/\mathscr{I}_3 \cong \mathrm{Ker}(\tilde{\omega}_1)$, $\mathscr{I}_1/\mathscr{I}_2 \cong \mathrm{Ker}(\tilde{\omega}_0)$ and $\mathscr{I}_0/\mathscr{I}_1 \cong \mathrm{Im}(\eta_0) \cong C(\mathbb{T}^2)$. Note that $C(Y_{i+1}) \otimes \mathscr{K}(\mathscr{H}) \subseteq \mathrm{Ker}(\tilde{\omega}_i) \subseteq \mathrm{Im}(\eta_{i+1}) \subseteq C(Y_{i+1}) \otimes \mathscr{B}(\mathscr{H})$ (for some L^2-space $\mathscr{H}$), where Y_k is homeomorphic to $\mathbb{T}^2$ if $k = 3$ or 0, and to the disjoint union of 2 copies of $\mathbb{T}^2$ if $k = 2$ or 1. If $C(Y_{i+1}) \otimes \mathscr{K}(\mathscr{H}) \neq \mathrm{Ker}(\tilde{\omega}_i)$, then we have non-trivial irreducible representations of $\mathrm{Ker}(\tilde{\omega}_i)/C(Y_{i+1}) \otimes \mathscr{K}(\mathscr{H})$ which will induce irreducible representations of $C(S_\mu U(3))$ not unitarily equivalent to any of the π's found in **[B]**. So we have $C(Y_{i+1}) \otimes \mathscr{K}(\mathscr{H}) = \mathrm{Ker}(\tilde{\omega}_i)$.

We summarize what we obtained about the structure of the C^*-algebra $C(S_\mu U(3))$ in the following theorem.

THEOREM. *The C^*-algebra $C(S_\mu U(3))$ of the twisted* SU(3) *group has the composition sequence*

$$\mathscr{I}_3 \subseteq \mathscr{I}_2 \subseteq \mathscr{I}_1 \subseteq \mathscr{I}_0 = C(S_\mu U(3))$$

such that

$$\mathscr{I}_0/\mathscr{I}_1 \cong C(\mathbb{T}^2), \quad \mathscr{I}_1/\mathscr{I}_2 \cong \mathscr{I}_2/\mathscr{I}_3 \cong 2C(\mathbb{T}^2) \otimes \mathscr{K}$$

and $\mathscr{I}_3 \cong C(\mathbb{T}^2) \otimes \mathscr{K}$.

We remark that the above decomposition of $C(S_\mu U(3))$ is compatible with the singular foliation of the Lie-Poisson SU(3) **[Lu-We]** by the symplectic leaves **[We]**. More precisely, there are six 2-parameter families (with parameters in $\mathbb{T}^2$) of symplectic leaves diffeomorphic to $\mathbb{C}^0$, $\mathbb{C}^1$, $\mathbb{C}^1$, $\mathbb{C}^2$, $\mathbb{C}^2$ and $\mathbb{C}^3$, respectively as pointed out by

A. Weinstein in a private communication. With each leaf of positive dimension quantized by the Weyl quantization **[Hö, Vo]**, it is likely that we can find a deformation quantization (in the sense of **[Ri1]**) of the Poisson SU(3) as we did for the case of Poisson SU(2) in **[S]**. In a sense as explained in **[S]**, $C(S_\mu U(3))$ can be regarded as a foliation C^*-algebra of the (singular) symplectic foliation on SU(3).

With some more effort to analyse the data obtained, we are able to describe the topology of the spectrum Y of $C(S_\mu U(3))$. In order to do so, we shall say that a copy of $\mathbb{T}^2$ approximates another copy of $\mathbb{T}^2$ in a topological space in type $\ldots$ if any sequence in the first $\mathbb{T}^2$ converges to any element in the second $\mathbb{T}^2$, and in type $\underline{\quad}$, $\overline{\quad}$, $>$ or $=$, if a sequence $z(n)$ in the first $\mathbb{T}^2$ converges to w in the second $\mathbb{T}^2$ if and only if $z(n)_2 \to w_2$, $z(n)_1 \to w_1$, $z(n)_1 z(n)_2 \to \overline{w}_2$ or $z(n)_1 z(n)_2 \to w_1 w_2$ respectively. Now clearly Y is a union of the above Y_k's, and by a more detailed analysis of the factorizability among η's than the one specified in (1), we can conclude that Y is a disjoint union of Y_0, Y_{11}, Y_{12}, Y_{21}, Y_{22} and Y_3 (each homeomorphic to $\mathbb{T}^2$) such that (i) Y_3 is open dense in Y in the way that Y_3 approximates Y_{21}, Y_{22}, Y_{11}, Y_{12} and Y_0 in type $\underline{\quad}$, $\overline{\quad}$, $\ldots$, $\ldots$, and $\ldots$, respectively, (ii) Y_{21} and Y_{22} are disjoint open sets with dense union $Y_2 = Y_{21} \cup Y_{22}$ in $Y \backslash Y_3$ such that Y_{21} approximates Y_{11}, Y_{12}, and Y_0 in type $\overline{\quad}$, $=$ and $\ldots$ respectively, and Y_{22} approximates $Y_{11} Y_{12}$ and Y_0 in type $\underline{\quad}$ $>$ and $\ldots$ respectively ($Y_{12} \cap \overline{Y}_{22} = \varnothing$), (iii) Y_{11} and Y_{12} are disjoint open sets with dense union $Y_1 = Y_{11} \cup Y_{12}$ in $Y \backslash (Y_3 \cup Y_2)$ such that Y_{11} and Y_{12} approximating Y_0 in type $=$ and $\overline{\quad}$ respectively.

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