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CURRENTS, METRICS AND MOISHEZON MANIFOLDS

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A compact complex manifold M is Moishezon if and only if there exists an integral closed positive $(1, 1)$-current ω such that $\omega \geq \varepsilon\sigma$ and ω is smooth outside an analytic subvariety.

1. Introduction. Given a Moishezon manifold M, it is well known (cf. [**Mo**], [**W**]) that there is a bimeromorphic morphism $\pi: \widetilde{M} \to M$ such that the manifold $\widetilde{M}$ is projective algebraic. Let $\tilde{\omega}$ be Kähler form on $\widetilde{M}$ with $[\tilde{\omega}] \in H^2(\widetilde{M}, \mathbf{Z})$. Then the pushforward current $\omega = \pi_*\tilde{\omega}$ is a d-closed current on M such that

(i) $[\omega] \in H^2(M, \mathbf{Z})$;

(ii) ω is smooth on $M-S$, where S is some proper analytic subset in M;

(iii) $\omega \geq \varepsilon\sigma$ in the sense of currents, where $\varepsilon > 0$ is some real number and σ is a fixed positive definite $(1, 1)$-form (not necessarily d-closed) on M.

Conversely, we shall prove the following

THEOREM 1.1. *Let M be a compact complex manifold of dimension n. Then M is Moishezon if and only if there exists a d-closed* $(1, 1)$*-current ω on M such that the conditions* (i), (ii) *and* (iii) *above are satisfied.*

In fact, the above theorem is a weak version of a general conjecture of Shiffman [**J**] which asked: whether a compact complex manifold N is Moishezon if and only if there exists a d-closed $(1, 1)$-current satisfying the conditions (i) and (iii) above. The conjecture is to generalize the well-known Kodaira embedding theorem in terms of currents and it is still unknown. Some partial results have been obtained [**J**]: if M is complex torus, Shiffman's conjecture is true; if S is a set of isolated points, Theorem 1.1 follows from an extension theorem of Miyaoka [**M**]; if S is special in some sense, Theorem 1.1 is also true. All of these results are proved by smoothing of currents technique, and depends on a fact that the top degree Chern number $(c_1([\omega])^n, M) > 0$. However, it is easy to find an example of a current ω satisfying (i), (ii) and (iii) but its top degree Chern number is negative. So the method of smoothing currents cannot prove Theorem 1.1.

Recently, Demailly introduced a very useful notion of singular Hermitian metric on a holomorphic line bundle **[D2]** and he proved many interesting results. One of them **[D2**, Proposition 4.2 (b)**]** is that if M is a projective algebraic manifold of n-dimension with a Kähler form σ and if L is a holomorphic line bundle over M, then L admits a singular Hermitian metric with $c(L) \geq \varepsilon\sigma$ if and only if the Kodaira dimension $\kappa(L) = n$. We observed that this result is in fact the special case of Shiffman's conjecture in which M is projective algebraic (Lemma 2.1). Thus we want to modify Demailly's idea to prove Theorem 1.1. The Demailly's proof is based on the standard L^2-estimate of $\overline{\partial}$ over Stein or projective algebraic manifolds. However, in our problem, M is only a compact complex manifold. By observing that $M-S$ is complete Kähler (Lemma 4.1), instead of using the standard L^2-estimate of $\overline{\partial}$, we then prove Theorem 1.1 by using a deep generalization of the L^2 estimate theorem by Demailly **[D1]** on complete Kähler manifolds with non-complete Kähler metric and with singular metric on the line bundle. By a similar method, a special case of Shiffman's conjecture when M is Kähler is also proved.

THEOREM 1.2. *Let M be a compact Kähler manifold of dimension n. Then M is projective algebraic if and only if there exists a d-closed* $(1, 1)$*-current ω on M such that the conditions* (i) *and* (iii) *are satisfied.*

We also study a class $\mathscr{H}$ of compact complex manifolds as suggested by Harvey and Lawson **[HL**, §5, problem 2**]**. We prove the following result.

THEOREM 1.3. *Let $X \in \mathscr{H}$. Then X is a Moishezon manifold iff it is projective algebraic. In particular, this holds for any analytic compact smooth family X of curves with Kähler base space.*

Finally we point out an interesting fact below. Its proof is easy from **[K]**, **[NS]**, **[N]**.

THEOREM 1.4. *Let M be any compact complex manifold. Then the statements are equivalent:*

(i) *M is a Moishezon manifold;*

(ii) *there is a proper analytic subset $S \subset M$ such that $M-S$ admits a complete Kähler-Einstein metric with negative Ricci curvature;*

(iii) *There is a proper analytic subset* $S \subset M$ *such that* $M - S$ *admits a complete Kähler-Einstein metric with negative Ricci curvature and with finite volume*;

(iv) *There is a proper analytic subset* $S \subset M$ *such that* $tM - S$ *admits a complete Kähler metric* g *with* $\mathrm{Ricci}(g) \le -g$.

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2. Singular metric on line bundles. Let L be a holomorphic line bundle over a complex manifold M. A *singular Hermitian metric* h on L **[D2]** is a metric which is given in any local trivialization $\theta: L|_U \to U \times C$ by

$$\|\xi\| = |\theta(\xi)|e^{-h_U(x)}, \qquad x \in U, \xi \in L_x,$$

where $h_U \in L^1_{\mathrm{loc}}(U)$ is an arbitrary function, called the *weight* of the metric with respect to the trivialization θ. If $\theta': L|_{U'} \to U' \times \mathbb{C}$ is another trivialization with the associated weight h'_U, and if $\rho \in \mathscr{O}^*(U \cap U')$ is the transition function, then $\theta'(\xi) = \rho(x)\theta(\xi)$ for $\xi \in L_x$, and $h'_U = h_U + \log|\rho|$ on $U \cap U'$. The curvature form of L is then given by the d-closed $(1,1)$-current $c(L, h) = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}h_U$ on U, which is independent of the choice of local trivialization. The de Rham cohomology class of $c(L, h)$ is the image of the first Chern class $c_1(L) \in H^2(X, \mathbf{Z})$ in $H^2_{DR}(X, \mathbf{R})$.

In order to relate any integral d-closed positive $(1,1)$-current to singular Hermitian metric, we need to have the following lemma in the type of Lefschetz' $(1,1)$-theorem.

Lemma 2.1. *Let* M *be a complex manifold of dimension* n *and let* ω *be a* d*-closed positive* $(1,1)$*-current on* M. *If the de Rham class* $[\omega] \in H^2(M, \mathbb{R})$ *is integral, then there exists a holomorphic line bundle* L *with a singular Hermitian metric* h *such that*

$$\omega = c(L, h).$$

Proof. Choose an open cover $\mathscr{U} = \{U_\alpha\}$ of M such that U_α are geometrically convex and then all finite intersections of the sets in $\mathscr{U}$ are contractible. Also assume that each U_α is chosen small enough so that there exists a plurisubharmonic function h_{U_α} on U_α satisfying

$$\omega = \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}h_{U_\alpha} = \frac{1}{2\pi}dd^c h_{U_\alpha} \quad \text{on } U_\alpha$$

where $d^c = \sqrt{-1}(\overline{\partial} - \partial)$; hence $dd^c = 2\sqrt{-1}\partial\sqrt{-1}\partial\overline{\partial}$.

It is sufficient to find transition functions $\{\rho_{\alpha\beta}\} \in Z^1(\mathscr{U}, \mathscr{O}^*)$ defining a holomorphic line bundle such that

$$h_{U_\beta} = h_{U_\alpha} + \log|\rho_{\alpha\beta}|. \tag{2.2}$$

In this case, h_{U_α} is the weight of the singular metric.

Put $u_j = \frac{1}{2\pi} h_{U_\alpha}$ and $u_{\alpha\beta} = u_\beta - u_\alpha$. Because $u_\beta - u_\alpha$ is pluriharmonic on $U_\alpha \cap U_\beta$, it implies that $u_\beta - u_j$ must be smooth; thus $u_{\alpha\beta} \in \mathscr{C}^\infty_{\mathbf{R}}(U_\alpha \cap U_\beta)$.

By exactly the same argument as in [SS, Lemma 2.36, p. 38], we can construct a family of transition functions $\{\rho_{\alpha\beta}\}$ satisfying (2.2). Here we only sketch the proof: since $dd^c u_{\alpha\beta} = 0$, we can choose $v_{\alpha\beta} = C^\infty_{\mathbb{R}}(U_\alpha \cap U_\beta)$ such that $dv_{\alpha\beta} = d^c u_{\alpha\beta}$. Then $c_{\alpha\beta\gamma} = v_{\beta\gamma} - v_{\alpha\gamma} + v_{\alpha\beta}$ defines an element $\{c_{\alpha\beta\gamma}\} \in Z^2(\mathscr{U}, \mathbf{R})$. By Leray isomorphism, $\{c_{\alpha\beta\gamma}\}$ corresponds to the cohomology class $[\omega]$. Since $\{c_{\alpha\beta\gamma}\}$ is integral, there is a 1-cochain $\{b_{\alpha\beta}\} \in C^1(\mathscr{U}, \mathbf{R})$ such that

$$c_{\alpha\beta\gamma} + b_{\beta\gamma} - b_{\alpha\gamma} + b_{\alpha\beta} = m_{\alpha\beta\gamma} \in \mathbf{Z}.$$

Let $f_{\alpha\beta} = u_{\alpha\beta} + \sqrt{-1}(u_{\alpha\beta} + b_{\alpha\beta})$, which is a holomorphic function such that

$$f_{\beta\gamma} - f_{\alpha\gamma} + f_{\alpha\beta} = \sqrt{-1} m_{\alpha\beta\gamma}.$$

Let $\rho_{\alpha\beta} = \exp(2\pi f_{\alpha\beta})$. Such $\{\rho_{\alpha\beta}\} \in Z^1(\mathscr{U}, \mathscr{O}^*)$ satisfies (2.2). Thus the lemma is proved. □

From the proof above, by the standard regularity theorem for elliptic operators, it is easy to obtain the following

COROLLARY 2.3. *Let M, ω be as in Lemma 2.1 and let ω be smooth on $M - S$ for some proper analytic subset $S \subset M$. Then there exists a holomorphic line bundle L over M with a singular metric h such that*

$$\omega = c(L, h)$$

and $h|_{M-S}$ is smooth, i.e., for any point in $x \in M$, there is a neighborhood U of x in M such that the weight h_U of the singular metric h is smooth on $U - S$.

Let L be a holomorphic line bundle over M admitting a singular metric h such that the curvature current $c(L, h) \geq 0$. For any $x \in M$, let h_U be the weight of the metric on a neighborhood U of x, we define the *Lelong number* with respect to the singular metric by (cf. **[D2]**)

$$v(h_U, x) = \liminf_{z \to x} \frac{h_U(z)}{\log|z-x|}. \tag{2.4}$$

Equivalently,

$$v(h_U, x) = \lim_{r \to 0} v(c(L, h), x, r)$$

where

$$v(c(L, h), x, r) = \frac{1}{(2\pi r^2)^{n-1}} \int_{B(x,r)} c(L, h) \wedge (\sqrt{-1}\partial\overline{\partial}|z|^2)^{n-1}.$$

So we can denote $v(h_U, x)$ to be $v(c(L, h), x)$. We define a set

$$E_c(c(L, h)) = \{x \in M; v(c(L, h), x) \geq c\}$$

which is an analytic subset by a well-known theorem of Siu **[Si]**.

LEMMA 2.5 (*cf.* **[D2**, *Lemma* 2.8]). *If ϕ is a plurisubharmonic function on M, then $e^{-2\phi}$ is integrable in a neighborhood of $x \in M$ if $v(\phi, x) < 1$, and $e^{-2\phi}$ is non-integrable on any neighborhood of x if $v(\phi, x) \geq n$.*

3. L^2 estimate for $\overline{\partial}$ over complete Kähler manifolds. In this section we review some results of Demailly **[D1]** and state a general L^2 estimate for $\overline{\partial}$ for line bundles with singular metric.

Let M be a complex manifold of dimension n with a Kähler metric ω. We shall use the same notation ω to denote the associated Kähler form. Denote $dV_\omega = \omega^n/n!$ to be the volume form of (X, ω). The form ω defines an operator on $\bigwedge^{p,q} T^*M$ by

$$\omega(\alpha) = \omega \wedge \alpha \in \bigwedge^{p+1,q+1} T^*M$$

and its adjoint operator Λ is defined by

$$\langle \Lambda\alpha, \beta \rangle = \langle \alpha, \omega(\beta) \rangle$$

for all $\alpha \in \bigwedge^{p,q} T^*M$, $\beta \in \bigwedge^{p-1,q-1} T^*M$. Here $\langle\ ,\ \rangle$ is the inner product given by ω.

Let L be a holomorphic line bundle over X. Then these operators ω and Λ can be extended to the space of L-valued (p, q)-forms, $\bigwedge^{p,q} T^*M \otimes L$, by the identity map id_L. In additional we suppose that (L, h) is a line bundle over M with a positive C^2 Hermitian metric h, i.e., its first Chern class $c_1(L, h) > 0$. For each integer q, $1 \leq q \leq n$, we define a bilinear form $c(L, h)_q$

$$c(L, h)_q(\alpha, \beta) = \langle 2\pi c_1(L, h)\Lambda\alpha, \beta\rangle$$

for all α, $\beta \in \bigwedge^{n,q} T^*M \otimes L$. Since $c_1(L, h) > 0$, it is known that $c(L, h)_q$ is positive, for all q [**D1**, Lemma 3.1]. For any forms $\alpha \in \bigwedge^{n,q} T^*M \otimes L$, one defines

$$|\alpha|_{c(L,h)_q} = \sup_\beta \left\{ \frac{|\langle \alpha, \beta\rangle|}{c(L, h)_q(\beta, \beta)} \right\}$$

where $0 \neq \beta$ runs through $\bigwedge^{n,q} T^*M \otimes L$. Notice that the number $|\alpha|_{c(L,h)_q}$ may be equal to infinity. In practice, in order to estimate the term $|\alpha|_{c(L,h)_q}$, we have the following result. If $c(L, h) \geq \lambda\omega \otimes \mathrm{Id}_L$, where $\lambda \geq 0$ is a measurable function on M, then for $\alpha \in \bigwedge^{n,q} T^*M \otimes L$, one has [**D1**, Lemma 3.2]

$$|\alpha|^2_{c(L,h)_q} \leq \frac{1}{q\lambda}|\alpha|^2. \tag{3.1}$$

LEMMA 3.2 [**D1**, *Theorem* 4.1]. *Let M be a complete Kähler manifold of dimension n. Let ω be a Kähler metric which is not necessarily complete. Let (L, h) be a holomorphic Hermitian line bundle over M with a C^2 positive Hermitian metric h. Then for any smooth L-valued (n, q)-form g on M with*

$$\bar\partial g = 0, \int_M |g|^2\, dV_\omega < \infty \quad \textit{and} \quad \int_X |g|^2_{c(L,h)_q}\, dV_\omega < \infty,$$

there exists a smooth L-valued $(n, q-1)$-form f on M such that

$$\bar\partial f = g, \quad \textit{and} \quad \int_M |f|^2\, dV_\omega \leq \int_X |g|^2_{c(L,h)_q}\, dV_\omega .$$

Notice that M is complete Kähler, i.e., M admits a complete Kähler metric g, but ω may not be equal to g. The norm $|\ |$ is defined with respect to ω and h.

Let M be a complete Kähler manifold of dimension n. Again let ω be a Kähler metric which is not necessarily complete. Let L be a holomorphic line bundle over M with a C^2 Hermitian metric h. Let ϕ be a function on M such that for any point $x \in M$, there is a neighborhood U of x in M such that the restriction of ϕ on U

$$\phi|_U = \phi_1 + \phi_2 \tag{3.3}$$

where ϕ_1 is a C^2 function on U and ϕ_2 is a plurisubharmonic function on U. The Lebesgue decomposition of the 0-order current $\sqrt{-1}\partial\overline{\partial}\phi$ gives

$$\sqrt{-1}\partial\overline{\partial}\phi = \sqrt{-1}(\partial\overline{\partial}\phi)_c + \sqrt{-1}(\partial\overline{\partial}\phi)_s$$

where the singular part $\sqrt{-1}(\partial\overline{\partial}\phi)_s$ is a positive $(1, 1)$-current, and the absolute continuous part $\sqrt{-1}(\partial\overline{\partial}\phi)_c$ is a semipositive $(1, 1)$-form with L^1_{loc} coefficients.

We define

$$c(L, e^{-\phi}h) = c(L, h) + \frac{\sqrt{-1}}{\pi}(\partial\overline{\partial}\phi)_c.$$

LEMMA 3.4 **[D1,** *Theorem* 5.1**]**. *Let M be a complete Kähler manifold of dimension n. Let ω be a Kähler metric which is not necessarily complete. Let L be a holomorphic line bundle over M with a C^2 Hermitian metric h. Let ϕ be a function which is locally the sum of a C^2 function and a plurisubharmonic function as in* (3.3). *Suppose $c(L, e^{-\phi}h) \geq 0$. Then for any smooth L-valued (n, q)-form g on M with*

$$\overline{\partial} g = 0 \quad \textit{and} \quad \int_M |g|^2_{c(L, e^{-\phi}h)_q} e^{-2\phi}\, dV_\omega < \infty,$$

there exists a smooth L-valued $(n, q-1)$-form f on M such that

$$\overline{\partial} f = g \quad \textit{and} \quad \int_M |f|^2 e^{-2\phi}\, dV_\omega \leq \int_M |g|^2_{c(L, e^{-\phi}h)_q} e^{-2\phi}\, dV_\omega.$$

where $|\ |$ is defined with respect to h and ω.

The above lemma leads to a general L^2-estimate for $\overline{\partial}$ for any holomorphic line bundle with singular metric as follows.

Let M, ω be as in Lemma 3.4 above. Let L be a holomorphic line bundle over M with a singular Hermitian metric h. Suppose

$$c(L, h) \geq 0$$

in the sense of currents. Now take and fix any smooth Hermitian metric h_0 and L; then on each open subset U such that $L|_U$ is trivial, the weight $h_{0,U}$ of h_0 is a smooth function on U. Define on each such U a function

$$\varphi_U = h_U - h_{0,U}. \tag{3.5}$$

It is easy to see that we have in fact defined a function φ on M globally such that

$$\varphi|_U = \varphi_U.$$

Since $c(L, h) \geq 0$, by the proof of Lemma 2.1, we see that any weight h_U of the metric h is plurisubharmonic, then the function φ is obviously locally a sum of a C^2-function and a plurisubharmonic function. Then from Lemma 3.4 we obtain

COROLLARY 3.6. *Let M, ω be as in Lemma 3.4. Let L be a holomorphic line bundle over M with a singular Hermitian metric h such that $c(L, h) \geq 0$. Suppose that h_0 is any smooth Hermitian metric on L and denote φ to be the function on M defined by* (3.5). *Then for any smooth L-valued (n, q)-form g on M with*

$$\bar{\partial} g = 0 \quad \text{and} \quad \int_M |g|^2_{c(L, e^{-\varphi} h_0)_q} e^{-2\varphi}\, dV_\omega < \infty,$$

there exists a smooth L-valued $(n, q-1)$-form f on M such that

$$\bar{\partial} f = g \quad \text{and} \quad \int_M |f|^2\, dV_\omega \leq \int_M |g|^2_{c(L, e^{-\varphi} h_0)_q} e^{-2\varphi}\, dV_\omega,$$

where $|\ |$ is defined by h and ω.

Proof. Apply Lemma 3.4 to (L, h_0) and φ, we know that for any g with $\bar{\partial} g = 0$ and $\int_M |g|^2_{c(L, e^{-\varphi} h_0)_q} e^{-2\varphi}\, dV_\omega < \infty$, there exists f such that

$$\bar{\partial} f = g \quad \text{and} \quad \int_M |f|^2_{h_0, \omega} e^{-2\varphi}\, dV_\omega \leq \int_M |g|^2_{c(L, e^{-\varphi} h_0)_q} e^{-2\varphi}\, dV_\omega.$$

Notice that $|f|^2_{h_0, \omega} e^{-2\varphi} = |f|^2_{h, \omega}$ by (3.5), and the corollary follows. □

REMARK 3.7. Suppose that the line bundle L has a singular metric h such that $c(L, h) \geq \varepsilon\omega$, for some constant $\varepsilon > 0$, i.e.,

$$c(L, h)(v, v) \geq \varepsilon\omega(v, v),$$

for any test form v. Since

$$\begin{aligned} c(L, e^{-\phi} h_0)(v, v) &= \left[c(L, h_0) + \left(\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\right)_c\right](v, v) \\ &= \left[c(L, h_0) + \frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\phi\right](v, v) \\ &= c(L, h)(v, v), \end{aligned}$$

we see that

$$c(L, e^{-\phi} h_0)(v, v) \geq \varepsilon\omega(v, v).$$

Therefore by (3.1) we have the estimate

$$|\alpha|^2_{c(L, e^{-\phi}h_0)_q} \leq \text{const}.|\alpha|^2$$

for any $\alpha \in \bigwedge^{n,q} T^*M \otimes L$.

4. Complete Kähler metric on $M - S$. In order to prove Theorem 1.1, we wish to apply Lemma 3.4 to $X - S$. In general, $X - S$ may not be complete Kähler, but together with the d-closed $(1, 1)$-current ω satisfying (ii) and (iii), we can construct a complete Kähler metric on $M - S$. The proof is analogous to [**D1**, Proposition 1.6].

LEMMA 4.1. *Let M be a compact complex manifold and $S \subset M$ a proper analytic subset. Let ω be a d-closed* $(1, 1)$*-current satisfying the conditions* (ii) *and* (iii) *in* §1. *Then $M - S$ admits a complete Kähler metric.*

Proof. By [**D1**, Proposition 1.4], for any complex manifold M and any analytic subset $S \subset M$, there exists a locally integrable function ψ on M such that ψ is smooth on $M - S$; $\psi(x) < -1$, for any $x \in M - S$; and $\psi(x) \to -\infty$ as x goes to S, and there exists a real continuous $(1, 1)$-form γ on X such that

(4.2) $\sqrt{-1}\partial\overline{\partial}\psi \geq \gamma$;

(4.3) if $\alpha > 0$ is a real number, $e^{-\alpha\psi}$ is non-integrable on a neighborhood of a point $s \in S$ where the codimension of the germ S_s satisfies

$$\operatorname{codim} S_s \geq \alpha.$$

Put $\tilde{\omega} = C\omega - \sqrt{-1}\partial\overline{\partial}\sqrt{-\psi}$, which is a smooth form on $M - S$ and is a current on M. We claim that we can choose the constant $C > 0$ large enough such that

$$(C-1)\omega + \frac{\sqrt{-1}\partial\overline{\partial}\psi}{2\sqrt{-\psi}} \geq 0 \quad \text{on } M - S. \tag{4.4}$$

In fact, since γ is continuous on M and $\omega \geq \varepsilon\sigma$ on M (cf. (iii) in §1), and since X is compact, we can find a constant number $C > 0$ such that

$$(C-1)\varepsilon\sigma \geq -\frac{\gamma}{2\sqrt{-\psi}} \quad \text{on } M.$$

Then

$$(C-1)\omega + \frac{\sqrt{-1}\partial\overline{\partial}\psi}{2\sqrt{-\psi}} \geq (C-1)\varepsilon\sigma + \frac{\gamma}{2\sqrt{-\psi}} \geq 0$$

on M in the sense of currents. Thus (4.4) is proved. From (4.4), it yields

$$\tilde{\omega} \geq \omega + 4\sqrt{-1}\partial(-\psi)^{1/4} \wedge \overline{\partial}(-\psi)^{1/4} \tag{4.5}$$

because

$$\sqrt{-1}\partial\overline{\partial}(-\sqrt{-\psi}) = \frac{\sqrt{-1}\partial\overline{\partial}\psi}{2\sqrt{-\psi}} + 4\sqrt{-1}\partial(-\psi)^{1/4} \wedge \overline{\partial}(-\psi)^{1/4}.$$

Then $\tilde{\omega}$ is complete Kähler by the same argument of [**D1**, proof of Proposition 1.6]. For the reader's convenience, we still give the proof: let δ (resp. $\tilde{\delta}$) be the geodesic distance associated to ω (resp. $\tilde{\omega}$). For any two $z_1, z_2 \in M$,

$$\delta(z_1, z_2) = \inf \int_0^1 \sqrt{\omega\left(\frac{du}{dt}, \sqrt{-1}\frac{du}{dt}\right)}\, dt$$

(similarly one defines $\tilde{\delta}(z_1, z_2)$) where u runs through the set of all C^1 curves $u: [0, 1] \to M - S$ with the ending points z_1 and z_2. By (4.5),

$$\begin{aligned}\tilde{\omega}\left(\frac{du}{dt}, \sqrt{-1}\frac{du}{dt}\right) &\geq \omega\left(\frac{du}{dt}, \sqrt{-1}\frac{du}{dt}\right) + 4\left|\partial\psi\left(\frac{du}{dt}\right)\right|^2 \\ &\geq \omega\left(\frac{du}{dt}, \sqrt{-1}\frac{du}{dt}\right) + \left|\frac{d(\psi \circ u)}{dt}\right|^2\end{aligned}$$

because

$$\frac{d(\psi \circ u)}{dt} = d\psi\left(\frac{du}{dt}\right) = 2\operatorname{Re}\partial\psi\left(\frac{du}{dt}\right).$$

Thus

$$\tilde{\delta}(z_1, z_2) \geq \sup(\delta(z_1, z_2), |\psi(z_1) - \psi(z_2)|).$$

Since ψ is exhaustive, and since a manifold admits a complete metric $\tilde{\omega}$ if and only if the closed balls defined by geodesic distance $\tilde{\delta}$ are always compact, we know that $\tilde{\omega}$ is complete Kähler on $M - S$. □

5. Proofs of Theorems 1.1 and 1.2.

LEMMA 5.1 [**D1**, *Lemma* 6.9]. *Let $\Omega \subset \mathbf{C}^n$ be an open subset, and let $Y \subset \Omega$ be an analytic subset. If w is a (p, q)-form with L^1_{loc} coefficients on Ω, and v is a $(p, q-1)$-form with L^2_{loc} coefficients on Ω such that $\overline{\partial}v = w$ on $\Omega - Y$ in the sense of currents, then $\overline{\partial}v = w$ on Ω in the sense of currents.*

Proof of Theorem 1.1. By Lemma 2.1 and Corollary 2.2, there exists a holomorphic line bundle L over M with a singular metric h such that $\omega = c(L, h)$ and h is smooth on $U - S$.

By Lemma 4.1, $M - S$ is a complete Kähler manifold. Since the restriction of the singular metric h on L is smooth over $M - S$, $L|_{M-S}$ has a smooth Hermitian $h = h|_{M-S}$. We consider ω as Kähler metric on $M - S$. Notice that ω is not necessarily complete.

Take and fix a point $x_0 \in M - S$. Because h is smooth on $M - S$, we see that the Lelong number is

$$v(c(L, h), x) = 0, \quad x = x_0 \text{ or } x \text{ near } x_0.$$

Let Ψ_0 be a smooth function on $M - \{x_0\}$ which is equal to $n \log|z - x_0|$ (in some coordinates) near x_0.

By the hypothesis (iii), $c(L, h) \geq \varepsilon\sigma$ on M, there is some m such that

$$mc(L, h) + \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\Psi_0 \geq m\varepsilon\sigma + \frac{\sqrt{-1}}{\pi}\partial\overline{\partial}\Psi_0 \geq \sigma. \tag{5.2}$$

Put $v_m = c(L, h) + (\sqrt{-1}/\pi)\partial\overline{\partial}\Psi_0$. The Lelong number

$$v(v_m, x_0) = n + 1 \quad \text{and} \quad v(v_m, z) < 1, \qquad \text{for } z \neq x_0 \text{ near } x_0.$$

Therefore, by Lemma 2.5, $e^{-2\Psi_0}$ is non-integrable near x_0.

Let $P(z)$ be an arbitrary polynomial of degree 1 in the given coordinates V of x_0. Fix a smooth cut-off function χ with compact support in V such that $\chi = 1$ near x_0. Fix a non-vanishing local holomorphic section $g \in H^0(V, K_M \otimes L^m)$.

Then $v = P\overline{\partial}\chi \otimes g$ is regarded as a smooth $\overline{\partial}$-closed L^m-valued $(n, 1)$-form on M and hence on $M - S$ such that

$$\int_{M-S} |v|^2 e^{-2\Psi_0}\, dV_\omega < \infty$$

where Ψ_0 is constructed as above, and $|\;|$ is defined by ω and by the smooth Hermitian metric h.

Then by (5.2), we apply Lemma 3.4 and Remark 3.7 to $M - S$, (L^m, h^m) and Ψ_0, and then there is a smooth L^m-valued $(n, 0)$-form u on $M - S$ such that

$$\overline{\partial}u = v \quad \text{and} \quad \int_{M-S} |u|^2 e^{-2\Psi_0}\, dV_\omega \leq \int_{M-S} |v|^2 e^{-2\Psi_0}\, dV_\omega < \infty.$$

Then we claim that u can be extended as a smooth L^m-valued $(n, 0)$-form on M. In fact, we apply Lemma 5.1 to prove it. Since v is a smooth L^m-valued $(n, 0)$-form on M, it is sufficient to show that $\forall x \in S$, there is a neighborhood $U = U_x$ of x in M such that

$$\int_{U-S} |u|_U^2\, dV_U < \infty$$

where dV_U is the Euclidean volume form on U with respect to a coordinate system and the norm $|\ |_U$ is with respect to dV_U and h. Recall $\omega \geq \varepsilon\sigma$, and Ψ_0 is smooth near S; it yields

$$\int_U |u|_U^2 dV_U = \int_{U-S} |u|_U^2 \, dV_U \leq \text{constant} \int_{U-S} |u|^2 e^{-2\Psi_0} \, dV_\omega < \infty .$$

The claim then is proved by the regularity theorem.

Also we claim that $|u(z)| = o(|z - x_0|)$ near x_0. In fact, it is true by the fact that $\int_{M-S} |u|^2 e^{-2\Psi_0} \, dV < \infty$, and that $e^{-2\Psi_0}$ is non-integrable near x_0, and that u is holomorphic near x_0.

Therefore

$$f := \chi P g - u \in H^0(M, K_M \otimes L^m)$$

has the prescribed 1-jet Pg at x_0. Thus the Kodaira dimension of $K_M \otimes L^m = n$. Hence M is Moishezon. □

Proof of Theorem 1.2. By the similar procedure as above, replacing $M - S$ by M, we can apply Corollary 3.6 and Remark 3.7 to know that M is Moishezon if and only if there is a $(1, 1)$-form ω satisfying (i) and (iii). Since M is Kähler, by Moishezon's theorem [**Mo**], it is equivalent to M being projective algebraic. Then Theorem 1.2 follows. □

6. Projectivity of a class of Moishezon manifolds. In 1983, Harvey and Lawson proved a characterization theorem for Kähler manifolds [**HL**], that is, a compact complex manifold is Kähler if and only if there exists no nontrivial positive current which is a bidimension $(1, 1)$-component of a boundary. They also raised several general problems. One of them [**HL**, §5, problem 2] is as follows: describe the class of compact complex manifolds which satisfy that if there exists a non-trivial positive current which is the bidimension $(1, 1)$-component of a boundary, then there exists a non-trivial positive smooth current which is the bidimension $(1, 1)$-component of a boundary. We denote this class by $\mathscr{H}$. The significance of this problem is that to test whether a given manifold in $\mathscr{H}$ is Kähler; it suffices to check the pointwise non-negative, smooth $(n-1, n-1)$-forms, to see if one is a boundary.

It is worth remarking that investigating the obstruction of a Moishezon manifold to be projective algebraic is an interesting problem in the theory of compact complex manifolds. Classically we know that there is no obstruction for compact complex surfaces (Chow-Kodaira [**CK**]) and for complex tori (Lefschetz [**W**]). Moishezon's theorem

[**Mo**] just means that this obstruction is equivalent to that the manifold is non-Kähler. Recently Peternell showed that this obstruction for 3-dimensional complex manifolds is a positive integral linear combination of irreducible curves which is homologous to zero.

As an important example in $\mathscr{H}$, we point out that if (X, Y, f) is any analytic compact smooth family of curves with Kähler base space Y, then X is in $\mathscr{H}$. Here we say that (X, Y, f) is an *analytic compact smooth family of curves* if X and Y are compact connected complex manifolds, and $f: X \to Y$ is a surjective holomorphic map which is everywhere of maximal rank such that each fiber $X_y = f^{-1}(y)$ is a connected smooth curve for any $y \in Y$. Notice that in this case f is a submersion. For any analytic compact smooth family of curves (X, Y, f) with Kähler base space Y, we know $X \in \mathscr{H}$ by [**HL**, Theorem $(17)^\infty$].

Proof of the Theorem 1.3. By applying Moishezon's theorem [**Mo**] that a Moishezon manifold is projective algebraic if and only if it is Kähler, it suffices to show: if X is a Moishezon manifold, then X is a Kähler manifold. Suppose X is non-Kähler. By the result of Harvey and Lawson [**HL**, Proposition (12) and Theorem (14)], we know that there exists a non-trivial positive current T_0 on X which is the bidimension $(1, 1)$-component of a boundary. Since $X \in \mathscr{H}$, there exists a non-trivial positive smooth current T which is the bidimension $(1, 1)$-component of a boundary. We can write

$$T = \partial S^{2,1} + \bar{\partial} S^{1,2},$$

where $S^{2,1}$ and $S^{1,2}$ are some currents of X of the bidimension $(2, 1)$ and $(1, 2)$, respectively. Notice that these $S^{2,1}$ and $S^{1,2}$ may not be smooth.

Since X is Moishezon, there exists a modification $\pi: \widetilde{X} \to X$ such that the manifold $\widetilde{X}$ is projective algebraic. Let $\tilde{\sigma}$ be a Kähler form on $\widetilde{X}$.

We claim that the push-forward current $\sigma := \pi_* \tilde{\sigma}$ is a d-closed $(1, 1)$-current on X satisfying the following property: for any point $a \in X$, there exists a neighborhood U of a in X with a local coordinates system $(U, z^1, \dots, z^n)$ and a positive constant C such that

$$\sigma - C \sum_{j=1}^{n} \sqrt{-1}\, dz^j \wedge d\bar{z}^j \geq 0 \tag{6.1}$$

on U in the sense of currents. In fact, for any point $a \in M$, we can find a neighborhood U of a in M with a local coordinate system $(U, z^1, \dots, z^n)$ and a constant $C > 0$ such that

$$\tilde{\sigma} - C\pi^* \left(\sum_{j=1}^{n} \sqrt{-1}\, dz^j \wedge d\overline{z}^j \right) \geq 0$$

on $\pi^{-1}(U)$ in the sense of currents. Then on U, we see $\pi_*\tilde{\sigma} - C\pi_*\pi^*(\sum_{j=1}^n \sqrt{-1}\, dz^j \wedge d\overline{z}^j) \geq 0$ in the sense of currents. So we have $\sigma = \pi_*\tilde{\sigma} \geq C\pi_*\pi^*(\sum_{j=1}^n \sqrt{-1}\, dz^j \wedge d\overline{z}^j) \geq C\sum_{j=1}^n \sqrt{-1}\, dz^j \wedge d\overline{z}^j$ on U in the sense of currents. The claim (6.1) is then proved.

We define the smoothing σ_ε as follows: Let $\{U_i\}_{1\leq i \leq}$ be any finite open covering of X and $\{\varphi_i\}_{1\leq i\leq q}$ be any partition of unity subordinate to $\{U_i\}_{1\leq i\leq q}$. Suppose that every U_i is a coordinate chart and that U_i is identified with a unit ball with center $0 \in \mathbf{C}^n$ with respect to the coordinate chart. On each U_i, since it is biholomorphic to the open unit ball, we can write

$$\sigma = \sqrt{-1}\partial\overline{\partial} f_i .$$

Because $f_i - f_j$ is pluriharmonic on $U_i \cap U_j$, $\forall i \neq j$, it implies that $f_j = f_i$ is smooth. Then we define a global d-closed smooth real $(1, 1)$-form P on X

$$P := \omega - \sqrt{-1}\partial\overline{\partial} \sum_{i=1}^{r} \varphi_i f_i$$

because $P|_{U_j} = \sqrt{-1}\partial\overline{\partial} \sum_{i=1}^r \varphi_i(h_j - h_i)$; i.e., we have $\sigma = P + \sqrt{-1}\sum_i \varphi_i f_i$. Then smoothing σ_ε of σ, a d-closed real $(1, 1)$-form on X, is defined by

$$\sigma_\varepsilon = P + \sqrt{-1}\partial\overline{\partial} \left(\sum_{i=1}^{q} \chi_{i,\varepsilon} * (\varphi_i f_i) \right) ,$$

where $\chi_{i,\varepsilon}$ is the standard approximation of identity defined on U_i.

Since $\sigma_\varepsilon - \sigma = \sqrt{-1}\partial\overline{\partial}\{\sum_{i=1}^q (\chi_{i,\varepsilon} * (\varphi_i f_i) - (\varphi_i f_i))\}$, it follows that

$$[\sigma] = [\sigma_\varepsilon] \in H^{1,1}(X, \mathbf{R}) \quad \text{and} \quad \sigma_\varepsilon \to \sigma, \text{ as } \varepsilon \to 0$$

in the sense of currents.

By the facts that X is compact and that T is smooth, we can make the following computation:

$$\begin{aligned}(\sigma, T) &= \lim_{\varepsilon\to 0}(\sigma_\varepsilon, T) = \lim_{\varepsilon\to 0}(\sigma_\varepsilon, \partial S^{2,1} + \overline{\partial} S^{1,2}) \\ &= \lim_{\varepsilon\to 0}(0 + 0) = 0,\end{aligned}$$

where we used the fact that σ_ε is d-closed.

On the other hand, the following claim leads to a contradiction and it completes the proof of the theorem:

$$(6.2) \qquad (\sigma, T) > 0.$$

In fact, let the open covering $\{U_i\}_{1\leq i\leq q}$ of X and a partition of unity $\{\varphi_i\}_{1\leq i\leq q}$ subordinate to $\{U_i\}_{1\leq i\leq q}$ be as before. Since T is non-trivial and σ satisfies the property (6.1), we assume that the form $\varphi_1 T \neq 0$ and that $\sigma \geq C\sum_{j=1}^n \sqrt{-1}\, dz_j \wedge d\overline{z}_j$ on U_1 for some positive constant number C in the sense of currents. Then

$$(\sigma, T) = \sum_{i=1}^{q} (\sigma, \varphi_i T).$$

We claim that for each i, $1 \leq i \leq q$, $(\sigma, \varphi_i T) \geq 0$. In fact, let $\chi_{i,\varepsilon}$ be the standard approximation of identity defined on U_i. Then $(\sigma, \varphi_i T) = \lim_{\varepsilon\to 0}(\chi_{i,\varepsilon} * \sigma, \varphi_i T)$. Notice that $\chi_{i,\varepsilon} * \sigma$ is a positive $C^\infty(1, 1)$-form on U_i for any $\varepsilon > 0$, and that $\varphi_i T$ is positive $(n-1, n-1)$-current on U_i; it follows that $(\chi_{i,\varepsilon} * \sigma, \varphi_i T)$ is non-negative. The claim then is verified by letting ε go to zero. Therefore, we have shown

$$(6.3) \qquad (\sigma, T) \geq (\sigma, \varphi_1 T).$$

For the positive current $\sigma - C\sum_{j=1}^n \sqrt{-1}\, dz_j \wedge d\overline{z_j}$ on U_1, by applying the same method, we know $(\sigma - C\sum_{j=1}^n \sqrt{-1} dz_j \wedge d\overline{z_j}, \varphi_1 T) \geq 0$, i.e.,

$$(6.4) \qquad (\sigma, \varphi_1 T) \geq C\left(\sum_{j=1}^{n} \sqrt{-1} dz_j \wedge d\overline{z_j}, \varphi_1 T\right).$$

Applying Wirtinger's Inequality as well as the argument in [**HL**, §4], we get

$$(6.5) \qquad \left(\sum_{j=1}^{n} \sqrt{-1} dz_j \wedge d\overline{z_j}, \varphi_1 T\right) = M(\varphi_1 T) > 0,$$

where $M(\varphi_1 T)$ is the mass of T. The claim (6.2) follows from (6.3), (6.4), (6.5) above. □

7. Moishezon manifolds and Kähler-Einstein metrics. We prove Theorem 1.4 now. The statements (ii) ⇒ (ii) ⇒ (iv) are trivial. It suffices to show (i) ⇒ (iii) and (iv) ⇒ (v).

Proof for (i) $\Rightarrow$ (iii). Suppose that M is a Moishezon manifold. Then there is a projective algebraic manifold $\widetilde{M}$, a proper surjective holomorphic mapping $\pi: \widetilde{M} \to M$, and an analytic set V such that the restriction mapping $\pi|_{\widetilde{M}-\widetilde{V}}: \widetilde{M} - \widetilde{V} \to M - V$ is bihomororphic, where $\widetilde{V} := \pi^{-1}(V)$. Choose a purely 1-codimensional analytic subset $\widetilde{S}$ on $\widetilde{M}$ such that $\widetilde{V} \subset \widetilde{S}$ and that $K_M \otimes [\widetilde{S}]$ is ample. By applying Hironaka's resolution of singularities if necessary, we assume without loss of generality that $\widetilde{S}$ is with simple normal crossings. Then by a result of Kobayashi [**K**, Theorem 1], we know that $\widetilde{M} - \widetilde{S}$ admits a complete Kähler-Einstein metric $\tilde{g}$ which is with negative Ricci curvature and with finite volume. Then we set $g := ((\pi|_{M-S})^{-1})^* \tilde{g}$.

Proof for (iv) $\Rightarrow$ (i). Suppose that there is an analytic subset $S \subset M$ such that $M - S$ admits a complete Kähler metric g with Ricci$(g) \leq -g$. By Hironaka's resolution of singularities again if necessary, we can assume S is a hypersurface. Then by the L^2 Riemann-Roch inequality proved by Nadel and Tsuji [**NS**] and by [**N**, Proposition 1.11], it implies

$$\begin{aligned}
&\liminf_{k\to+\infty} \frac{1}{k^n} \dim H^0(M, K_K^{\otimes k} \times [S]^{\otimes(k-1)}) \\
&\quad \geq \liminf_{k\to+\infty} \frac{1}{k^n} \dim H^0_{(2)}(j, K_{M-S}^{\otimes k}) \\
&\quad \geq \frac{1}{n!} \int_{M-S} c_1(K_{M-S})^n \geq \frac{1}{n!} \int_{M-S} g^n > 0.
\end{aligned}$$

Since $H^0(M, K_M^{\otimes k} \otimes [S]^{\otimes(k-1)}) \subset H^0(M, (K_M \otimes [S])^k)$, we then see

$$\begin{aligned}
&\dim H^0(M, (K_M \otimes [S])^{\otimes k}) \\
&\quad \geq \frac{k^n}{n!} \int_{M-S} g^n + O(k^n) = \left(\frac{1}{n!} \int_{M-S} g^n + \frac{O(k^m)}{k^n}\right) k^n.
\end{aligned}$$

This yields (i). □

Note added in proof. Recently, Shiffman's conjecture has been proved completely. See: S. Ji and B. Shiffman, *Properties of compact complex manifolds carrying closed positive currents*, to appear in J. Geom. Anal.

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