

SZEGÖ MAPS AND HIGHEST WEIGHT REPRESENTATIONS

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Let G be a connected noncompact simple Lie group with finite center and let K be a maximal compact subgroup of G . Assume the space G/K is Hermitian symmetric. We associate to each irreducible representation τ of K a principal series representation $W(\tau)$ and a G -equivariant Szegő-type integral operator S_τ such that S_τ maps the K -finite vectors in $W(\tau)$ onto an irreducible highest weight \mathfrak{g} -module $L(\tau)$. Of primary concern here are those representations τ which are reduction points. For such τ , we construct certain systems \mathcal{D}_τ of G -equivariant differential operators and then utilize \mathcal{D}_τ to establish the infinitesimal irreducibility of the image of S_τ .

1. Introduction. Let G be a connected noncompact simple Lie group with finite center and let K be a maximal compact subgroup of G . Assume the space G/K is Hermitian symmetric. The main purpose of this article is to realize each irreducible highest weight representation of G as the image of a G -equivariant quotient map defined on principal series representations. To make this more precise, recall that each irreducible highest weight representation π_τ of G is parametrized by an irreducible unitary representation τ of K . Let $C^\infty(G, \tau)$ denote the space of τ -covariant C^∞ -functions on G . We associate to τ a principal series representation $W(\tau)$ and a Szegő map $S_\tau: W(\tau) \rightarrow C^\infty(G, \tau)$ having the property that the K -finite vectors in $W(\tau)$ are mapped onto an irreducible \mathfrak{g} -module equivalent to the derived action of π_τ . In the case of discrete series and limits thereof, this type of result was proved by Knapp and Wallach [16] in the general setting where G is a semisimple equirank Lie group with finite center. The main result here is that the irreducibility of the image of S_τ persists for all highest weight representations. Moreover, for certain τ called reduction points, the irreducibility of $\text{Image}(S_\tau)$ is proved by showing this space is annihilated by a system \mathcal{D}_τ of G -equivariant differential operators. The system \mathcal{D}_τ somewhat parallels the role of the Schmid operator in the Knapp and Wallach result.

The realization of distinguished representations as irreducible images of quotient maps is a recurring theme in the literature which

appears in somewhat different contexts. For example, Okamoto [19] realizes both discrete series representations and limits thereof in this way when (G, K) is a Hermitian symmetric pair. By inducing from a maximal parabolic, Inoue [13] identifies generalized limits of discrete series representations as Hardy-type spaces and views the Szegő mapping as projection onto the irreducible Hardy space. For semisimple equirank G , both Knapp and Wallach [16] and Blank [1] realize discrete series representations as the image of such quotient mappings. Limits of complementary series are obtained by Gilbert, Kunze, Stanton and Tomas [9] in an analogous way. We refer also to [5], [6] and [8].

The main feature contrasting our results and the results cited above is that we work in the Hermitian symmetric setting with the set of all highest weight representations of G . In particular, unitarity is not assumed. Our results are most significant in the case of exceptional highest weight representations. By this we mean the representations π_τ such that τ lies in the set \widehat{K}_r of reduction points. To explain the notion of reduction point, we briefly recall a standard realization of the highest weight representation space (cf. §5). Define left translation on the space $C^\infty(G, \tau; \mathfrak{p}_-)$ of functions on G which are smooth, vector-valued, annihilated by left invariant vector fields $r(X)$, $X \in \mathfrak{p}_-$, and which satisfy a transformation property by τ . Then $\tau \in \widehat{K}_r$ if and only if the subspace $C^\infty_\circ(G, \tau; \mathfrak{p}_-)$ of K -finite vectors forms a reducible \mathfrak{g} -module.

This characterization of \widehat{K}_r reveals why it is natural to focus on those highest weight representations parametrized by τ in \widehat{K}_r . For given an irreducible unitary representation τ of K , Theorem (6.1) [16] gives appropriate inducing parameters (σ, ν) so that the corresponding S_τ maps the nonunitary principal series $W(\sigma, \nu)$ G -equivariantly and nontrivially into $C^\infty(G, \tau; \mathfrak{p}_-)$. If $\tau \notin \widehat{K}_r$, then $C^\infty_\circ(G, \tau; \mathfrak{p}_-)$ is irreducible and thus the image of the K -finite vectors in $W(\sigma, \nu)$ is an irreducible \mathfrak{g} -module. If $\tau \in \widehat{K}_r$ however, the space $C^\infty_\circ(G, \tau; \mathfrak{p}_-)$ is reducible and yet the infinitesimal irreducibility of the image of S_τ persists. To prove this result we turn to gradient-type differential operators.

A gradient-type differential operator ∂ is a G -equivariant homogeneous differential operator $\partial: C^\infty(G, \tau) \rightarrow C^\infty(G, \xi)$ (cf. §7). For $\tau \in \widehat{K}_r$, we construct a finite system $\mathcal{D}_\tau = \{\partial_1, \dots, \partial_r\}$ of such operators and show that the subspace of $C^\infty_\circ(G, \tau; \mathfrak{p}_-)$ annihilated by all the ∂_i is irreducible. In the case where τ corresponds to a unitariz-

able highest weight representation, it is shown in [3] that \mathcal{D}_τ can be chosen to consist of a single differential operator. This distinguished operator, called a covariant differential operator, arises as the dual of a \mathfrak{g} -mapping between generalized Verma modules. The irreducibility of the kernel in that case is immediate from duality and a special submodule property resulting from unitarity (cf. [2], [3]). Here we avoid duality arguments and prove the irreducibility of $\text{Ker}(\mathcal{D}_\tau) \cap C^\infty(G, \tau; \mathfrak{p}_-)$ directly.

From the above comments it is clear that the proof of the infinitesimal irreducibility of the image of S_τ reduces to showing that S_τ maps into the kernel of \mathcal{D}_τ . It is natural to attempt to prove S_τ maps into $\text{Ker}(\mathcal{D}_\tau)$, $\tau \in \widehat{K}_r$, by combining an induction argument on the degree of ∂_i with the calculational techniques found in the proof of (6.1) [16]. This approach forces one to deal with the fact that the kernel function s_τ defining S_τ is initially expressed in terms of an Iwasawa decomposition while the operator \mathcal{D}_τ is expressed in terms of a Cartan decomposition. This fact, coupled with the indefinite order of the operators in \mathcal{D}_τ , leads to some computational problems which we find to be intractable.

To circumvent these difficulties, we utilize a particularly useful reformulation of s_τ in terms of a factor of automorphy J_τ . This reformulation is well-suited for our purposes since the factor J_τ is defined in terms of the Cartan decomposition. The precise connection between s_τ and J_τ involves a distinguished point b in the Bergmann-Shilov boundary and is given explicitly in §4. A connection of this kind had been previously observed by Inoue [13] in the context of “generalized” limits of discrete series. There the image of the Szegő map S_τ is viewed as a Hardy space H^2 and the associated G -action is the restriction of a holomorphically induced multiplier representation T to H^2 .

In our context, the multiplier representation T plays an important role as well. This is because the proof that the image of S_τ is annihilated by each ∂_i in \mathcal{D}_τ (cf. §8) reduces to showing $\partial_i s_\tau = 0$, $\partial_i \in \mathcal{D}_\tau$. Since both s_τ and T are defined in terms of J_τ , the function $\partial_i s_\tau$ can be conveniently expressed in terms of the derived action of T . The precise formulation is given in §6.

The authors thank Professor Ray A. Kunze for helpful discussions regarding quotient maps and Professor Larry Smolinsky for illuminating conversations on covering spaces. We also express our appreciation to the referee for offering suggestions that greatly enhanced the clarity

and scope of this paper. In particular, our original version dealt only with unitary highest weight representations. The referee suggested that our differential operator techniques may extend to the nonunitary setting as well. We found this to indeed be true and have subsequently dropped the unitarity assumption from the final version.

2. Preliminaries. Let G denote a connected noncompact simple Lie group with finite center. Let K be a maximal compact subgroup and assume G/K is a Hermitian symmetric space. Choose a compact Cartan subgroup $T \subset K$ and let \mathfrak{g}_0 , \mathfrak{k}_0 and \mathfrak{t}_0 denote the Lie algebras of G , K and T . Fix a Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$. We adopt the convention that the removal of the subscript 0 denotes complexification. By our assumptions there exists a decomposition $\mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{t} \oplus \mathfrak{p}_-$ where \mathfrak{p}_+ and \mathfrak{p}_- are $\text{ad}(\mathfrak{k})$ -invariant abelian subalgebras of \mathfrak{p} . The subalgebra \mathfrak{t} is a Cartan subalgebra of \mathfrak{g} and we let Φ be the roots corresponding to $(\mathfrak{g}, \mathfrak{t})$. Let Φ_c and Φ_n denote the set of compact and noncompact roots, respectively. We denote the root space corresponding to $\alpha \in \Phi$ by \mathfrak{g}_α and choose a positive system of roots Φ^+ so that $\mathfrak{p}_+ = \bigoplus_{\alpha \in \Phi_n^+} \mathfrak{g}_\alpha$ and $\mathfrak{p}_- = \bigoplus_{\alpha \in \Phi_n^-} \mathfrak{g}_\alpha$ where $\Phi_n^+ = \Phi^+ \cap \Phi_n$ and $\Phi_n^- = (-\Phi^+) \cap \Phi_n$.

If θ denotes Cartan involution and B denotes the Killing form, then we choose root vectors $E_\alpha \in \mathfrak{g}_\alpha$, $\alpha \in \Phi$, so that $B(E_\alpha, E_{-\alpha}) = 2/(\alpha, \alpha)$ and $\theta \bar{E}_\alpha = -E_{-\alpha}$. The bar here is conjugation of \mathfrak{g} with respect to \mathfrak{g}_0 and (\cdot, \cdot) is the standard inner product on the real space of linear functionals on \mathfrak{t} taking purely imaginary values on \mathfrak{t}_0 . We put $H_\alpha = [E_\alpha, E_{-\alpha}]$.

Let r be the split rank of G . We inductively define a maximal set $\{\gamma_1, \dots, \gamma_r\}$ of strongly orthogonal positive noncompact roots by setting $\gamma_1 =$ largest root of Φ and $\gamma_{j+1} =$ largest root in Φ_n^+ which is strongly orthogonal to $\gamma_1, \dots, \gamma_j$. Then $E_{\gamma_j} + E_{-\gamma_j}$ lies in \mathfrak{p}_0 for each $1 \leq j \leq r$ and

$$(2.1) \quad \mathfrak{a}_0 = \bigoplus_{1 \leq j \leq r} \mathbb{R}(E_{\gamma_j} + E_{-\gamma_j})$$

is a maximal abelian subspace of \mathfrak{p}_0 . Let $R = R(\mathfrak{a}_0)$ denote the set of restricted roots with respect to \mathfrak{a}_0 and let R^+ denote the positive roots with respect to the lexicographic ordering induced by the ordered basis $\{E_{\gamma_1} + E_{-\gamma_1}, \dots, E_{\gamma_r} + E_{-\gamma_r}\}$ of \mathfrak{a}_0 . If \mathfrak{n}_0 denotes the sum of positive restricted root spaces, then $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}_0 \oplus \mathfrak{n}_0$ gives an Iwasawa decomposition of \mathfrak{g}_0 with corresponding decomposition $G = KAN$

of G . For $g \in G$, we write

$$(2.2) \quad g = \kappa(g)e^{H(g)}n(g)$$

where $\kappa(g) \in K$, $H(g) \in \mathfrak{a}_0$ and $n(g) \in N$.

We now recall the definition of the principal series representations of G . Let M denote the centralizer of \mathfrak{a}_0 in K and let σ be an irreducible unitary representation of M with representation space H_σ . Set $2\rho = \sum_{\mu \in R^+} (\dim \mu)\mu$ and let ν denote a complex-valued linear function on \mathfrak{a}_0 . We induce the representation $\sigma \otimes e^\nu \otimes 1$ of the minimal parabolic subgroup $P = MAN$ to a representation of G . Let $C(\nu, \sigma)$ denote the space of continuous functions $f: G \rightarrow H_\sigma$ satisfying the property

$$(2.3) \quad f(gman) = e^{-(\nu+\rho)H(a)}\sigma(m)^{-1}f(g).$$

Let $W(\nu, \sigma)$ denote the completion of $C(\nu, \sigma)$ with respect to the norm

$$(2.4) \quad \|f\|^2 = \int_K |f(k)|^2 dk.$$

The representation $L = L(\nu, \sigma)$ of G on $W(\nu, \sigma)$ is given by left translation: $(L(g)f)(x) = f(g^{-1}x)$, $x \in G$. Then for each $g \in G$ the operator $L(g)$ is bounded and for each $f \in W(\nu, \sigma)$, the mapping $g \rightarrow L(g)f$ is continuous. The parameters are arranged so that L is unitary if ν is imaginary.

3. Quotient maps. We briefly discuss some basic facts about quotient maps which are needed for our purposes. We refer the reader to [8] and [18] for a more comprehensive exposition and some explicit examples. Let (τ, V_τ) (resp. (σ, H_σ)) be an irreducible unitary representation of K (resp. M) on the complex vector space V_τ (resp. H_σ). Let $\text{Hom}_M(H_\sigma, V_\tau)$ denote the complex space of linear maps $C: H_\sigma \rightarrow V_\tau$ satisfying $C\sigma(m) = \tau(m)C$ for all $m \in M$. For $C \in \text{Hom}_M(H_\sigma, V_\tau)$ we define $B_C: V_\tau \rightarrow W(\nu, \sigma)$ by

$$(3.1) \quad B_C v(g) = e^{-(\nu+\rho)H(g)}C^*\tau(\kappa(g))^{-1}v, \quad v \in V_\tau.$$

Then B_C is a continuous mapping into $C(\nu, \sigma)$ which satisfies $L(k)B_C = B_C\tau(k)$ for all $k \in K$. The adjoint map $B_C^*: W(\nu, \sigma) \rightarrow V_\tau$ is easily seen to be

$$(3.2) \quad B_C^* f = \int_K \tau(k)Cf(k) dk.$$

Let $C(G, \tau)$ denote the space of continuous functions $f: G \rightarrow V_\tau$ satisfying $f(gk) = \tau(k)^{-1}f(g)$ for all $g \in G$ and $k \in K$. Following

Kunze [18], we define the quotient map $S_C: W(\nu, \sigma) \rightarrow C(G, \tau)$ by $S_C f(x) = B_C^* L(x^{-1})f$. For $f \in W(\nu, \sigma)$ and $x \in G$, then

$$(3.3) \quad S_C f(x) = \int_K \tau(k) C f(xk) dk.$$

It is clear that left translation $L(g)$, $g \in G$, leaves the space $C(G, \tau)$ invariant and we have $L(g)S_C = S_C L(g)$. Moreover, the equation $S_C B_C v(1) = B_C^* B_C v$, for $v \in V_\tau$, implies that the \mathbb{C} -linear map $C \rightarrow S_C$ is injective.

For each $x \in G$ and function $f \in C(\nu, \sigma)$, the function $F_x: K \rightarrow H_\sigma$ defined by $F_x(k) = \tau(k) C f(xk)$ is right M -invariant and so we may apply the integral formula [15, p. 170]

$$\int_K F(k) dk = \int_K e^{-2\rho H(g^{-1}k)} F(\kappa(g^{-1}k)) dk, \quad g \in G,$$

to the function F_x and write $S_C f(x)$ in (3.3) as

$$\begin{aligned} S_C f(x) &= \int_K e^{-2\rho H(g^{-1}k)} F_x(\kappa(g^{-1}k)) dk \\ &= \int_K e^{-2\rho H(g^{-1}k)} \tau(\kappa(g^{-1}k)) C f(x\kappa(g^{-1}k)) dk \\ &= \int_K e^{-2\rho H(g^{-1}k)} \tau(\kappa(g^{-1}k)) \\ &\quad \times e^{-(\nu+\rho)H(x\kappa(g^{-1}k))} C f(\kappa(x\kappa(g^{-1}k))) dk. \end{aligned}$$

If we let $x = g$ and observe that $\kappa(g\kappa(g^{-1}k)) = k$ and $H(g\kappa(g^{-1}k)) = -H(g^{-1}k)$ we obtain

$$(3.4) \quad (S_C f)(g) = \int_K e^{-(\rho-\nu)H(g^{-1}k)} \tau(\kappa(g^{-1}k)) C f(k) dk.$$

Define $s_C: G \rightarrow \text{Hom}(H_\sigma, V_\tau)$ by $s_C(g) = e^{-(\rho-\nu)H(g^{-1})} \tau(\kappa(g^{-1})) C$ for $g \in G$. Then

$$(3.5) \quad (S_C f)(g) = \int_K s_C(k^{-1}g) f(k) dk.$$

For a fixed representation (σ, H_σ) of M , let σ_ν denote the representation of $P = MAN$ on H_σ given by

$$(3.6) \quad \sigma_\nu(man) = e^{(\nu-\rho)H(a)} \sigma(m).$$

Let $K(\nu, \sigma, \tau)$ denote the space of functions $s: G \rightarrow \text{Hom}(H_\sigma, V_\tau)$ satisfying the transforming property

$$(3.7) \quad s(pgk) = \tau(k)^{-1} s(g) \sigma_\nu(p)^{-1}$$

for all $p \in P$, $g \in G$ and $k \in K$. We call $K(\nu, \sigma, \tau)$ the *space of kernel functions associated to (ν, σ, τ)* .

(3.8) PROPOSITION. *The map $\Gamma: \text{Hom}_M(H_\sigma, V_\tau) \rightarrow K(\nu, \sigma, \tau)$ given by $\Gamma(C) = s_C$ is an isomorphism of $\text{Hom}_M(H_\sigma, V_\tau)$ onto $K(\nu, \sigma, \tau)$.*

Proof. We first show that s_C satisfies (3.7). Using the Iwasawa decomposition $G = ANK$, write $g \in G$ as $g = e^{H_1(g)}(g)n_1(g)\kappa_1(g)$ where $H_1(g) \in \mathfrak{a}_\sigma$, $n_1(g) \in N$ and $\kappa_1(g) \in K$. One has

$$\begin{aligned} g &= (g^{-1})^{-1} = (\kappa(g^{-1})e^{H(g^{-1})}n(g^{-1}))^{-1} \\ &= n(g^{-1})^{-1}e^{-H(g^{-1})}\kappa(g^{-1})^{-1} \\ &= e^{-H(g^{-1})}(e^{H(g^{-1})}n(g^{-1})^{-1}e^{-H(g^{-1})})\kappa(g^{-1})^{-1}. \end{aligned}$$

Since A normalizes N , the two Iwasawa decompositions are related by the formulas $H_1(g) = -H(g^{-1})$, $n_1(g) = e^{H(g^{-1})}n(g^{-1})^{-1}e^{-H(g^{-1})}$ and $\kappa_1(g) = \kappa(g^{-1})^{-1}$. Hence $s_C(g) = e^{(\rho-\nu)H_1(g)}\tau(\kappa_1(g))^{-1}C$. Since MA normalizes N we have $\kappa_1(mang) = m\kappa_1(g)$ and $e^{H_1(mang)} = ae^{H_1(g)} = e^{H(a)}e^{H_1(g)}$ for all $man \in MAN$ and $g \in G$. Then

$$\begin{aligned} s_C(mang) &= \tau(k^{-1})s_C(mang) \\ &= \tau(k)^{-1}e^{(\rho-\nu)H_1(mang)}\tau(\kappa_1(mang))^{-1}C \\ &= \tau(k)^{-1}e^{(\rho-\nu)H(a)}e^{(\rho-\nu)H_1(g)}\tau(\kappa_1(g))^{-1}\tau(m)^{-1}C \\ &= \tau(k)^{-1}s_C(g)\sigma_\nu(man)^{-1}. \end{aligned}$$

This shows that Γ maps into $K(\nu, \sigma, \tau)$. If $s \in K(\nu, \sigma, \tau)$, then (3.7) implies $s(1) = s(m^{-1}1m) = \tau(m)^{-1}s(1)\sigma(m)$ so that $s(1) \in \text{Hom}_M(H_\sigma, H_\tau)$. Moreover, $\Gamma(s(1))(ank) = \tau(k)^{-1}e^{(\rho-\nu)H(a)}s(1) = s(ank)$ and since $G = ANK$, it follows that Γ is onto. Finally, the equation $\Gamma(C)(1) = C$ implies Γ is injective and so the proof is complete.

Let $\lambda = \lambda(\tau)$ be the highest weight of (τ, V_τ) and let $\phi_\lambda \in V_\tau$ denote a nonzero highest weight vector. We define a representation (σ, H_σ) of M as follows:

$$(3.9) \quad \begin{cases} H_\sigma = \text{span}_{\mathbb{C}}\{\tau(m)\phi_\lambda : m \in M\} & \text{and} \\ \sigma \text{ is the restriction of } \tau \text{ to } M \text{ on } H_\sigma. \end{cases}$$

Since we are in the Hermitian symmetric setting we know by [17] and Proposition 5.5 [16; p. 176] that (σ, H_σ) is an irreducible representation of M .

Let $W(\tau)$ denote the nonunitary principal series representation with $\sigma = \sigma(\tau)$ chosen as in (3.9) and $\nu = \nu(\tau)$ chosen to satisfy

$$(3.10) \quad (\nu - \rho)(E_{\gamma_j} + E_{-\gamma_j}) = \frac{2(\lambda, \gamma_j)}{(\gamma_j, \gamma_j)}, \quad 1 \leq j \leq r.$$

The inclusion map $C_\tau: H_\sigma \rightarrow V_\tau$ is clearly a nonzero element of $\text{Hom}_M(H_\sigma, V_\tau)$ and we let S_τ (resp. s_τ) denote the associated quotient map (resp. kernel function). We call S_τ the *Szegö map associated to τ* . Note that since the map $C \rightarrow S_C$ is injective, S_τ is not the zero map.

4. Factors of automorphy and kernel functions. We begin by recalling some standard facts on covering groups. The details that are omitted here may be found in [12] and [20]. Since G has finite center it covers a group G° which admits a faithful matrix representation. (One may take for example $G^\circ = G/Z$, where Z is the center of G . Then G° is isomorphic to the adjoint group $\text{Ad}(G)$.) Let $\gamma: G \rightarrow G^\circ$ denote the covering map and \mathcal{E} the kernel of γ . Then \mathcal{E} is contained in the center of G . The image K° , A° and N° of K , A and N is an Iwasawa decomposition of G° and A and N are diffeomorphic to A° and N° , respectively. Furthermore, since the center of G is contained in the center of K , the restriction $\gamma_K: K \rightarrow K^\circ$ is a covering map with kernel \mathcal{E} . Let $K_\mathbb{C}^\circ$, $K_\mathbb{C}$ and $G_\mathbb{C}^\circ$ denote the complexifications of K° , K and G° , respectively. Since K° , K and G° admit faithful finite dimensional representations, we can and do identify each group with its image in the complexification. Furthermore, we identify $K_\mathbb{C}^\circ$ with the connected component of $G_\mathbb{C}^\circ$ whose Lie algebra is \mathfrak{k} and note that $\gamma_K: K \rightarrow K_\mathbb{C}^\circ$ extends to a covering map, again denoted by γ_K , of $K_\mathbb{C}$ onto $K_\mathbb{C}^\circ$. The kernel of this extension is also \mathcal{E} . By identifying G° (resp. $K_\mathbb{C}^\circ$) with the quotient G/\mathcal{E} (resp. $K_\mathbb{C}/\mathcal{E}$), we assume the differential of γ (resp. γ_K) at the identity element 1 of G (resp. $K_\mathbb{C}$) to be the identity map. In particular, if $\exp: \mathfrak{g}_\circ \rightarrow G$ (resp. $\exp_\circ: \mathfrak{g}_\circ \rightarrow G^\circ$) denotes the exponential map, then $\gamma(\exp(X)) = \exp_\circ(X)$ for all $X \in \mathfrak{g}_\circ$.

Recall the Harish-Chandra realization [10] of G/K as a bounded domain in \mathfrak{p}_+ . If $P_\pm = \exp_\circ(\mathfrak{p}_\pm)$ in $G_\mathbb{C}^\circ$, then the map $\mathfrak{p}_+ \times K_\mathbb{C}^\circ \times \mathfrak{p}_- \rightarrow G_\mathbb{C}^\circ$ defined by $(a, k, b) \rightarrow \exp_\circ(a)k \exp_\circ(b)$ is a holomorphic diffeomorphism onto a dense open set $\Omega = P_+K_\mathbb{C}^\circ P_-$ of $G_\mathbb{C}^\circ$. We uniquely write each $x \in \Omega$ as the product

$$(4.1) \quad x = \pi_+(x)\pi_\circ(x)\pi_-(x), \quad \text{where } \pi_\pm(x) \in P_\pm \text{ and } \pi_\circ(x) \in K_\mathbb{C}^\circ.$$

One knows $G^\circ \subset \Omega$ and the map $\zeta: \Omega \rightarrow \mathfrak{p}_+$ defined by $\zeta(x) = \log(\pi_+(x))$ induces a holomorphic diffeomorphism of $G^\circ/K^\circ = G/K$ onto $\zeta(G^\circ)$. The set $D = \zeta(G^\circ)$ is a bounded domain in \mathfrak{p}_+ and we identify D with G/K . In particular, the coset K in G/K is identified with the origin 0 in \mathfrak{p}_+ .

For each $(g, z) \in G_{\mathbb{C}}^\circ \times \mathfrak{p}_+$ for which $g \exp_o(z) \in \Omega$ we set

$$(4.2) \quad g.z = \log \pi_+(g \exp_o(z)) \quad \text{and}$$

$$(4.3) \quad j_o(g, z) = \pi_o(g \exp_o(z)).$$

It is known that if $g \in G^\circ$ and $z \in \overline{D}$ then $g \exp(z) \in \Omega$. Consequently, (4.2) defines a G_o -action on \overline{D} (the closure of D in \mathfrak{p}_+). This action lifts to G through the covering map and we denote it in the same way: $g.z = \gamma(g).z$, $g \in G$ and $z \in \overline{D}$. The map j_o satisfies

$$(4.4) \quad j_o(k, z) = k \quad \text{for all } (k, z) \in K_{\mathbb{C}}^\circ \times \mathfrak{p}_+,$$

$$(4.5) \quad j_o(p, z) = 1 \quad \text{for all } (p, z) \in P_+ \times \mathfrak{p}_+,$$

$$(4.6) \quad j_o(g_1 g_2, z) = j_o(g_1, g_2.z) j_o(g_2, z),$$

where (4.6) holds for all $g_1, g_2 \in G_{\mathbb{C}}^\circ$ and $z \in \mathfrak{p}_+$ for which $g_1 g_2 \exp(z)$ and $g_2 \exp(z)$ both lie in Ω .

(4.7) **PROPOSITION.** *The map $j_o: G^\circ \times \overline{D} \rightarrow K_{\mathbb{C}}^\circ$ lifts uniquely to a continuous map $j: G \times \overline{D} \rightarrow K_{\mathbb{C}}$ satisfying*

$$(1) \quad \gamma_K(j(g, z)) = j_o(\gamma(g), z), \text{ for all } (g, z) \in G \times \overline{D},$$

$$(2) \quad j(k, z) = k, \text{ for all } (k, z) \in K \times \overline{D},$$

$$(3) \quad j(g_1 g_2, z) = j(g_1, g_2.z) j(g_2, z), \text{ for all } g_1, g_2 \in G, z \in \overline{D},$$

$$(4) \quad j \text{ is } C^\infty \text{ on } G \text{ and holomorphic on } D.$$

Proof. A norm $|\cdot|$ can be defined on \mathfrak{p}_+ so that $D = \{X \in \mathfrak{p}_+ : |X| < 1\}$ (cf. [14]). Since D is connected and simply connected, so is \overline{D} . Since AN is simply connected, the map-lifting theorem states that there is a unique continuous map $j: AN \times \overline{D} \rightarrow K_{\mathbb{C}}$ satisfying $j(1, 0) = 1$ and $\gamma_K(j(an, z)) = j_o(\gamma(an), z)$. Now extend j to $G \times \overline{D}$ by $j(kan, z) = k j(an, z)$. Such an extension is clearly continuous. Moreover,

$$\begin{aligned} \gamma_K(j(kan, z)) &= \gamma_K(k) \gamma_K(j(an, z)) \\ &= \gamma_K(k) j_o(\gamma(an), z) = j_o(\gamma_K(k) \gamma(an), z) = j_o(\gamma(kan), z). \end{aligned}$$

Thus j satisfies (1).

By (4.5) $j_\circ(1, z) = 1$ for all $z \in \overline{D}$ so that by (1), $z \rightarrow j(1, z)$ is a continuous \mathcal{E} -valued function on \overline{D} and hence a constant function. Since $j(1, 0) = 1$, we have $j(1, z) = 1$ for all $z \in \overline{D}$. This implies (2).

Let $g, h \in G$ and $z \in \overline{D}$. By (4.6) and (1) we have $j(gh, z) = j(g, h \cdot z)j(h, z)F(g, h, z)$, where F is a continuous \mathcal{E} -valued function on $G \times G \times \overline{D}$. Since G and \overline{D} are connected, it follows that F is constant. But $F(1, 1, 0) = j(1, 0)^{-1} = 1$ so that (3) follows.

Property (4) follows from the fact that j_\circ is C^∞ on G° and holomorphic on D . Finally, a continuous function $j: G \times \overline{D} \rightarrow K_C$ satisfying (1)–(4) is unique. This completes the proof.

We define the Cayley transform $c \in G_C^\circ$ to be

$$(4.8) \quad c = \exp_\circ \left(\sum_{i=1}^r \frac{\pi}{4} (E_{\gamma_i} - E_{-\gamma_i}) \right).$$

One easily finds

$$\exp \left(-\operatorname{ad} \left(\frac{\pi}{4} (E_{\gamma_i} - E_{-\gamma_i}) \right) \right) (E_{\gamma_j} + E_{-\gamma_j}) = \begin{cases} -H_{\gamma_i} & \text{if } j = i, \\ E_{\gamma_j} + E_{-\gamma_j} & \text{for } j \neq i. \end{cases}$$

Consequently we have

$$(4.9) \quad \operatorname{Ad}(c^{-1})(E_{\gamma_j} + E_{-\gamma_j}) = -H_{\gamma_j}, \quad 1 \leq j \leq r.$$

Let $\mathfrak{t}^- = \bigoplus_{i=1}^r \mathbb{R}H_{\gamma_i}$. Then $\mathfrak{t}^- \subset \mathfrak{t}$ and $i\mathfrak{t}^- \subset \mathfrak{t}_\circ$. Recall the Iwasawa decomposition $\mathfrak{g}_\circ = \mathfrak{k}_\circ \oplus \mathfrak{a}_\circ \oplus \mathfrak{n}_\circ$ chosen in §2.

(4.10) LEMMA. *We have*

- (i) $\operatorname{Ad}(c^{-1})\mathfrak{a}_\circ = \mathfrak{t}^-$ and
- (ii) $\operatorname{Ad}(c^{-1})\mathfrak{n}_\circ \subset \sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha}$.

Proof. Part (i) follows from (4.9). To prove (ii), let \mathfrak{h}_\circ^- denote the subspace of \mathfrak{t}_\circ annihilated by $\{\gamma_j: 1 \leq j \leq r\}$. Then clearly $\mathfrak{h}_\circ^- \subset \mathfrak{m}_\circ$ and $\mathfrak{t}_\circ = \mathfrak{h}_\circ^- \oplus i\mathfrak{t}^-$. It follows that \mathfrak{h}_\circ^- is maximal abelian in \mathfrak{m}_\circ . The abelian subalgebra $\mathfrak{h}_\circ = \mathfrak{h}_\circ^- \oplus i\mathfrak{a}_\circ$ is a Cartan subalgebra of the compact form $\mathfrak{u}_\circ = \mathfrak{k}_\circ \oplus i\mathfrak{p}_\circ$ and we form the root space decomposition of $\mathfrak{g} = \mathfrak{u} = \mathfrak{h} \oplus \sum_{\delta \in \Delta(\mathfrak{g}, \mathfrak{h})} \mathfrak{u}_\delta$ with respect to \mathfrak{h} . By the definition of \mathfrak{h}_\circ^- , $\operatorname{Ad}(c^{-1})$ is the identity on \mathfrak{h}_\circ^- . From (i) we have $\operatorname{Ad}(c^{-1})(\mathfrak{h}_\circ^- \oplus i\mathfrak{a}_\circ) = \mathfrak{h}_\circ^- \oplus i\mathfrak{t}^- = \mathfrak{t}_\circ$. If $\operatorname{Ad}(c)^t$ denotes the transpose of $\operatorname{Ad}(c)$, then it

follows $\text{Ad}(c)^t: \Delta(\mathfrak{g}, \mathfrak{h}) \rightarrow \Phi$ is a bijection and $\text{Ad}(c^{-1})u_\delta = \mathfrak{g}_{\text{Ad}(c)^t\delta}$ for all $\delta \in \Delta(\mathfrak{g}, \mathfrak{h})$.

Let w_μ denote the root space corresponding to the restricted root μ . Since $\mathfrak{n}_o = \bigoplus_{\mu > 0} w_\mu$, it suffices to show that

$$w_\mu \subset \text{Ad}(c) \left(\sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha} \right)$$

for each positive μ . The restricted roots R are the nonzero restrictions of the roots $\Delta(\mathfrak{g}, \mathfrak{h})$ to \mathfrak{a}_o . For $\mu \in R$, let $\Delta(\mu) = \{\delta \in \Delta(\mathfrak{g}, \mathfrak{h}): \delta \text{ restricted to } \mathfrak{a}_o \text{ is } \mu\}$ and let $U_\mu = \bigoplus_{\delta \in \Delta(\mu)} u_\delta$. Since $w_\mu = \mathfrak{g}_o \cap U_\mu$, it suffices to show $u_\delta \subset \text{Ad}(c)(\sum_{\alpha \in \Phi^+} \mathfrak{g}_{-\alpha})$ for all $\delta \in \Delta(\mu)$ with $\mu > 0$. Since $u_\delta = \text{Ad}(c)(\mathfrak{g}_{\text{Ad}(c)^t\delta})$, we need only show the root $\text{Ad}(c)^t\delta$ lies in Φ^- , or equivalently, that $\text{Ad}(c)^t\delta$ cannot lie in Φ^+ . First note that since μ is positive with respect to the lexicographic ordering induced by $\{E_{\gamma_1} + E_{-\gamma_1}, \dots, E_{\gamma_r} + E_{-\gamma_r}\}$, if $i \in \{1, \dots, r\}$ is the smallest integer for which $\mu(E_{\gamma_i} + E_{-\gamma_i}) \neq 0$ then $\mu(E_{\gamma_i} + E_{-\gamma_i}) > 0$. Consequently, for this integer i , $(\text{Ad}(c)^t\delta)(H_{\gamma_i}) = \delta(\text{Ad}(c)H_{\gamma_i}) = -\delta(E_{\gamma_i} + E_{-\gamma_i}) = -\mu(E_{\gamma_i} + E_{-\gamma_i}) < 0$ by (4.9). Let $\pi: \mathfrak{t}' \rightarrow (\mathfrak{t}')'$ denote restriction to \mathfrak{t}' . From Lemma 16 [10; p. 588] one knows that

$$\begin{aligned} \pi(\Phi^+) &= \{0\} \cup \{\tfrac{1}{2}\pi(\gamma_j): 1 \leq j \leq r\} \\ &\cup \{\tfrac{1}{2}\pi(\gamma_i \pm \gamma_j): 1 \leq i < j \leq r\} \cup \{\pi(\gamma_j): 1 \leq j \leq r\}. \end{aligned}$$

(See also [13; p. 82].) In particular, the nonzero elements $\pi(\alpha)$, $\alpha \in \Phi^+$, have the property that if $i \in \{1, \dots, r\}$ is the smallest integer for which $\pi(\alpha)(H_{\gamma_i}) \neq 0$, then $\pi(\alpha)(H_{\gamma_i}) > 0$. Since $\pi(\text{Ad}(c)^t\delta) \neq 0$, it follows that $\text{Ad}(c)^t\delta$ cannot lie in Φ^+ . This completes the proof of (ii).

Let $T_{\mathbb{C}}^\circ$ and $B_{\mathbb{C}}^\circ$ (resp. $T_{\mathbb{C}}$ and $B_{\mathbb{C}}$) be the connected subgroups of $K_{\mathbb{C}}^\circ$ (resp. $K_{\mathbb{C}}$) whose Lie algebras are \mathfrak{t} and $\bigoplus_{\alpha \in \Phi_c^+} \mathfrak{g}_{-\alpha}$. We then have from (4.10) the following result.

$$(4.11) \quad \text{COROLLARY. (1) } c^{-1}A^\circ c \subset T_{\mathbb{C}}^\circ \text{ and (2) } c^{-1}N^\circ c \subset B_{\mathbb{C}}^\circ P_-.$$

The strong orthogonality of the γ_j along with an explicit calculation

in $\mathrm{SL}(2, \mathbb{C})$ gives

$$(4.12) \quad \exp \left(\sum_{i=1}^r t(E_{\gamma_i} - E_{-\gamma_i}) \right) \\ = \exp \left(\tan(t) \sum_{i=1}^r E_{\gamma_i} \right) \exp \left(-\ln(\cos(t)) \sum_{i=1}^r H_{\gamma_i} \right) \\ \times \exp \left(-\tan(t) \sum_{i=1}^r E_{-\gamma_i} \right),$$

for $t \in \mathbb{R}$. In particular if $t = \frac{\pi}{4}$ then $c \in \Omega = P_+ K_{\mathbb{C}}^{\circ} P_-$ and $c \cdot 0 = \log \pi_+(c) = \sum_{i=1}^r E_{\gamma_i}$. Consequently we have $G^{\circ}c = K^{\circ}A^{\circ}N^{\circ}c = K^{\circ}c(c^{-1}A^{\circ}c)(c^{-1}N^{\circ}c) \subset K^{\circ}cK_{\mathbb{C}}^{\circ}P_-$. Since $c \in \Omega$, we conclude $G^{\circ}c \subset \Omega$. Thus if $b = \sum_{i=1}^r E_{\gamma_i}$, then $g \cdot b$ is defined for each $g \in G^{\circ}$ and by extension for each $g \in G$.

(4.13) **LEMMA.** *The stability subgroup G_b of $b = \sum_{i=1}^r E_{\gamma_i}$ in G contains MAN .*

Proof. To show $an \cdot b = b$ for $an \in AN$, it suffices to show that $c^{-1}anc \cdot 0 = 0$ for $an \in A^{\circ}N^{\circ}$. But this follows from (4.11) and the fact that $K_{\mathbb{C}}^{\circ}P_- \cdot 0 = 0$. To show $m \cdot b = b$ we first observe that $k \cdot z = \mathrm{Ad}(k)z$ for all $k \in K$ and $z \in \mathfrak{p}_+$. Since \mathfrak{p}_+ and \mathfrak{p}_- are $\mathrm{Ad}(K)$ -invariant, $M \subset K$ and $(b + \bar{b}) \in \mathfrak{a}_0$, we have $b + \bar{b} = \mathrm{Ad}(m)(b + \bar{b}) = \mathrm{Ad}(m)(b) + \mathrm{Ad}(m)(\bar{b})$ so that $m \cdot b = \mathrm{Ad}(m)(b) = b$. This completes the proof.

(4.14) **REMARKS.** (1) In general, MAN is properly contained in G_b . (2) By the cocycle property (4.7.3) and by (4.13) the function $g \rightarrow j(g, b)$, $g \in G_b$, is a homomorphism.

Throughout the remainder of the paper we assume that τ is an irreducible unitary representation of K on the space V_{τ} with highest weight λ and we fix a nonzero highest weight vector ϕ_{λ} . The representation τ extends to a holomorphic representation of $K_{\mathbb{C}}$ on V_{τ} . We define the *factor of automorphy* J_{τ} by $J_{\tau}(g, z) = \tau(j(g, z))$ for $(g, z) \in G \times \bar{D}$. By (4.14.2) the map $\omega: G_b \rightarrow \mathrm{GL}(V_{\tau})$ given by $\omega(g) = J_{\tau}(g, b)^{*^{-1}}$, $g \in G_b$, defines a representation of G_b and hence of MAN .

Choose the representation $\sigma = \sigma(\tau)$ as in (3.9) and the linear functional $\nu = \nu(\tau)$ satisfying (3.10), i.e. $(\nu - \rho)(E_{\gamma_j} + E_{-\gamma_j}) = 2(\lambda, \gamma_j)/(\gamma_j, \gamma_j) = \lambda(H_{\gamma_j})$ for $1 \leq j \leq r$. Keeping in mind these

choices for σ and ν , recall the definition (3.6) of the corresponding irreducible representation σ_ν of MAN on H_σ .

(4.15) LEMMA. *For each $g \in MAN$, the operator $\omega(g)$ leaves the subspace H_σ invariant. Moreover, the restriction of $\omega(g)$, $g \in MAN$, to H_σ is $\sigma_\nu(g)$.*

Proof. Recall that H_σ is the span of $\tau(m)\phi_\lambda$, $m \in M$. Since M commutes with A and normalizes N , the lemma will follow if we show $\omega(g)\phi_\lambda = \sigma_\nu(g)\phi_\lambda$ for all g in M , A , and N . Now for $m \in M$, $\omega(m)\psi = \sigma_\nu(m)\psi$ for $\psi \in H_\sigma$ by (4.7.2).

We now show $\omega(a)\phi_\lambda = \sigma_\nu(a)\phi_\lambda$. Let $a \in A^\circ$. Then by (4.6) and (4.13), one has $j_\circ(c^{-1}ac, 0) = j_\circ(c^{-1}, c \cdot 0)j_\circ(a, b)j_\circ(c, 0)$. By (4.11) and (4.12), $j_\circ(c^{-1}ac, 0)$ and $j_\circ(c, 0)$ are in $T_\mathbb{C}^\circ$. Since $j_\circ(c^{-1}, c \cdot 0) = j_\circ(c, 0)^{-1}$ and $T_\mathbb{C}^\circ$ is abelian, it follows that $j_\circ(a, b) = j_\circ(c^{-1}ac, 0) = c^{-1}ac$. If $a = \exp(\sum a_i(E_{\gamma_i} + E_{-\gamma_i}))$ ($a_i \in \mathbb{R}$), then $j_\circ(\gamma(a), b) = c^{-1}\gamma(a)c = \exp_\circ(\sum -a_i H_{\gamma_i}) \in T_\mathbb{C}^\circ$ by (4.9). From this it follows that $j(a, b) = \exp(\sum -a_i H_{\gamma_i})$. We now obtain

$$\begin{aligned} \omega(a)\phi_\lambda &= J_\tau(a, b)^{* -1} \phi_\lambda \\ &= \tau \left(\exp \left(- \sum_i a_i H_{\gamma_i} \right) \right)^{* -1} \phi_\lambda \\ &= \tau \left(\exp \left(\sum_i a_i H_{\gamma_i} \right) \right) \phi_\lambda \\ &= e^{\sum_i a_i \lambda(H_{\gamma_i})} \phi_\lambda \\ &= e^{\sum_i a_i (\nu - \rho)(E_{\gamma_i} + E_{-\gamma_i})} \phi_\lambda \quad \text{by (3.10)} \\ &= e^{(\nu - \rho)H(a)} \phi_\lambda \\ &= \sigma_\nu(a)\phi_\lambda. \end{aligned}$$

We now show $\omega(n)\phi_\lambda = \sigma_\nu(n)\phi_\lambda$ for all $n \in N$. As in the preceding paragraph, we have $j_\circ(n, b) = j_\circ(c, 0)j_\circ(c^{-1}nc, 0)j_\circ(c, 0)^{-1}$, for $n \in N^\circ$. By (4.11), $c^{-1}Nc \in B_\mathbb{C}^\circ P_-$ so that $j_\circ(c^{-1}nc, 0) \in B_\mathbb{C}^\circ$. Since $j_\circ(c, 0) \in T_\mathbb{C}^\circ$ and $T_\mathbb{C}^\circ$ normalizes $B_\mathbb{C}^\circ$ it follows that $j_\circ(n, b) \in B_\mathbb{C}^\circ$. Therefore, for $n \in N$, $j(n, b) \in \gamma_K^{-1}(B_\mathbb{C}^\circ)$ by (4.7.1). On the other hand, $j(n, b)$ is in the connected component of the identity of $\gamma_K^{-1}(B_\mathbb{C}^\circ)$ because $j(1, b) = 1$, N is connected and $n \rightarrow j(n, b)$ is continuous. Thus $j(n, b) \in B_\mathbb{C}$. But then $\omega(n)\phi_\lambda = J_\tau(n, b)^{* -1} \phi_\lambda = \tau(j(n, b))^{* -1} \phi_\lambda = \phi_\lambda$ since ϕ_λ is a highest weight vector. However $\sigma_\nu(n)\phi_\lambda = \phi_\lambda$ for $n \in N$. This completes the proof.

(4.16) **PROPOSITION.** *Let $C_\tau: H_\sigma \rightarrow V_\tau$ denote inclusion. Then $s_\tau(g) = J_\tau(g^{-1}, b)^{* -1} C_\tau$ for all $g \in G$.*

Proof. Let $J(g) = J_\tau(g^{-1}, b)^{* -1} C_\tau$. By (3.8) it suffices to show that J satisfies transforming property (3.7) and $J(1) = s_\tau(1)$. We have

$$\begin{aligned} J(man g) &= J_\tau(g^{-1}(man)^{-1}, b)^{* -1} C_\tau \\ &= J_\tau(g^{-1}, (man)^{-1} \cdot b)^{* -1} J_\tau((man)^{-1}, b)^{* -1} C_\tau \quad \text{by (4.7.3)} \\ &= J_\tau(g^{-1}, b)^{* -1} \omega((man)^{-1}) C_\tau \quad \text{by (4.13)} \\ &= J(g) \sigma_\nu(man)^{-1} \quad \text{by (4.15)} \end{aligned}$$

for all $g \in G$ and $man \in MAN$. By (4.7.2) we also have $J(gk) = J_\tau(k^{-1}g^{-1}, b)^{* -1} C_\tau = \tau(k^{-1})^{* -1} J_\tau(g^{-1}, b)^{* -1} C_\tau = \tau(k^{-1})J(g)$ for all $g \in G$ and $k \in K$. Finally, observe that $J(1) = C_\tau = s_\tau(1)$.

5. Holomorphic-type functions on G . The purpose of this section is to show that the Szegő operator S_τ defined in §3 takes its values in a standard realization of a highest weight representation space.

With τ as in §4, let $C^\infty(G, \tau)$ denote the subspace of C^∞ functions in $C(G, \tau)$. On $C^\infty(G, \tau)$ we define left-invariant vector fields $r(X)$ for $X \in \mathfrak{g}_0$ by $(r(X)f)(g) = \frac{d}{dt} f(g \exp(tX))|_{t=0}$ where $g \in G$. For $X = X_1 + iX_2$ with $X_j \in \mathfrak{g}_0$, $j = 1, 2$, we set $r(X) = r(X_1) + ir(X_2)$. Put

$$(5.1) \quad C^\infty(G, \tau; \mathfrak{p}_-) = \{f \in C^\infty(G, \tau) : r(X)f = 0 \text{ for all } X \in \mathfrak{p}_-\}.$$

Clearly $C^\infty(G, \tau; \mathfrak{p}_-)$ is $L(G)$ -invariant. We call $C^\infty(G, \tau; \mathfrak{p}_-)$ the space of holomorphic-type functions on G determined by τ .

Note that from (4.7.3) and (4.7.2) one has

$$\begin{aligned} J_\tau(g^{-1}k, b)^{* -1} C_\tau &= J_\tau(g^{-1}, k \cdot b)^{* -1} J_\tau(k, b)^{* -1} C_\tau \\ &= J_\tau(g^{-1}, k \cdot b)^{* -1} \tau(k) C_\tau. \end{aligned}$$

It then follows from (3.5) and (4.16) that the Szegő map S_τ may be written in the form

$$(5.2) \quad S_\tau f(g) = \int_K J_\tau(g^{-1}, k \cdot b)^{* -1} \tau(k) C_\tau f(k) dk, \\ f \in W(\tau), g \in G.$$

(5.3) **PROPOSITION.** *The Szegő operator S_τ maps $W(\tau)$ into $C^\infty(G, \tau; \mathfrak{p}_-)$.*

Proof. Since the kernel function s_τ is C^∞ , it follows that S_τ maps $W(\tau)$ into $C^\infty(G, \tau)$. By G -equivariance, it suffices to show that $(r(X)S_\tau f)(1) = 0$ for all $f \in W(\tau)$ and $X \in \mathfrak{p}_-$. However, from (5.2) it suffices to show $r(X)J_\tau(g, z)^{* -1}|_{g=1} = 0$ for all $z \in \overline{D}$. By the assumptions of §4, the differentials of γ and γ_K are the identity maps. Since $\gamma_K(j(g, z)) = j_\circ(\gamma(g), z)$, for all $(g, z) \in G \times \overline{D}$, it follows that for each $z \in \overline{D}$, $r(X)j(g, z)|_{g=1} = r(X)j_\circ(\gamma(g), z)|_{g=1}$ for all $X \in \mathfrak{g}$. But for each $z \in \overline{D}$, there exists a neighborhood N_z of the identity in $G_\mathbb{C}^\circ$ on which $g \rightarrow j_\circ(g, z)$ is holomorphic. For $Y \in \mathfrak{p}_+$ we consequently have $r(Y)j_\circ(g, z)|_{g=1} = \frac{d}{dt}j_\circ(\exp(tY), z)|_{t=0} = 0$ since $j_\circ(p, z)$ is the identity for all $(p, z) \in P_+ \times \mathfrak{p}_+$ by (4.5). Thus for $X \in \mathfrak{p}_-$ we find $r(X)J_\tau(g, z)^{* -1}|_{g=1} = (r(\overline{X})J_\tau(g, z)^{-1})^*|_{g=1} = 0$. This completes the proof.

(5.4) **REMARKS.** (1) The proof of (5.3) given here depends on the results thus far developed and reflects the special nature of the Hermitian symmetric setting. Proposition (5.3) is however an immediate consequence of the more general Theorem (6.1) [16]. One only needs to check that parameters may be chosen in a consistent way.

(2) It is shown in [21] that the choice of imbedding parameters of irreducible highest weight $(\mathfrak{g}, \mathfrak{k})$ -modules in principal series is unique. Thus the parameters chosen in §3 for the Szegő map S_τ are the only parameters possible.

6. Holomorphically induced multiplier representations. In order to apply certain G -equivariant differential operators to the kernel function s_τ , we first need to compute the action of left-invariant vector fields by elements of \mathfrak{p}_+ on the factor of automorphy J_τ . The factor J_τ also appears as a multiplier in the definition of the holomorphic representation T of G induced by τ . This section shows that the desired \mathfrak{p}_+ -action can be conveniently expressed in terms of the derived action of T .

We begin by reviewing some basic facts concerning holomorphically induced multiplier representations. We refer the reader to [10] and [15] for more details.

Let (τ, V_τ) be an irreducible unitary representation of K and let $O(D, V_\tau)$ denote the space of V_τ -valued holomorphic functions on D . The associated multiplier representation T of G on $O(D, V_\tau)$ is defined by the formula

$$(6.1) \quad (T(g)F)(\zeta) = J_\tau(g^{-1}, \zeta)^{-1}F(g^{-1} \cdot \zeta),$$

where $F \in O(D, V_\tau)$, $g \in G$ and $\zeta \in D$. The K -finite vectors in $O(D, V_\tau)$ are the polynomial functions. Since D is open in \mathfrak{p}_+ , we may identify the K -finite vectors with the \mathfrak{g} -module \mathbb{V}_τ of V_τ -valued polynomials on \mathfrak{p}_+ . We will again denote the derived representation of \mathfrak{g} on \mathbb{V}_τ by T . To give the \mathfrak{g} -action explicitly, we first define the differential operator $\delta(x)$, $x \in \mathfrak{p}_+$, as follows: for $f \in \mathbb{V}_\tau$ and $x \in \mathfrak{p}_+$ set

$$(6.2) \quad \delta(x)f(z) = \frac{d}{dt}f(z + tx)|_{t=0}, \quad z \in \mathfrak{p}_+.$$

Then for $f \in \mathbb{V}_\tau$ one has

$$(6.3) \quad \begin{aligned} T(x)f(z) &= -\delta(x)f(z), & x \in \mathfrak{p}_+, \\ T(x)f(z) &= d\tau(x)f(z) - \delta([x, z])f(z), & x \in \mathfrak{k}, \\ T(x)f(z) &= d\tau([x, z])f(z) \\ &\quad - \frac{1}{2}\delta([[x, z], z])f(z), & x \in \mathfrak{p}_-. \end{aligned}$$

Let \mathbb{V}_τ^j denote the space of homogeneous polynomials of degree j in \mathbb{V}_τ . Then it is clear from (6.3) that

$$(6.4) \quad \begin{aligned} T(x) : \mathbb{V}_\tau^j &\rightarrow \mathbb{V}_\tau^{j-1}, & x \in \mathfrak{p}_+, \\ T(x) : \mathbb{V}_\tau^j &\rightarrow \mathbb{V}_\tau^j, & x \in \mathfrak{k}, \\ T(x) : \mathbb{V}_\tau^j &\rightarrow \mathbb{V}_\tau^{j+1}, & x \in \mathfrak{p}_-, \end{aligned}$$

for all integers $j \geq 0$. (Here it is understood that $\mathbb{V}_\tau^{-1} = 0$.)

For $\phi \in V_\tau$, let $1_\phi \in \mathbb{V}_\tau^\circ$ denote the constant function with value ϕ . Recall from §4 that λ (resp. ϕ_λ) denotes the highest weight (resp. highest weight vector) for (τ, V_τ) . Let $L(\tau)$ denote the \mathfrak{g} -module generated by the constant functions \mathbb{V}_τ° . Then (6.4) implies every nontrivial submodule of \mathbb{V}_τ contains $L(\tau)$ and thus $L(\tau)$ is irreducible. Clearly $L(\tau)$ is a highest weight module with highest weight vector 1_{ϕ_λ} and highest weight λ .

We now define a Hermitian product on \mathbb{V}_τ which makes the $T(K)$ -action unitary. Normalize Lebesgue measure dv on \mathfrak{p}_+ so that

$$\int_{\mathfrak{p}_+} e^{-|v|^2} dv = 1$$

where $|\cdot|$ is the norm given by the standard Hermitian inner product on \mathfrak{g} . Put $dm(v) = e^{-|v|^2} dv$ and set

$$(6.5) \quad (f, g) = \int_{\mathfrak{p}_+} (f(v), g(v)) dm(v), \quad f, g \in \mathbb{V}_\tau.$$

Note that with this normalization, the inner product on the space of constant functions \mathbb{V}_τ° agrees with the inner product on V_τ . Also note that if $f \in \mathbb{V}_\tau$, then from (6.1) it follows that

$$(6.6) \quad T(k)f(z) = \tau(k)f(\text{Ad}(k^{-1})z), \quad k \in K, z \in \mathfrak{p}_+.$$

Hence, $T(k)$, $k \in K$, is unitary with respect to the inner product in (6.5) and each subspace \mathbb{V}_τ^j is invariant under $T(K)$. Finally, observe that for $k \in K$, $T(k)$ is equivalent to $\tau(k)$ on \mathbb{V}_τ° , i.e.

$$(6.7) \quad T(k)1_\phi = 1_{\tau(k)\phi}, \quad k \in K.$$

Let ε denote the conjugate linear antiautomorphism of $U(\mathfrak{g})$ induced by the map $X \rightarrow -X$ on the real form \mathfrak{g}_\circ of \mathfrak{g} . Equipping $S(\mathfrak{p}_\pm) = U(\mathfrak{p}_\pm) \subset U(\mathfrak{g})$ with the usual grading, we let $S_j(\mathfrak{p}_\pm)$ denote the homogeneous elements of degree j . Since $\varepsilon(\mathfrak{p}_\pm) = \mathfrak{p}_\mp$, note that $\varepsilon(S_j(\mathfrak{p}_\pm)) = S_j(\mathfrak{p}_\mp)$ for all $j \geq 0$.

(6.8) LEMMA. *Let $\phi, \psi \in V_\tau$ and $Y, Z \in S_j(\mathfrak{p}_+)$ for $j \geq 1$. Then*

$$(1_\phi, T(Y)T(\varepsilon(Z))1_\psi) = (T(Z)T(\varepsilon(Y))1_\phi, 1_\psi).$$

Proof. The proof proceeds by induction on j . For brevity, we suppress the use of T . Let $Y, Z \in S_1(\mathfrak{p}_+)$. Then since \mathfrak{p}_+ annihilates \mathbb{V}_τ° , we have

$$\begin{aligned} (1_\phi, Y\varepsilon(Z)1_\psi) &= (1_\phi, ([Y, \varepsilon(Z)] + \varepsilon(Z)Y)1_\psi) \\ &= (1_\phi, [Y, \varepsilon(Z)]1_\psi) \\ &= (\varepsilon([Y, \varepsilon(Z)])1_\phi, 1_\psi) \quad \text{since } [Y, \varepsilon(Z)] \in \mathfrak{k} \\ &= ([Z, \varepsilon(Y)]1_\phi, 1_\psi) \\ &= (Z\varepsilon(Y)1_\phi, 1_\psi). \end{aligned}$$

Now suppose $j > 1$. By linearity, we may assume both Z and Y are monomials in $S_j(\mathfrak{p}_+)$. Write $Y = Y'y$ where $y \in \mathfrak{p}_+$ and $Y' \in S_{j-1}(\mathfrak{p}_+)$. We have

$$\begin{aligned} (1_\phi, Y\varepsilon(Z)1_\psi) &= (1_\phi, Y'(y\varepsilon(Z) - \varepsilon(Z)y)1_\psi) \\ &= (1_\phi, Y'[y, \varepsilon(Z)]1_\psi). \end{aligned}$$

It is readily checked that $[y, \varepsilon(Z)] = t + \sum_i s_i k_i$ for $t, s_i \in S_{j-1}(\mathfrak{p}_-)$

and $k_i \in \mathfrak{k}$. Since $\varepsilon(\mathcal{S}_{j-1}(\mathfrak{p}_-)) \subset \mathcal{S}_{j-1}(\mathfrak{p}_+)$, induction gives

$$\begin{aligned}
(1_\phi, Y\varepsilon(Z)1_\psi) &= (1_\phi, Y't1_\psi) + \sum_i (1_\phi, Y's_i1_{\tau(k_i)\psi}) \quad \text{by (6.7)} \\
&= (1_\phi, Y'\varepsilon(\varepsilon(t))1_\psi) + \sum_i (1_\phi, Y'\varepsilon(\varepsilon(s_i))1_{\tau(k_i)\psi}) \\
&= (\varepsilon(t)\varepsilon(Y')1_\phi, 1_\psi) + \sum_i (\varepsilon(s_i)\varepsilon(Y')1_\phi, 1_{\tau(k_i)\psi}) \\
&= (\varepsilon(t)\varepsilon(Y')1_\phi, 1_\psi) + \sum_i (\varepsilon(k_i)\varepsilon(s_i)\varepsilon(Y')1_\phi, 1_\psi) \\
&= \left(\varepsilon \left(t + \sum_i s_i k_i \right) \varepsilon(Y')1_\phi, 1_\psi \right) \\
&= ([Z, \varepsilon(y)]\varepsilon(Y')1_\phi, 1_\psi) \\
&= (Z\varepsilon(y)\varepsilon(Y')1_\phi, 1_\psi) - (\varepsilon(y)Z\varepsilon(Y')1_\phi, 1_\psi).
\end{aligned}$$

By (6.4) we know $Z\varepsilon(Y')1_\phi = 0$. Since $\varepsilon(y)\varepsilon(Y') = \varepsilon(Y)$, the proof is complete.

(6.9) LEMMA. *Let $\phi, \psi \in V_\tau$. Then for all integers $j \geq 1$,*

$$(\phi, (T(\bar{z})^j 1_\psi)(y)) = ((T(\bar{y})^j 1_\phi)(z), \psi) \quad \text{for all } z, y \in \mathfrak{p}_+.$$

Proof. Set

$$f_1(z, y) = (\phi, (T(\bar{z})^j 1_\psi)(y)) \quad \text{and} \quad f_2(z, y) = ((T(\bar{y})^j 1_\phi)(z), \psi).$$

For $i = 1, 2$, note that $f_i(z, y)$ is a homogeneous polynomial in z of degree j and a homogeneous polynomial of degree j in \bar{y} . It suffices to show therefore that all j th order partial derivatives of f_1 and f_2 agree. Choose $z_1, \dots, z_j, y_1, \dots, y_j$ in \mathfrak{p}_+ and set $d = \prod_{i=1}^j \delta(z_i) \prod_{k=1}^j \delta(y_k)$ where $\delta(z_i)$ (resp. $\delta(y_k)$) differentiates with respect to the z (resp. y) variable. Using (6.3) and the linearity of T we find

$$\begin{aligned}
df_1 &= (-1)^j j! (\phi, T(Y)T(\bar{Z})1_\psi(0)) \quad \text{and} \\
df_2 &= (-1)^j j! (T(Z)T(\bar{Y})1_\phi(0), \psi),
\end{aligned}$$

where $T(Y) = \prod_{k=1}^j T(y_k)$ and $T(Z) = \prod_{k=1}^j T(z_k)$. Consequently, $df_1 = j!(1_\phi, T(Y)T(\varepsilon(Z))1_\psi)$ and $df_2 = j!(T(Z)T(\varepsilon(Y))1_\phi, 1_\psi)$. By (6.8), we conclude $df_1 = df_2$. Since $z_1, \dots, z_j, y_1, \dots, y_j$ are arbitrary in \mathfrak{p}_+ it follows that $f_1 = f_2$. This completes the proof.

(6.10) PROPOSITION. *If $\phi \in V_\tau$ and $y \in \bar{D}$, then*

$$r(z)^j J_\tau(g^{-1}, y)^{* -1} \phi|_{g=1} = (T(\bar{y})^j 1_\phi)(z)$$

for all $z \in \mathfrak{p}_+$ and integers $j \geq 1$.

Proof. Let $\phi, \psi \in V_\tau$. Then for $z \in \mathfrak{p}_+$ and $y \in \bar{D}$ we have

$$\begin{aligned} (r(z)^j J_\tau(g^{-1}, y)^{* -1} \phi|_{g=1}, \psi) &= r(z)^j (J_\tau(g^{-1}, y)^{* -1} \phi, \psi)|_{g=1} \\ &= r(z)^j (\phi, J_\tau(g^{-1}, y)^{-1} \psi)|_{g=1} \\ &= (\phi, r(\bar{z})^j J_\tau(g^{-1}, y)^{-1} \psi|_{g=1}) \\ &= (\phi, r(\bar{z})^j T(g) 1_\psi(y)|_{g=1}) \\ &= (\phi, T(\bar{z})^j 1_\psi(y)) \\ &= ((T(\bar{y})^j 1_\phi)(z), \psi) \quad \text{by (6.9)}. \end{aligned}$$

Since $\psi \in V_\tau$ is arbitrary, the proof of the proposition is complete.

We conclude this section with a brief discussion of the G -isomorphism of the spaces $O(D, V_\tau)$ and $C^\infty(G, \tau; \mathfrak{p}_-)$. In particular, we give a characterization of the unique irreducible \mathfrak{g} -module in $C^\infty(G, \tau; \mathfrak{p}_-)$ which is used in connection with the differential operators defined in §7.

For $f \in O(D, V_\tau)$, define the function ΘF on G by the formula

$$\Theta F(g) = J_\tau(g, 0)^{-1} F(g \cdot 0).$$

It is readily checked that $\Theta F \in C^\infty(G, \tau; \mathfrak{p}_-)$ and $\Theta T(g) = L(g)\Theta$. The mapping Θ is invertible with inverse given by

$$\Theta^{-1} f(\zeta) = J_\tau(g, 0) f(g), \quad \zeta \in D,$$

where $g \in G$ satisfies $g \cdot 0 = \zeta$.

Let $C_0^\infty(G, \tau; \mathfrak{p}_-)$ denote the K -finite vectors in $C^\infty(G, \tau; \mathfrak{p}_-)$. Then $\Theta(V_\tau) = C_0^\infty(G, \tau; \mathfrak{p}_-)$ and we put $L(\tau)^\sim = \Theta(L(\tau))$.

(6.11) PROPOSITION. *An element $f \in C_0^\infty(G, \tau; \mathfrak{p}_-)$ lies in $L(\tau)^\sim$ if and only if the function $z \rightarrow (r(z)^j f)(1)$, $z \in \mathfrak{p}_+$, lies in $L(\tau)$ for each $j \geq 0$.*

Proof. Let $F \in V_\tau$ and $z \in \mathfrak{p}_+$. By the proof of (5.3) one has

$$\begin{aligned} (r(z)^j \Theta F)(1) &= (d^j/dt^j) F(\exp(tz) \cdot 0)|_{t=0} \\ &= (d^j/dt^j) F(tz)|_{t=0} = \nabla^j F(0)(z, \dots, z) \end{aligned}$$

where $\nabla^j F(0)$ denotes the j th derivative of F at 0. By Taylor's theorem, $F(z) = \sum_{j=0}^{\infty} \nabla^j F(0)(z, \dots, z)/j! = \sum_{j=0}^{\infty} (r(z)^j \Theta F)(1)/j!$. Setting $f = \Theta F$ gives

$$\Theta^{-1} f(z) = \sum_{j=0}^{\infty} \frac{1}{j!} (r(z)^j f)(1), \quad \text{for } f \in C_0^\infty(G, \tau; \mathfrak{p}_-).$$

Since $\Theta^{-1}(L(\tau)^\sim) = L(\tau)$ and $L(\tau)$ is graded, the proposition now follows.

7. Gradient-type differential operators. Gradient-type differential operators are distinguished differential operators which intertwine the G -actions on spaces of K -covariant functions on G (cf. [3], [11]). First order gradient-type operators were used in connection with certain unitary representations of $SU(p, q)$ [4]. For our purposes we require systems of such operators. These systems may be viewed as G -equivariant analogues of the systems given in (2.9) of [2]. Here we associate to certain representations τ of K a finite system \mathcal{D}_τ of gradient-type operators and show that the K -finite vectors in the kernel of this system is an irreducible highest weight \mathfrak{g} -module. In the context of unitarity, it was shown in [3] that \mathcal{D}_τ can be chosen to consist of a single operator.

Let \widehat{K} denote the set of irreducible unitary representations of K . For (τ, V_τ) in \widehat{K} , let \mathfrak{p}_+ act trivially on the \mathfrak{k} -module V_τ and form the generalized Verma module $\mathcal{N}(\tau) = U(\mathfrak{g}) \otimes_{U(\mathfrak{k} \oplus \mathfrak{p}_+)} V_\tau$. Identifying \mathfrak{p}_- with the dual of \mathfrak{p}_+ gives a \mathfrak{k} -isomorphism $\mathcal{N}(\tau) \rightarrow \mathbb{V}_\tau$. Denote the \mathfrak{g} -structure on \mathbb{V}_τ arising from $\mathcal{N}(\tau)$ by juxtaposition. The \mathfrak{g} -module $\mathcal{N}(\tau)$ contains a unique maximal submodule; let M_τ denote its image in \mathbb{V}_τ . Define the subset \widehat{K}_r of \widehat{K} by

$$(7.1) \quad \widehat{K}_r = \{\tau \in \widehat{K} : M_\tau \neq 0\}.$$

Since \widehat{K}_r corresponds to the representations of K for which $\mathcal{N}(\tau)$ is reducible, we call \widehat{K}_r the set of *reduction points*. This terminology is taken from the Enright, Howe and Wallach classification of unitary highest weight modules (cf. [7]). Those $\tau \in \widehat{K}_r$ for which $L(\tau)$ is unitarizable appear in that classification as either the endpoint of a half-line or as isolated equally spaced points.

The \mathfrak{g} -structure on \mathbb{V}_τ arising from $\mathcal{N}(\tau)$ is the conjugate dual of T given in (6.3). To make this terminology more precise, we recall the inner product (6.5) on \mathbb{V}_τ . The two \mathfrak{g} -structures on \mathbb{V}_τ are then related by the formula

$$(7.2) \quad (T(\varepsilon(x))f, h) = (f, xh), \quad x \in U(\mathfrak{g}), f, h \in \mathbb{V}_\tau,$$

where ε is the anti-automorphism given in §6. It follows from (7.2) that the orthogonal complement of a \mathfrak{g} -module is a $T(\mathfrak{g})$ -module and vice versa. In particular, note that $L(\tau) = M_\tau^\perp$ is properly contained in \mathbb{V}_τ if and only if $\tau \in \widehat{K}_r$.

Using the Killing form, we identify the dual of \mathfrak{p}_- with \mathfrak{p}_+ . Let $\{X_i\}$ denote a basis of \mathfrak{p}_- and let $\{X'_i\} \subset \mathfrak{p}_+$ denote the dual basis. Following [3], we define the operator $\nabla: C^\infty(G, \tau) \rightarrow C^\infty(G, \tau \otimes \text{Ad}|_{\mathfrak{p}_-})$ by the formula

$$(7.3) \quad \nabla f(g) = \sum_i r(X'_i) f(g) \otimes X_i.$$

Then ∇ is easily seen to be independent of the basis $\{X_i\}$ and clearly intertwines the left G -actions. For each integer $j \geq 1$, $\nabla^j f(g)$ is a symmetric j -tensor and thus may be regarded as homogeneous polynomial of degree j on \mathfrak{p}_+ . One readily checks the formula

$$(\nabla^j f(g))(z) = r(z)^j f(g), \quad g \in G, z \in \mathfrak{p}_+.$$

Let $W \subset \mathbb{V}_\tau$ denote an irreducible $T(K)$ -space and let τ_W denote the restriction of $T(K)$ to W . Since the space of homogeneous polynomials of a fixed degree is $T(K)$ -invariant, there exists a unique integer $j \geq 0$ such that $W \subset \mathbb{V}_\tau^j$. Let P_W denote orthogonal projection of \mathbb{V}_τ^j onto W . We then have

$$(7.4) \quad \tau_W(k)P_W = P_W T(k), \quad k \in K.$$

We refer to the differential operator $\partial_W: C^\infty(G, \tau) \rightarrow C^\infty(G, \tau_W)$ defined by

$$(7.5) \quad \partial_W f(g, \cdot) = P_W(r(\cdot)^j f(g))$$

as a *gradient-type differential operator*.

We henceforth assume that $\tau \in \widehat{K}_r$. Since \mathbb{V}_τ , viewed as a $S(\mathfrak{p}_-)$ -module, is Noetherian, it follows that the submodule M_τ is finitely generated over $S(\mathfrak{p}_-)$. Let $\{m_i: 1 \leq i \leq r\}$ be generators of M_τ over $S(\mathfrak{p}_-)$ and let W denote the K -invariant space generated by $\{m_i\}$. Decompose W into K -irreducible spaces W_1, \dots, W_s and put $\partial_i = \partial_{W_i}$. We call $\mathcal{D}_\tau = \{\partial_1, \dots, \partial_s\}$ a *system of gradient-type differential operators associated to τ* . We remark that the notation for a system \mathcal{D}_τ does not reflect its dependence on the choice of generators $\{m_i\}$. However, the next result shows that the kernel of such a system \mathcal{D}_τ in $C_\infty^\infty(G, \tau; \mathfrak{p}_-)$ is in fact independent of the generators.

(7.6) **PROPOSITION.** *Let $\tau \in \widehat{K}_r$ and let \mathcal{D}_τ be any system of gradient-type differential operators associated to τ . Then the kernel of \mathcal{D}_τ in $C^\infty(G, \tau; \mathfrak{p}_-)$ is $L(\tau)^\sim$.*

Proof. Let \mathcal{D}_τ be any system of gradient-type differential operators associated to τ . Suppose $f \in \text{Ker}(\mathcal{D}_\tau) \cap C^\infty(G, \tau; \mathfrak{p}_-)$. Then $\partial_i f = 0$ for all $i = 1, \dots, s$. With the notation of the preceding paragraph, let j_i denote the positive integer for which $W_i \subset \mathbb{V}_\tau^{j_i}$. We then have $(r(\cdot)^{j_i} f(g), W_i) = 0$, $1 \leq i \leq s$. Since $\mathbb{V}_\tau^{j_i}$ is orthogonal to \mathbb{V}_τ^m for $j_i \neq m$, we have $(r(\cdot)^{j_i} f(g), W_i) = 0$, $1 \leq i \leq s$ and $j_i \geq 0$. In particular, since $\{m_1, \dots, m_r\} \subset \bigoplus_{i=1}^s W_i$ we have

$$(7.7) \quad (r(\cdot)^j f(g), m_i) = 0, \quad 1 \leq i \leq r, j \geq 0.$$

Now let $y \in \mathfrak{p}_+$. By (6.2) and (6.3) one has

$$(7.8) \quad r(y)r(z)^j f(g) = \frac{-1}{j+1} T(y)(r(z)^{j+1} f(g)), \quad z \in \mathfrak{p}_+.$$

Let $x \in S_k(\mathfrak{p}_-)$. Since $\deg(m_i) > 0$, if $k \geq j$ then $(r(\cdot)^j f(g)|x m_i) = 0$. If $k < j$, then we have

$$\begin{aligned} (r(\cdot)^j f(g)|x m_i) &= (T(\varepsilon(x))(r(\cdot)^j f(g)), m_i) && \text{by (7.2)} \\ &= C(r(\varepsilon(x))r(\cdot)^{j-k} f(g), m_i), \quad C \in \mathbb{R}, && \text{by (7.8)} \\ &= Cr(\varepsilon(x))r(\cdot)^{j-k} f(g), m_i) = 0 && \text{by (7.7)}. \end{aligned}$$

Now M_τ is the linear span of elements of the form $x m_i$ and $L(\tau) = M_\tau^\perp$. It follows that for each $j \geq 0$ and $g \in G$, the function $z \rightarrow r(z)^j f(g)$, $z \in \mathfrak{p}_+$, lies in $L(\tau)$. Letting $g = 1$, we conclude by (6.11) that $f \in L(\tau)^\sim$. Thus $\text{Ker}(\mathcal{D}_\tau) \cap C^\infty(G, \tau; \mathfrak{p}_-) \subset L(\tau)^\sim$.

Since $L(\tau)^\sim$ is irreducible, it remains to show that the \mathfrak{g} -module $\text{Ker}(\mathcal{D}_\tau) \cap C^\infty(G, \tau; \mathfrak{p}_-)$ is nontrivial. Let $B_\tau: V_\tau \rightarrow W(\tau)$ denote the map defined in §3 which corresponds to the inclusion map $C_\tau: H_\sigma \rightarrow V_\tau$. The vector $S_\tau B_\tau \phi_\lambda$ is clearly a nonzero K -finite vector. The explicit calculation in the proof of (8.1) shows in particular that $S_\tau B_\tau \phi_\lambda \in \text{Ker}(\mathcal{D}_\tau)$. Consequently, the space $\text{Ker}(\mathcal{D}_\tau) \cap C^\infty(G, \tau; \mathfrak{p}_-)$ equals $L(\tau)^\sim$.

8. Irreducibility of the image of S_τ . In §7 we let \widehat{K}_r denote the irreducible unitary representations of K such that the irreducible \mathfrak{g} -module $L(\tau)$ generated by the constant polynomials \mathbb{V}_τ^0 is properly contained in \mathbb{V}_τ . The relevancy of the set \widehat{K}_r of reduction points is apparent for the following reason. If τ is not a reduction point, then the subspace $C^\infty(G, \tau; \mathfrak{p}_-)$ of K -finite vectors is canonically isomor-

phic to $L(\tau)$. Since the (nonzero) image of S_τ lies in $C^\infty(G, \tau; \mathfrak{p}_-)$ by (5.3), it follows from the G -invariance of S_τ that the space of K -finite vectors in the image of S_τ is $C^\infty(G, \tau; \mathfrak{p}_-)$. Thus the K -finite vectors in the image of S_τ form an irreducible highest weight space with highest weight $\lambda = \lambda(\tau)$. The main theorem here extends this result to the case where τ is a reduction point. The proof of this result utilizes the system \mathcal{D}_τ defined in section seven.

Recall from §3 that $W(\tau)$ denotes the nonunitary principal series representation with imbedding parameters σ and ν chosen according to (3.9) and (3.10). As in §7, we let $\mathcal{D}_\tau = \{\partial_1, \dots, \partial_s\}$ be any system associated to $\tau \in \widehat{K}_r$.

(8.1) **THEOREM.** *The Szegő operator S_τ associated to $\tau \in \widehat{K}_r$ maps $W(\tau)$ into the kernel of \mathcal{D}_τ and the K -finite vectors in $W(\tau)$ are mapped \mathfrak{g} -equivariantly onto an irreducible highest weight module with highest weight $\lambda = \lambda(\tau)$.*

Proof. By (5.3), S_τ maps into $C^\infty(G, \tau; \mathfrak{p}_-)$. We now show that each $\partial_i \in \mathcal{D}_\tau$ annihilates the image of S_τ . Let $\partial_i \in \mathcal{D}_\tau$. With the notation of §7, we have $\partial_i f(g, \cdot) = P_i(r(\cdot)^{j_i} f(g))$ where $P_i: \mathbb{V}_\tau^{j_i} \rightarrow W_i$ denotes projection. Since ∂_i and S_τ are both G -equivariant, it suffices to show that $(\partial_i S_\tau f)(1) = 0$ for all f in $W(\tau)$.

Let $c_g: G \rightarrow G$ denote conjugation by $g: c_g(x) = gxg^{-1}$. Then for $k \in K$ and integers $j \geq 1$, one has

$$(8.2) \quad r(z)^j (f \circ c_k) = (r(\text{Ad}(k)z)^j f) \circ c_k, \quad f \in C^\infty(G).$$

For f in $W(\tau)$ we compute

$$\begin{aligned} (\partial_i S_\tau f)(1) &= \int_K \partial_i (s_\tau(k^{-1}g))|_{g=1} f(k) dk \\ &= \int_K \partial_i (s_\tau(k^{-1}gkk^{-1}))|_{g=1} f(k) dk \\ &= \int_K \partial_i (\tau(k)(s_\tau \circ c_{k^{-1}})(g))|_{g=1} f(k) dk \quad \text{by (3.7)} \\ &= \int_K P_i \tau(k) (r(\cdot)^{j_i} (s_\tau \circ c_{k^{-1}})(g))|_{g=1} f(k) dk \\ &= \int_K P_i (\tau(k) (r(\text{Ad}(k^{-1})\cdot)^{j_i} s_\tau)(g))|_{g=1} f(k) dk, \quad \text{by (8.2)} \\ &= \int_K P_i (T(k) (r(\cdot)^{j_i} s_\tau)(g))|_{g=1} f(k) dk, \quad \text{by (6.6)} \\ &= \int_K \tau_i(k) \partial_i (s_\tau(g))|_{g=1} f(k) dk \quad \text{by (7.4),} \end{aligned}$$

where $\tau_i(k)$ denotes the restriction of $T(k)$ to W_i . It suffices to show that for each $\phi \in H_\sigma$ we have $\partial_i(s_\tau(g)\phi)|_{g=1} = 0$. But by Propositions (4.16) and (6.10) we find

$$\partial_i(s_\tau(g)\phi)|_{g=1} = \partial_i(J_\tau(g^{-1}, b)^{* -1}\phi)|_{g=1} = P_i(T(\bar{b})^{j_i}1_\phi).$$

Since $T(\bar{b})^{j_i}1_\phi \in L(\tau)$ and P_i projects onto $W_i \subset M_\tau = L(\tau)^\perp$, we conclude $P_i(T(\bar{b})^{j_i}1_\phi) = 0$ for all $\phi \in H_\sigma$. This proves the image of S_τ lies in $\text{Ker}(\mathcal{D}_\tau)$. By (7.6), S_τ maps the space of K -finite vectors in $\mathcal{W}(\tau)$ to a nonzero \mathfrak{g} -module in $\text{Ker}(\mathcal{D}_\tau) \cap C^\infty(G, \tau; \mathfrak{p}_-) = L(\tau)^\sim$. Since $L(\tau)^\sim$ is irreducible, the theorem now follows.

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Received November 8, 1990 and in revised form December 15, 1991. The first named author was supported in part by the Louisiana Education Quality Support Fund 86-LBR-016-04 and LEQSF (1991–94)-RD-A-04. The second named author was supported in part by a research grant from Baylor University.

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