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A PHRAGMÉN-LINDELÖF THEOREM

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Let Ω be an unbounded and connected domain in E^n. Consider on $\Omega \times (0, \infty)$ the parabolic equation

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, \nabla u) = B(x, t, u, \nabla u).$$

Under proper conditions a theorem of Phragmén-Lindelöf type is proved for generalized solutions of the equation.

Introduction. The classical Phragmén-Lindelöf principle gives an important property of harmonic functions defined on a plane sector domain. That has been generalized not only to generalized solutions of quasi-linear elliptic equations in more general unbounded and connected domains (see **[1]**–**[5]**), but also to the ones of quasi-linear parabolic equations in divergence form which have their principal parts only **[6]**. In this paper the result is extended to generalized solutions of the equation (1). We prove the result by an argument based on the technique of Moser [7] and Ladyženskaja-Ural′ceva **[8]**. We have not seen any reference discussing such behavior for solutions of parabolic equations except **[6]** where the simpler situation of the equation (1), namely $B \equiv 0$, is considered.

The paper is organized as follows. In §1 the main result is mentioned and in §2 several lemmata are given as preliminaries. Finally, a full proof of our theorem is stated in §3.

1. Main result. Let Ω be an unbounded and connected domain in the n-dimensional Euclidean space E^n. Denote by $\partial\Omega$ the boundary of Ω. On $\Omega \times (0, \infty)$ we consider the following equation:

$$u_t - \operatorname{div} \mathbf{A}(x, t, u, \nabla u) = B(x, t, u, \nabla u) \tag{1}$$

where $A(x, t, u, \xi)$ and $B(x, t, u, \xi)$ are defined on $\Omega \times (0, \infty) \times E^1 \times E^n$, continuous with respect to u and ξ for fixed x and t, measurable with respect to x and t for fixed u and ξ, and satisfying the following structural conditions:

$$\begin{aligned} \xi \cdot \mathbf{A}(x, t, u, \xi) &\geq \kappa_0 |\xi|^2, \\ |\mathbf{A}(x, t, u, \xi)| &\leq \kappa_1 |\xi|, \\ |B(x, t, u, \xi)| &\leq b(x, t)|\xi|, \end{aligned} \tag{2}$$

where $\kappa_1 \geq \kappa_0 > 0$, $b(x, t) \in L_\infty(\Omega \times (0, \infty))$ and

$$|b(x, t)| = O(|x|^{-1}) \text{ (uniformly for } t) \quad \text{as } |x| \to \infty. \tag{3}$$

We need the supposition on Ω: there exist some $x_0 \in \partial\Omega$ and a $\theta \in (0, 1)$ such that

$$\begin{aligned} \operatorname{meas}(\Omega \cap \{B(x_0, \rho_0) \backslash B(x_0, \rho_1)\}) \\ \leq \theta \operatorname{meas}\{B(x_0, \rho_0) \backslash B(x_0, \rho_1)\} \end{aligned} \tag{4}$$

for any $\rho_0 > \rho_1 > 0$, where $\operatorname{meas} e$ denotes the Lebesgue measure of the set e in E^n and

$$B(x_0, \rho) = \{x \in E^n, |x - x_0| < \rho\}.$$

For $G \subset E^n$, $W_2^1(G)$ and $\overset{\circ}{W}{}_2^1(G)$ stand for the usual Sobolev spaces. Let X be a Banach space formed by measurable functions defined on G with respect to the norm $\|\cdot\|_X$. Denote $L_p(0, T, X)$ the Banach space formed by the mapping from $[0, T]$ into X with norm $\|u\|_{L_p(0,T,X)}$ defined by

$$\|u\|_{L_p(0,T,X)} = \left(\int_0^T \|u\|_X^p \, dx\right)^{1/p} \quad \left(= \operatorname*{ess\,sup}_{t \in (0,T)} \|u\|_X \text{ if } p = \infty\right).$$

Similarly, the space $C(0, T, X)$ etc. can also be defined.

The function u is called a generalized solution of the equation (1) if for any $T > 0$ and for arbitrary $G \subset \Omega$ and $G \subset\subset E^n$,

$$u \in C(0, T, L_2(G)) \cap L_2(0, T, W_2^1(G)) \tag{5}$$

and the following holds:

$$\begin{aligned} (1)' \quad & \int_0^t \int_G \{-v_t u + \nabla v \cdot A(x, t, u, \nabla u) - vB(x, t, u, \nabla u)\} \, dx \, dt \\ & \qquad + \int_G v(x, t) u(x, t) \Big|_{t=0}^{t=t} dx = 0, \end{aligned}$$

$$\forall t \in (0, T), \quad v \in W_2^1(0, T, L_2(G)) \cap L_2(0, T, \overset{\circ}{W}{}_2^1(G))$$

where $u(x, 0)$ is a given initial value of u.

As the main result we have

THEOREM. *Suppose that the conditions* (2)–(4) *are satisfied and the generalized solution* u *of the equation* (1) *satisfies*

$$u^+ = \max(u, 0) = 0 \quad \text{on } \partial\Omega \times (0, \infty) \quad \text{and} \quad u^+|_{t=0} = 0. \tag{6}$$

If there exists an $R > 0$ *such that* $M(R) > 0$, *then*

$$M(\rho) \to \infty \quad \text{as } \rho \to \infty$$

where

$$M(\rho) = \operatorname*{ess\,sup}_{Q(\rho)} u(x, t), \qquad Q(\rho) = \{\Omega \cap B(x_0, \rho)\} \times (0, \rho^2).$$

As an immediate consequence we have

COROLLARY. *If the* u *in the theorem is bounded from above, then* $u \le 0$ *on* $\Omega \times (0, \infty)$.

REMARK. The results of the theorem and corollary and the proof given in §3 below are also true for subsolutions of the equation (1). As the definition u is a subsolution if besides (5) it satisfies the following:

$$\int_{t'}^{t''} \int_G \{-v_t u + \nabla v \cdot \mathbf{A}(x, t, u, \nabla u) - vB(x, t, u, \nabla u)\}\, dx\, dt + \int_G v(x, t)u(x, t)\Big|_{t=t'}^{t=t''} dx \le 0,$$

$$\forall (t', t'') \subset (0, T), \quad v \in W_2^1(0, T, L_2(G)) \cap L_2(0, T, \overset{\circ}{W}{}_2^1(G)) \text{ and } v \ge 0.$$

2. Preliminaries.

LEMMA 1. *Suppose* G *is a bounded domain in* E^n, $T > 0$ *is a definite value and* u *satisfies* (5) *and* (1)$'$. *If there exists a constant* $M > 0$ *such that*

$$(u - M)^+ \in L_2(0, T, \overset{\circ}{W}{}_2^1(G)) \quad \text{and} \quad (u - M)^+\big|_{t=0} = 0 \tag{7}$$

then

$$\operatorname*{ess\,sup}_{G\times(0, T)} u(x, t) \le M. \tag{8}$$

Proof. If the statement were not true, there would be a

$$M' = \operatorname*{ess\,sup}_{G\times(0, T)} u > M \quad (M' = \infty \text{ is not exclusive}).$$

By (7), we have for any $k \in (M, M')$

$$(u - k)^+ \in L_2(0, T, \overset{\circ}{W}{}_2^1(G)) \quad \text{and} \quad (u - k)^+\big|_{t=0} = 0.$$

Hence it follows by the imbedding inequality in $L_2(0, T, \overset{\circ}{W}{}_2^1(G))$ that

$$\left(\int_0^T \int_G |(u-k)^+|^q \, dx \, dt\right)^{2/q} \leq C(n) |||(u-k)^+|||_{G\times(0,T)}$$

where $q = 2(1+2/n)$ and

$$|||(u-k)^+|||_{G\times(0,T)} = \operatorname*{ess\,sup}_{G\times(0,T)} \int_G |(u-k)^+|^2 \, dx + \int_0^T \int_G |\nabla(u-k)^+|^2 \, dx \, dt.$$

We assume temporarily that $(u-k)^+ \in W_2^1(0, T, L_2(G))$; then $v = (u-k)^+$ can be taken as a test function. Substituting v into $(1)'$ and integrating by parts with respect to t, we have by the use of (2) that

$$\tag{9} \begin{aligned} &\int_G |(u-k)^+|^2 \, dx + \int_0^t \int_G |\nabla(u-k)^+|^2 \, dx \, dt \\ &\quad \leq C \int_0^t \int_G b(x, t)(u-k)^+ |\nabla(u-k)^+| \, dx \, dt, \end{aligned}$$

where the constant $C > 0$ depends only on n and κ_0. However, we cannot guarantee $(u-k)^+ \in W_2^1(0, T, L_2(G))$ when u is the function in Lemma 1. What we have to do now is to extend $(u-k)^+$ to $G \times (-\infty, 0)$ by letting $(u-k)^+ = 0$ and instead of v we take

$$v' = \frac{1}{h} \int_t^{t+h} (u-k)^+ \, d\tau$$

as the test function. Repeating the above process again we obtain (9) by letting $h \to 0$ in the last result.

Since the two terms on the left-hand side of (9) are all non-negative, each of them does not exceed that on the right-hand side. Taking their supremums for $t \in (0, T)$, we have

$$\tag{10} |||(u-k)^+|||_{G\times(0,T)} \leq C \int_0^T \int_G (u-k)^+ |\nabla(u-k)^+| \, dx \, dt,$$

where we absorb the $\|b(x, t)\|_{L_\infty}$ into the constant C. Considering that the effective integral domain in (10) is only $\{G \times (0, T)\} \cap$

$\{k < u < M'\}$, we then have by Hölder inequality that

$$(11)\qquad \int_0^T \int_G (u-k)^+ |\nabla (u-k)^+|\, dx\, dt \le \varepsilon(k, M') \left(\int_0^T \int_G |(u-k)^+|^q \, dx\, dt \right)^{1/q} \cdot \left(\int_0^T \int_G |\nabla (u-k)^+|^2 \, dx\, dt \right)^{1/2} \le C(n)\varepsilon(k, M') |||(u-k)^+|||_{G\times(0,T)}$$

where

$$\varepsilon(k, M') = \left(\int_0^T \int_{G\cap\{k<u<M'\}} dx\, dt \right)^{1/(n+2)}.$$

Combining (10) with (11) we get

$$(12)\qquad 1 \le C(n)\varepsilon(k, M'),$$

where the constant $C(n) > 0$ is independent of k. So, we have $\varepsilon(k, M') \to 0$ as $k \to M'$ because

$$\iint_{\{G\times(0,T)\}\cap\{k<u<M'\}} dx\, dt \to 0 \quad \text{as } k \to M'.$$

Hence, the contradiction is obtained by (12). □

For simplicity we write $B(\rho) = B(0, \rho)$.

Lemma 2. *Suppose $\rho_0 > \rho_1 > 0$, $S \subset B(\rho_0)\backslash B(\rho_1)$ and*

$$\operatorname{meas} S \ge \theta \operatorname{meas}\{B(\rho_0)\backslash B(\rho_1)\}, \qquad \theta \in (0, 1).$$

Suppose $u \in W_p^1(B(\rho_0)\backslash B(\rho_1))$, $p \ge 1$ and $u = 0$ on S. Then

$$\int_{B(\rho_0)\backslash B(\rho_1)} |u|^p \, dx \le C\left(n, p, \theta, \frac{\rho_0}{\rho_1}\right) \rho_0^p \int_{B(\rho_0)\backslash B(\rho_1)} |\nabla u|^p \, dx.$$

Lemma 2 is a variety of Theorem 3.6.5, in Morrey **[9]** *and it can be proved by the same method.*

Lemma 3 **[10]**. *Let $f(t)$ be a non-negative bounded function defined for $0 \le r' \le t \le r$. If*

$$f(t) \le A(s-t)^{-\alpha} + B + \theta f(s), \qquad \forall r' \le t < s \le r$$

where A, B, α, θ are non-negative constants and $\theta \in (0, 1)$, then there exists a constant C depending only on α and θ such that

$$f(\rho) \le C(A(R-\rho)^{-\alpha} + B), \qquad \forall r' \le \rho < R \le r.$$

3. Proof of the theorem. Without loss of generality, let x_0 be the origin. We can rewrite the condition (3) as

$$(3)' \qquad |b(x, t)| \le K|x|^{-1} \quad \text{as } |x| \ge 1,$$

where K is a positive constant.

Let $\rho \ge \max(R, 1)$, $0 \le \rho_2 < \rho_1 < \rho_0 \le \rho$ and let $\zeta(x) = \zeta(|x|)$ be a piecewise linear and continuous function of $|x|$ satisfying

$$(13) \qquad \zeta(x) = \begin{cases} 0, & \text{as } |x| \le 2\rho - \rho_1 \text{ or } |x| \ge 4\rho + \rho_1, \\ 1, & \text{as } 2\rho - \rho_2 \le |x| \le 4\rho + \rho_2. \end{cases}$$

Then

$$|\nabla \zeta(x)| \le (\rho_1 - \rho_2)^{-1}.$$

The function u in the theorem as the generalized solution satisfying (5) and (6) is locally bounded from above on $(\Omega \cup \partial\Omega) \times (0, \infty)$ **[11]**. Therefore

$$M(\rho) = \operatorname*{ess\,sup}_{Q(\rho)} u(x, t) < \infty, \qquad Q(\rho) = \{\Omega \cap B(\rho)\} \times (0, \rho^2).$$

On $Q(5\rho)$ let

$$(14) \qquad \begin{aligned} w(x, t) &= \ln \frac{M(5\rho) + \varepsilon}{M(5\rho) + \varepsilon - u^+}, \qquad \varepsilon > 0, \\ v(x, t) &= \frac{\zeta^2(x)(w-k)^+}{M(5\rho) + \varepsilon - u^+}, \qquad k \ge 0. \end{aligned}$$

Because of the boundedness of u on $Q(5\rho)$, we have

$$(15) \qquad \begin{aligned} &w \in L_2(0, 25\rho^2 \cdot W_2^1(\Omega \cap B(5\rho)) \cap L_\infty(Q(5\rho)), \\ &w = 0 \quad \text{on } \{\partial\Omega \cap B(5\rho)\} \times (0, 25\rho^2) \cup \{t = 0\} \end{aligned}$$

and

$$v \in L_2(0, 25\rho^2, \overset{\circ}{W}{}_2^1(\Omega \cap B(5\rho))) \cap L_\infty(Q(5\rho)), \qquad v\big|_{t=0} = 0.$$

Suppose $v \in W_2^1(0, 25\rho^2, L_2(\Omega \cap B(5\rho)))$ (otherwise, we add a limit process to arrive at the same result). Such v can be taken as a test function. Substituting it into $(1)'$ yields

$$(16)\quad 0=\int_0^t\int_{\Omega\cap B(5\rho)}\left\{\zeta^2\left(\tfrac{1}{2}[(w-k)^+]^2\right)_t + \left[\frac{\zeta^2\nabla(w-k)^+}{M(5\rho)+\varepsilon-u^+} + \frac{\zeta^2(w-k)^+\nabla u^+}{(M(5\rho)+\varepsilon-u^+)^2} + \frac{(w-k)^+2\zeta\nabla\zeta}{M(5\rho)+\varepsilon-u^+}\right]\cdot\mathbf{A} + \frac{\zeta^2(w-k)^+B}{M(5\rho)+\varepsilon-u^+}\right\}dx\,dt,$$
$$t\in(0,25\rho^2).$$

By virtue of the appearance of $\zeta(x)$ and $(w-k)^+$ in (16) the effective integral domain is only

$$(17)\qquad \{\Omega\cap(B(4\rho+\rho_1)\backslash B(2\rho-\rho_1))\times(0,t)\}\cap\{w>k\},$$

on which $u^+>0$ because of (14). By the use of (2) it follows from (16) that

$$\frac{1}{2}\int_{\Omega\cap B(5\rho)}\zeta^2[(w-k)^+]^2\,dx + \kappa_0\int_0^t\int_{\Omega\cap B(5\rho)}(\zeta^2|\nabla(w-k)^+|^2+\zeta^2(w-k)^+|\nabla(w-k)^+|^2)\,dx\,dt \le \int_0^t\int_{\Omega\cap B(5\rho)}(w-k)^+[2\zeta|\nabla\zeta|\kappa_1+\zeta^2 b(x,t)]|\nabla(w-k)^+|\,dx\,dt.$$

With the aid of Young's inequality it follows from the inequality above that

$$(18)\quad \int_{\Omega\cap B(5\rho)}\zeta^2[(w-k)^+]^2\,dx+\int_0^t\int_{\Omega\cap B(5\rho)}\zeta^2|\nabla(w-k)^+|^2\,dx\,dt \le C\int_0^t\int_{\Omega\cap B(4\rho+\rho_1)\backslash B(2\rho-\rho_1)}(w-k)^+[|\nabla\zeta|^2+\zeta^2|b(x,t)|^2]\,dx\,dt \le C\left(\frac{1}{(\rho_1-\rho_2)^2}+\frac{1}{\rho^2}\right)\int_0^t\int_{\Omega\cap B(4\rho+\rho_1)\backslash B(2\rho-\rho_1)}(w-k)^+\,dx\,dt,$$

where the last inequality in (18) is obtained by the fact that $(3)'$ holds on the effective integral domain (17) and the constant $C>0$ depends only on n, κ_0, κ_1 and K. Extend w by taking $w(x, t) = 0$ as $x \notin \Omega$. We have from (4)

$$\operatorname{meas}(\{B(4\rho + \rho_1)\backslash B(2\rho - \rho_1)\} \cap \{(w - k)^+ = 0\}) \\ \geq (1 - \theta) \operatorname{meas}\{B(4\rho + \rho_1)\backslash B(2\rho - \rho_1)\}.$$

For $p = 1, 2$ applying Lemma 2 to $(w-k)^+$ on $B(4\rho+\rho_1)\backslash B(2\rho-\rho_1)$, we obtain

$$(19)' \quad \int_{\Omega\cap B(4\rho+\rho_1)\backslash B(2\rho-\rho_1)} (w - k)^+ \, dx \\ \leq C(n, \theta)\rho \int_{\Omega\cap B(4\rho+\rho_1)\backslash B(2\rho-\rho_1)} |\nabla(w - k)^+| \, dx$$

and

$$(19)'' \quad \int_{\Omega\cap B(4\rho+\rho_1)\backslash B(2\rho-\rho_1)} [(w - k)^+]^2 \, dx \\ \leq C(n, \theta)\rho^2 \int_{\Omega\cap B(4\rho+\rho_1)\backslash B(2\rho-\rho_1)} |\nabla(w - k)^+|^2 \, dx$$

respectively. It follows from (18) and $(19)'$ that

$$(20) \int_{\Omega\cap B(5\rho)} \zeta^2[(w - k)^+]^2 \, dx + \int_0^t \int_{\Omega\cap B(5\rho)} \zeta^2 |\nabla(w - k)^+|^2 \, dx \, dt \\ \leq C \left[\frac{1}{(\rho_1 - \rho_2)^2} + \frac{1}{\rho^2}\right] \rho \\ \cdot \int_{\Omega\cap B(4\rho+\rho_1)\backslash B(2\rho-\rho_1)} |\nabla(w - k)^+| \, dx \, dt \\ \leq C \left[\frac{1}{(\rho_1 - \rho_2)^2} + \frac{1}{\rho^2}\right]^2 \rho^2 \int_{\Omega\cap B(4\rho+\rho_1)\backslash B(2\rho-\rho_1)} \chi(k) \, dx \, dt \\ + \frac{1}{4} \int_{\Omega\cap B(4\rho+\rho_1)\backslash B(2\rho-\rho_1)} |\nabla(w - k)^+|^2 \, dx \, dt$$

where the constant $C > 0$ depends only on n, κ_0, κ_1, K and θ, and $\chi(k)$ is the characteristic function of the set $\{w > k\}$. Taking the supremum in (20) for $t \in (0, \rho^2)$ we get

$$\begin{aligned}(21)\qquad &\operatorname*{ess\,sup}_{t\in(0,\rho^2)}\int_{\Omega\cap B(5\rho)}\zeta^2[(w-k)^+]^2\,dx\\&\quad+\int_0^{\rho^2}\int_{\Omega\cap B(5\rho)}\zeta^2|\nabla(w-k)^+|^2\,dx\,dt\\&\le C\left[\frac{1}{(\rho_1-\rho_2)^2}+\frac{1}{\rho^2}\right]\rho^2\\&\quad\cdot\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_1)\setminus B(2\rho-\rho_1)}\chi(k)\,dx\,dt\\&\quad+\frac{1}{2}\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_1)\setminus B(2\rho-\rho_1)}|\nabla(w-k)^+|^2\,dx\,dt.\end{aligned}$$

According to the definition of $\zeta(x)$ it is obvious that

$$\begin{aligned}(22)\qquad &\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_2)\setminus B(2\rho-\rho_2)}|\nabla(w-k)^+|^2\,dx\,dt\\&\le\int_0^{\rho^2}\int_{\Omega\cap B(5\rho)}\zeta^2|\nabla(w-k)^+|^2\,dx\,dt.\end{aligned}$$

On account of C being independent of ρ_1 and ρ_2 and the arbitrariness of ρ_1 and ρ_2 in $0\le\rho_2<\rho_1\le\rho$, combining (22) with (21) and applying Lemma 3 we obtain

$$\begin{aligned}(23)\qquad &\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_2)\setminus B(2\rho-\rho_2)}|\nabla(w-k)^+|^2\,dx\,dt\\&\le C\left[\frac{1}{(\rho_1-\rho_2)^2}+\frac{1}{\rho^2}\right]^2\rho^2\\&\quad\cdot\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_1)\setminus B(2\rho-\rho_1)}\chi(k)\,dx\,dt,\end{aligned}$$

where the constant $C>0$ is independent of ρ_1, ρ_2 and ρ. Therefore, if $0\le\rho_1<\rho_0\le\rho$, it follows from (23) by replacing ρ_1 and ρ_2 by ρ_0 and ρ_1 respectively that

$$\begin{aligned}(24)\qquad &\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_1)\setminus B(2\rho-\rho_1)}|\nabla(w-k)^+|^2\,dx\,dt\\&\le C\left[\frac{1}{(\rho_0-\rho_1)^2}+\frac{1}{\rho^2}\right]^2\rho^2\\&\quad\cdot\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_0)\setminus B(2\rho-\rho_0)}\chi(k)\,dx\,dt.\end{aligned}$$

From (15) we have

$$(25)\quad \left(\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_2)\setminus B(2\rho-\rho_2)} |(w-k)^+|^q\,dx\,dt\right)^{2/q}$$
$$\le C(n)|||\zeta(x)(w-k)^+|||_{\{\Omega\cap B(4\rho+\rho_1)\setminus B(2\rho-\rho_1)\}\times(0,\rho^2)}$$
$$\le C(n)\left\{\operatorname*{ess\,sup}_{t\in(0,\rho^2)}\int_{\Omega\cap B(5\rho)}\zeta^2[(w-k)^+]^2\,dx\right.$$
$$+\int_0^{\rho^2}\int_{\Omega\cap B(5\rho)}\zeta^2|\nabla(w-k)^+|^2\,dx\,dt$$
$$\left.+\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_1)\setminus B(2\rho-\rho_1)}|\nabla\zeta|^2|(w-k)^+|^2\,dx\,dt\right\}.$$

Collecting (19)″, (21), (24) and (25), it follows that

$$\left(\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_2)\setminus B(2\rho-\rho_2)} |(w-k)^+|^q\,dx\,dt\right)^{2/q}$$
$$\le C\left[\frac{1}{(\rho_1-\rho_2)^2}+\frac{1}{\rho^2}\right]^2\rho^2\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_1)\setminus B(2\rho-\rho_1)}\chi(k)\,dx\,dt$$
$$+C\left[\frac{1}{(\rho_0-\rho_1)^2}+\frac{1}{\rho^2}\right]^2\rho^2\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_0)\setminus B(2\rho-\rho_0)}\chi(k)\,dx\,dt,$$

where $C>0$ depends only on n, κ_0, κ_1, K and θ. In particular, let $0\le\rho''=\rho_2<\rho_0=\rho'<\rho$ and $\rho_1=\frac{1}{2}(\rho'+\rho'')$. The inequality above can be rewritten as follows:

$$(26)\quad \left(\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho'')\setminus B(2\rho-\rho'')} |(w-k)^+|^q\,dx\,dt\right)^{2/q}$$
$$\le C\left[\frac{1}{(\rho'-\rho'')^2}+\frac{1}{\rho^2}\right]^2\rho^2$$
$$\cdot\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho')\setminus B(2\rho-\rho')}\chi(k)\,dx\,dt.$$

Take for $\nu=0,1,2,\ldots$

$$\rho_\nu=\rho/2^\nu,\quad k_\nu=H-H/2^\nu\qquad (H>0 \text{ will be special}),$$
$$A_\nu=\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_\nu)\setminus B(2\rho-\rho_\nu)}\chi(k_\nu)\,dx\,dt.$$

Since the constant C in (26) is independent of ρ', ρ'' and k, replace ρ', ρ'' by ρ_ν, $\rho_{\nu+1}$, and k by k_ν, it follows from (26) that

$$\begin{aligned}&(k_{\nu+1}-k_\nu)^2 A_{\nu+1}^{2/q}\\ &\quad\le \left(\int_0^{\rho^2}\int_{\Omega\cap B(4\rho+\rho_{\nu+1})\setminus B(2\rho-\rho_{\nu+1})} |(w-k_\nu)^+|^q\,dx\,dt\right)^{2/q}\\ &\quad\le C\left[\frac{1}{(\rho_\nu-\rho_{\nu+1})^2}+\frac{1}{\rho^2}\right]^2 \rho^2 A_\nu, \qquad \nu=0,1,2,\dots,\end{aligned}$$

namely,

$$\begin{aligned}A_{\nu+1}^{2/q} &\le C\left(\frac{2^{\nu+1}}{H}\right)^2\left[\left(\frac{2^{\nu+1}}{\rho}\right)^2+\frac{1}{\rho^2}\right]^2\rho^2 A_\nu\\ &\le C2^8\cdot 2^{6\nu}(H\rho)^{-2}A_\nu, \qquad \nu=0,1,2,\dots.\end{aligned} \tag{27}$$

For $\nu=0$ we have

$$A_0=\int_0^{\rho^2}\int_{\Omega\cap B(5\rho)\setminus B(\rho)}\chi(0)\,dx\,dt\le \operatorname{meas}B(5)\rho^{n+2}. \tag{28}$$

As long as we assume $H>0$ so large that

$$\left(\frac{C\cdot 2^8}{H}\right)^{1+2/(n+2)}[\operatorname{meas}B(5)]^{2/(n+2)}\le\delta, \qquad 2^{6(1+2/(n+2))}\delta^{2/(n+2)}=1, \tag{29}$$

from (27), (28) and (29) it can be shown by induction that

$$A_\nu\le\delta^\nu A_0, \qquad \nu=1,2,\dots.$$

Let $\nu\to\infty$; then

$$\int_0^{\rho^2}\int_{\Omega\cap B(4\rho)\setminus B(2\rho)}\chi(H)\,dx\,dt=0,$$

which implies

$$\operatorname*{ess\,sup}_{\{\Omega\cap B(4\rho)\setminus B(2\rho)\}\times(0,\rho^2)} w\le H.$$

According to the definition of w we have

$$\operatorname*{ess\,sup}_{\{\Omega\cap B(4\rho)\setminus B(2\rho)\}\times(0,\rho^2)} u^+\le[M(5\rho)+\varepsilon](1-e^{-H}).$$

Let $\varepsilon\to 0$; then

$$\operatorname*{ess\,sup}_{\{\Omega\cap B(4\rho)\setminus B(2\rho)\}\times(0,\rho^2)} u^+\le M(5\rho)(1-e^{-H}).$$

It follows from Lemma 1 that

$$(30)\qquad M(\rho) = \operatorname*{ess\,sup}_{\{\Omega\cap B(\rho)\}\times(0,\rho^2)} u \le \operatorname*{ess\,sup}_{\{\Omega\cap B(3\rho)\}\times(0,\rho^2)} u$$
$$\le \operatorname*{ess\,sup}_{\{\Omega\cap B(4\rho)\setminus B(2\rho)\}\times(0,\rho^2)} u^+ \le M(5\rho)(1-e^{-H}).$$

We see from (29) that H is determined by constants C and n; hence, H is independent of ρ.

Now, suppose $\rho_0 = \max(R, 1)$. For any $\rho \ge \rho_0$ there exists an integer ν such that $5^\nu \rho_0 \le \rho < 5^{\nu+1}\rho_0$. Iterating by (30) we get

$$\begin{aligned} M(\rho) &\ge M(5^\nu \rho_0) \ge (1-e^{-H})^{-\nu} M(\rho_0) \\ &\ge (1-e^{-H}) M(\rho_0)(1-e^{-H})^{-\log_5(\rho/\rho_0)} \\ &= (1-e^{-H}) M(\rho_0)(\rho/\rho_0)^\lambda \ge (1-e^{-H}) M(R)(\rho/\rho_0)^\lambda, \end{aligned}$$

$$\lambda = \log_5(1-e^{-H})^{-1} > 0, \qquad \rho \ge \rho_0.$$

Thus, $M(\rho) \to \infty$ as $\rho \to \infty$ whenever $M(R) > 0$. The proof of the theorem is completed.

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