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BETWEEN THE UNITARY AND SIMILARITY ORBITS OF NORMAL OPERATORS

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D. A. Herrero has defined the $(\mathscr{U}+\mathscr{K})$-orbit of an operator T acting on a Hilbert space $\mathscr{H}$ to be $(\mathscr{U}+\mathscr{K})(T) = \{R^{-1}TR\colon R$ invertible of the form unitary plus compact$\}$. In this paper, we characterize the norm closure in $\mathscr{B}(\mathscr{H})$ of such an orbit in three cases: firstly, when T is normal; secondly when T is compact; and thirdly, when T is the unilateral shift. Some consequences of these characterizations are also explored.

1. Introduction. Let $\mathscr{H}$ be a complex, separable, infinite dimensional Hilbert space and denote by $\mathscr{B}(\mathscr{H})$ the set of bounded linear operators acting on $\mathscr{H}$. As usual, $\mathscr{K}(\mathscr{H})$ will denote the unique two-sided ideal of compact operators. There are many interesting ways of partitioning the set $\mathscr{B}(\mathscr{H})$ into equivalence classes. We mention two in particular.

Given $T \in \mathscr{B}(\mathscr{H})$, we define the *unitary orbit* of T as $\mathscr{U}(T) = \{U^*TU\colon U \in \mathscr{B}(\mathscr{H})$ a unitary operator$\}$. Then an operator $A \in \mathscr{U}(T)$ if its action on $\mathscr{H}$ is geometrically identical to that of T. Equivalently, one can think of A as T itself acting on an isomorphic copy of $\mathscr{H}$.

Another much studied class is the *similarity orbit* of $T \in \mathscr{B}(\mathscr{H})$, namely $\mathscr{S}(T) = \{S^{-1}TS\colon S \in \mathscr{B}(\mathscr{H})$ an invertible operator$\}$. This notion of equivalence ignores the geometry of the Hilbert space, and concentrates on the underlying vector space structure.

In general, neither of these sets need be closed. This is in contrast to finite dimensional Hilbert spaces, where $\mathscr{U}(T)$ is always closed while $\mathscr{S}(T)$ is closed if and only if T is similar to a normal matrix [**Her 1**, p. 14]. It is therefore interesting to describe the norm closure of these orbits, a program which for unitary orbits was undertaken by D. W. Hadwin [**Had**], using a result of D. Voiculescu [**Voi**], and for similarity orbits was done by C. Apostol, L. Fialkow, D. Herrero, and D. Voiculescu [**AFHV**].

One can also turn one's attention to the Calkin algebra $\mathscr{A}(\mathscr{H}) = \mathscr{B}(\mathscr{H})/\mathscr{K}(\mathscr{H})$ and consider both unitary and similarity orbits there. Indeed, one of the major results along these lines is the classification of the unitary orbits of normal elements of $\mathscr{A}(\mathscr{H})$ by L. Brown, R. G. Douglas and P. Fillmore **[BDF]**.

Denoting the canonical map from $\mathscr{B}(\mathscr{H})$ to $\mathscr{A}(\mathscr{H})$ by π, their theorem states that if N and M are in $\mathscr{B}(\mathscr{H})$ and are *essentially normal*, that is, if $\pi(N)$ and $\pi(M)$ are normal in $\mathscr{A}(\mathscr{H})$, then $\pi(N)$ and $\pi(M)$ are unitarily equivalent if and only if

(i) $\sigma(\pi(N)) = \sigma(\pi(M))$ in $\mathscr{A}(\mathscr{H})$; and

(ii) $\operatorname{ind}(\lambda I - N) = \operatorname{ind}(\lambda I - M)$ for all $\lambda \in \rho_{\mathrm{sF}}(N)$, where $\operatorname{ind} T = \operatorname{nul} T - \operatorname{nul} T^*$ is the *Fredholm index* of an operator $T \in \mathscr{B}(\mathscr{H})$, and $\rho_{\mathrm{sF}}(T)$ is the *semi-Fredholm domain* of T. More precisely, $\rho_{\mathrm{sF}}(T) = \{\lambda \in \mathbb{C}\colon \operatorname{ran} T$ is closed and either $\operatorname{nul} T < \infty$ or $\operatorname{nul} T^* < \infty\}$, and its complement is denoted by $\sigma_{\mathrm{fre}}(T)$. We also let $\sigma_{\mathrm{e}}(T)$ denote $\sigma(\pi(T))$, the *essential spectrum* of T. (Note: here, $\operatorname{nul} T = \dim\ker T$.)

In this paper we propose to study the closure of an orbit which lies between the unitary and similarity orbits of an operator as defined above, and is related to the unitary orbit of elements of the Calkin algebra. In studying various classes of operators in the past (for example, biquasitriangular operators or the closure of the set of nilpotent operators), much profit has been gained by observing that these classes were invariant under the action of similarity transformations. From this, spectral invariants have been deduced which produced useful characterizations of these classes (cf. **[Voi 2]**, **[AFV]**).

It is our feeling that the $(\mathscr{U} + \mathscr{K})$-orbits defined below may play a similar role when studying classes of operators closed under unitary plus compact transformations but not under general similarity transformations. The motivating example here is the class of quasidiagonal operators and certain of its subclasses. These classes behave very badly under similarity transformations (cf. **[Her 3]**). In fact, this orbit first made its appearance in **[Her 2]** in relation to a question concerning quasidiagonal operators. We define this orbit as follows.

First, for a Hilbert space $\mathscr{H}$, let $(\mathscr{U} + \mathscr{K})(\mathscr{H}) = \{R \in \mathscr{B}(\mathscr{H})\colon R$ is invertible in $\mathscr{B}(\mathscr{H})$ and R is of the form unitary plus compact$\}$. Then, for $T \in \mathscr{B}(\mathscr{H})$, let

$$(\mathscr{U} + \mathscr{K})(T) = \{R^{-1}TR\colon R \in (\mathscr{U} + \mathscr{K})(\mathscr{H})\}.$$

We write $T \cong_{u+k} S$ if $S \in (\mathscr{U} + \mathscr{K}(T))$, and note that $\cong_{u+k}$ is an equivalence relation on $\mathscr{B}(\mathscr{H})$. Clearly $\mathscr{U}(T) \subseteq (\mathscr{U} + \mathscr{K})(T) \subseteq \mathscr{S}(T)$, and the same obviously holds for their closures. In general these orbits need not coincide, although in finite dimensions,

$\overline{(\mathscr{U}+\mathscr{K})(T)} = \overline{\mathscr{S}(T)}$ is clear, since all invertibles *are* of the form unitary plus compact.

In §3 we extend this to the case of compact operators, showing that $\overline{(\mathscr{U}+\mathscr{K})(T)} = \overline{\mathscr{S}(T)}$ for all $T \in \mathscr{K}(\mathscr{H})$. In this case, something even stronger is true, namely $\overline{\mathscr{S}(T)} = \overline{(I+\mathscr{K})(T)} = \{R^{-1}TR\colon R$ invertible in $\mathscr{B}(\mathscr{H})$, R of the form identity plus compact$\}$.

In §2 we shall describe the closure of the $(\mathscr{U}+\mathscr{K})$-orbits of normal operators in $\mathscr{B}(\mathscr{H})$. We shall show that with one other condition, the list of necessary conditions for membership in $\overline{(\mathscr{U}+\mathscr{K})(N)}$, $N \in \mathscr{B}(\mathscr{H})$ normal, as given in **[Her 2**, p. 481] is complete, and also constitutes a list of sufficient conditions.

We would like to thank the referee for several useful comments, and in particular, for pointing out a fatal flaw in our original proof of Theorem 2.13.

2. The normal case.

2.1. In restricting our attention to the case of normal operators, we can make a number of observations which simplify our task. For instance, if we begin with a normal operator N, then not only is $\overline{(\mathscr{U}+\mathscr{K})(N)}$ contained in the set of essentially normal operators, but in fact $T \in \overline{(\mathscr{U}+\mathscr{K})(N)}$ implies $T \in (\mathscr{N}+\mathscr{K})(\mathscr{H}) = \{A \in \mathscr{B}(\mathscr{H})\colon A$ of the form normal plus compact$\}$. (This follows from the fact that the latter is norm-closed, by **[BDF]**.) Moreover, $\pi(T)$ and $\pi(N)$ must be unitarily equivalent in the Calkin algebra, and so again by **[BDF]** we can conclude that there exists an operator $K \in \mathscr{K}(\mathscr{H})$ and a unitary $U \in \mathscr{B}(\mathscr{H})$ such that $T = U^*NU + K$. Our question then becomes: "How does one absorb the compact perturbation of U^*NU into a similarity transform $R^{-1}NR$ of N, where R itself is of the form unitary plus compact?" The answer lies in the following series of approximations to the main theorem.

2.2. NOTATION. Following **[Her 1]**, for $T \in \mathscr{B}(\mathscr{H})$ and Δ a clopen (i.e. closed and open) subset of $\sigma(T)$, we denote by $E(\Delta; T)$ the corresponding Riesz idempotent and the range of $E(\Delta; T)$ is denoted by $\mathscr{H}(\Delta; T)$. If $\Delta = \{\lambda\}$ is a singleton and $\dim \mathscr{H}(\Delta; T)$ is finite, then λ is called a *normal eigenvalue* of T. The set of all normal eigenvalues of T is denoted by $\sigma_0(T)$.

2.3. THEOREM. *Let N be a normal operator whose spectrum $\sigma(N)$ is a perfect subset of $\mathbb{C}$. Let $\mathscr{T} = \{T \in \mathscr{B}(\mathscr{H})\colon T \in \overline{(\mathscr{N}+\mathscr{K})(\mathscr{H})}$, $\sigma_e(T) = \sigma_e(N) = \sigma(N)$ and $\sigma_0(T) = \varnothing\}$. Then $\overline{(\mathscr{U}+\mathscr{K})(N)} = \mathscr{T}$.*

Proof. First we show that $\mathscr{T} \subseteq \overline{(\mathscr{U}+\mathscr{K})(N)}$. From the Weyl-von Neumann-Berg-Sikonia Theorem **[Brg]**, **[Sik]**, we may assume that N is a diagonal operator with respect to an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ and that each eigenvalue of N is repeated with infinite multiplicity. To see why we may do so, first observe that $(\mathscr{U}+\mathscr{K})$-orbits are closed under the following transitivity relation: namely, that if $B \in \overline{(\mathscr{U}+\mathscr{K})(A)}$ and $C \in \overline{(\mathscr{U}+\mathscr{K})(B)}$ for operators A, B and C, then $C \in \overline{(\mathscr{U}+\mathscr{K})(A)}$ (the proof is not difficult). But then given any normal N' with $\sigma(N') = \sigma_e(N')$, the above theorem implies the existence of a diagonal operator N as above with $\sigma(N) = \sigma_e(N) = \sigma(N')$ and $\overline{\mathscr{U}(N)} = \overline{\mathscr{U}(N')}$. But then $\overline{(\mathscr{U}+\mathscr{K})(N)} = \overline{(\mathscr{U}+\mathscr{K})(N')}$.

Suppose $T \in \mathscr{T}$ and let $\varepsilon > 0$. From **[Her 1**, Thm. 3.48], we conclude that there exists $K \in \mathscr{K}(\mathscr{H})$ with $\|K\| < \varepsilon$ such that $\sigma(T+K) = \sigma_e(T+K) = \sigma_e(N) = \sigma(N)$. As such, it follows from **[BDF]** that $T + K = U^*NU + L_1$ for some unitary U and some $L_1 \in \mathscr{K}(\mathscr{H})$. In other words, $T+K \in \mathscr{U}(N+L)$ where $L = UL_1U^* \in \mathscr{K}(\mathscr{H})$. We need only show, therefore, that $N + L \in \overline{(\mathscr{U}+\mathscr{K})(N)}$.

Let P_n be the orthogonal projection onto $\operatorname{span}\{e_i\}_{i=1}^{n}$. Then $\{F_n = P_nLP_n\}_{n=1}^{\infty}$ is a sequence of finite rank operators satisfying $\lim_{n\to\infty} \|L - F_n\| = 0$. Moreover, it is not hard to see that $N + F_n \cong N \oplus G_n$ ($\cong$ denotes unitary equivalence), where $G_n = P_nNP_n + F_n$. Let V_n be the unitary such that $N+F_n = V_n^*(N \oplus G_n)V_n$. By dropping down to a suitable subsequence, we may assume that $\sigma(N \oplus G_n) = \sigma(N + F_n) \subseteq (\sigma(N))_{1/n} = \{\lambda \in \mathbb{C}\colon \operatorname{dist}(\lambda, \sigma(N)) < \frac{1}{n}\}$. This simply uses the upper semicontinuity of the spectrum with respect to $\sigma(N+L) = \sigma(N)$ and $\sigma(N+F_n)$, $n \geq 1$.

Thus we can perturb G_n by at most $\frac{1}{n}$ to obtain a new matrix G_n' satisfying $\sigma(G_n') \subseteq \sigma(N)$ (i.e. consider G_n in upper triangular form and simply shift the eigenvalues over). Then $\|G_n' - G_n\| < \frac{1}{n}$ implies $\|(N + F_n) - V_n^*(N \oplus G_n')V_n\| < \frac{1}{n}$ and so $V_n^*(N \oplus G_n')V_n$ also converges to $N + L$.

We can now use a technique similar to that found in **[Her 2]** to show that $N \oplus G_n' \in \overline{(\mathscr{U}+\mathscr{K})(N)}$, which will clearly be sufficient. We may assume that G_n' is upper triangular, say

$$G_n' = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & g_{ij} & \\ & & \ddots & \\ & 0 & & \\ & & & \lambda_m \end{bmatrix}.$$

Consider $D_0 = \operatorname{diag}\{\lambda_i\}_{i=1}^m$, where $\sigma(G'_n) = \{\lambda_i\}_{i=1}^m$ including multiplicity. Let $\delta > 0$. Using the fact that $\sigma(N)$ is perfect, at a cost of some $f(\delta)$ (with $\lim_{\delta\to 0} f(\delta) = 0$), we can perturb each λ_i to a $\lambda'_i \in \sigma(N)$ so that $\min_{i\neq j} |\lambda'_i - \lambda'_j| \geq \delta$. Let $D_n = \operatorname{diag}\{\lambda'_i\}_{i=1}^m$. It now follows from elementary linear algebra that there exists an invertible matrix S_δ so that

$$D_n = S_\delta^{-1} \begin{bmatrix} \lambda'_1 & & & \\ & \lambda'_2 & & g_{ij} \\ & & \ddots & \\ & 0 & & \\ & & & \lambda'_m \end{bmatrix} S_\delta .$$

Thus $\|G'_n - S_\delta D_n S_\delta^{-1}\| = \|\operatorname{diag}\{\lambda_i - \lambda'_i\}_{i=1}^m\| < f(\delta)$. Now $N \cong N \oplus D_n$, and so

$$\|(N\oplus G'_n) - (I\oplus S_\delta)(N\oplus D_n)(I\oplus S_\delta^{-1})\| = \|0\oplus(G'_n - S_\delta D_n S_\delta^{-1})\| < f(\delta).$$

Letting δ tend to 0 we obtain $N \oplus G'_n \in \overline{(\mathscr{U}+\mathscr{K})(N)}$, for all $n \geq 1$. But then $\lim_{n\to\infty} N \oplus G'_n = N + L \in \overline{(\mathscr{U}+\mathscr{K})(N)}$. From before, we conclude $T + K \in \overline{(\mathscr{U}+\mathscr{K})(N)}$. Finally, since $\varepsilon > 0$ was arbitrary, $T \in \overline{(\mathscr{U}+\mathscr{K})(N)}$.

To show that $\overline{(\mathscr{U}+\mathscr{K})(N)} \subseteq \mathscr{T}$, let $T \in \overline{(\mathscr{U}+\mathscr{K})(N)}$. From above, $T \in (\mathscr{N}+\mathscr{K})(\mathscr{H})$ and $\sigma_e(T) = \sigma_e(N)$. That $\sigma_0(T) = \varnothing$ is an immediate consequence of the upper semicontinuity of the spectrum, using the fact that $\sigma(N)$ is perfect. Thus $T \in \mathscr{T}$, completing the proof. □

2.4. In what follows we shall be looking at upper triangular operator matrices whose strictly upper triangular parts are compact. One of the main tools we shall use is Rosenblum's Theorem, which we now state.

ROSENBLUM'S THEOREM [*cf.* **Her 1**, *Cor.* 320]. *Let* $A, B \in \mathscr{B}(\mathscr{H})$ *and consider* $\tau_{AB}\colon \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{H})$, $X \mapsto AX - XB$. *Then*

(i) $\sigma(\tau_{AB}) = \sigma(A) - \sigma(B)$;

(ii) *if* $\sigma(A)\cap\sigma(B) = \varnothing$ *then there exists a Cauchy domain* Ω *such that* $\sigma(A) \subseteq \Omega$, $\sigma(B)\cap\overline{\Omega} = \varnothing$, *and for* $Z \in \mathscr{B}(\mathscr{H})$, $\tau_{AB}^{-1}(Z) = -\frac{1}{2\pi i}\int_{\partial\Omega}(\lambda I - A)^{-1} Z (\lambda I - B)^{-1}\, d\lambda$.

For our purpose, we shall also need the following observation:

It is not hard to see that we can actually choose Ω to be an *analytic* Cauchy domain. Now suppose Z is a compact operator. Then it becomes evident from the definition of the integral on the right of the above equation as a limit of "Riemann sums" that $\tau_{AB}^{-1}(Z)$ must also be compact, as each approximating sum is. Using this observation, we obtain the following version of the Rosenblum-Davis-Rosenthal Corollary:

2.5. COROLLARY. *Let $\mathscr{H}_A$ and $\mathscr{H}_B$ be two complex, separable Hilbert spaces and let $A \in \mathscr{B}(\mathscr{H}_A)$, $B \in \mathscr{B}(\mathscr{H}_B)$ and $Z \in \mathscr{B}(\mathscr{H}_B, \mathscr{H}_A)$, Z compact. Assume that $\sigma(A) \cap \sigma(B) = \varnothing$. Then there exists $R \in (\mathscr{U} + \mathscr{K})(\mathscr{H}_A \oplus \mathscr{H}_B)$ such that*

$$A \oplus B = R^{-1} \begin{bmatrix} A & Z \\ 0 & B \end{bmatrix} R.$$

Proof. By the preceding remarks, if $\dim \mathscr{H}_A = \dim \mathscr{H}_B = \infty$, then we can choose $\tau_{AB}^{-1}(-Z) = X$ compact. Note that $R = \left[\begin{smallmatrix} I & X \\ 0 & I \end{smallmatrix}\right]$ then does the job. The case where $\dim \mathscr{H}_A < \infty$ or $\dim \mathscr{H}_B < \infty$ is handled precisely as in [**Her 1**, Cor. 3.22], only noting the fact that in this case the similarities are already of the form $U + K$ with U unitary and K compact. □

2.6. COROLLARY. *Suppose*

$$T = \begin{bmatrix} A_1 & & & \\ & A_2 & Z_{ij} & \\ & & \ddots & \\ & 0 & & \\ & & & A_n \end{bmatrix}$$

is an operator matrix acting in the usual way on the direct sum $\bigoplus_{i=1}^n \mathscr{H}_i$ of Hilbert spaces $\mathscr{H}_i$, $1 \leq i \leq n$. Suppose also that each Z_{ij}, $1 \leq i < j \leq n$, is a compact operator and that $\sigma(A_i) \cap \sigma(A_j) = \varnothing$, $1 \leq i \neq j \leq n$. Then $T \cong_{u+k} \bigoplus_{i=1}^n A_i$.

Proof. Induction. □

2.7. Along similar lines, we can also obtain some information for the case when the spectra of the diagonal elements of the operator matrix T are not disjoint. We note that Al-Musallam has independently obtained this result in his thesis [**Al-M**], and that the proof there is similar to the one below. We include it for completeness.

PROPOSITION. *Let $\mathscr{H}_A$ and $\mathscr{H}_B$ be Hilbert spaces and let $T = \left[\begin{smallmatrix} A & Z \\ 0 & B \end{smallmatrix}\right]$ with respect to $\mathscr{H}_A \oplus \mathscr{H}_B$. Suppose Z is compact. Then $A \oplus B \in \overline{(\mathscr{U} + \mathscr{K})(T)}$.*

Proof. Without loss of generality, we may assume $\|T\| \leq 1$.

If $\mathscr{H}_A$ is finite dimensional, then let $\varepsilon > 0$ and set $R_\varepsilon = \varepsilon I_A \oplus I_B$ where I_A (resp. I_B) is the identity operator in $\mathscr{B}(\mathscr{H}_A)$ (resp. $\mathscr{B}(\mathscr{H}_B)$). Then $R_\varepsilon \in (\mathscr{U}+\mathscr{K})(\mathscr{H}_A \oplus \mathscr{H}_B)$ and $R_\varepsilon T R_\varepsilon^{-1} = \left[\begin{smallmatrix} A & \varepsilon Z \\ 0 & B \end{smallmatrix}\right]$. Thus by letting ε tend to 0 we get $A \oplus B \in \overline{(\mathscr{U}+\mathscr{K})(T)}$.

If $\mathscr{H}_A$ is infinite dimensional, then as in **[BD]** we can obtain a tridiagonal representation of A with respect to a decomposition $\mathscr{H}_A = \bigoplus_{n=1}^{\infty} \mathscr{H}_n$ where $\dim \mathscr{H}_n < \infty$ for all $n \geq 1$, and moreover we are free to choose $\mathscr{H}_1$ arbitrarily. Let $0 < \varepsilon < 1/\sqrt{2}$ and choose $\mathscr{H}_1$ large enough so that $\|Z - P(\mathscr{H}_1)Z\| < \varepsilon^3$, where $P(\mathscr{H}_1)$ is the orthogonal projection of $\mathscr{H}_A$ onto $\mathscr{H}_1$. Then we may write

$$T = \begin{bmatrix} \ddots & \ddots & & & \vdots \\ & A_{44} & A_{43} & & Z_4 \\ \ddots & A_{34} & A_{33} & A_{32} & Z_3 \\ & & A_{23} & A_{22} \quad A_{21} & Z_2 \\ & & & A_{12} \quad A_{11} & Z_1 \\ \dots & 0 & 0 & 0 & B \end{bmatrix}$$

and note that

$$\left\| \begin{bmatrix} \vdots \\ Z_4 \\ Z_3 \\ Z_2 \end{bmatrix} \right\| < \varepsilon^3.$$

Now $\|T\| \leq 1$ implies $\|A\| \leq 1$ and hence $\|A_{ij}\| \leq 1$ for all $i, j \geq 1$. We claim that we can find a finite sequence $\frac{\varepsilon}{2} = \delta_1 < \delta_2 < \cdots < \delta_m = 1$, where m is a positive integer and $m \leq 4/\varepsilon^2 + 1 < 5/\varepsilon^2$ such that

(1) $\|A_{i\ i+1} - (\delta_i I_i) A_{i\ i+1} (\delta_{i+1} I_{i+1})^{-1}\| < \varepsilon$; and

(2) $\|A_{i+1\ i} - (\delta_{i+1} I_{i+1}) A_{i+1\ i} (\delta_i I_i)^{-1}\| < \varepsilon$,

and where I_i is the identity operator in $\mathscr{B}(\mathscr{H}_i)$, $1 \leq i$.

To see this, consider the following: Let $\delta_1' = \frac{\varepsilon}{2}$. For $1 \leq i \leq 5/\varepsilon^2$, let $\delta_{i+1}' = \delta_i'(1 + \frac{\varepsilon}{2})$, so that $\delta_{i+1}' - \delta_i' = \delta_i'(\frac{\varepsilon}{2}) \geq \delta_1'(\frac{\varepsilon}{2}) \geq \varepsilon^2/4$. Thus there exists $m \leq 4/\varepsilon^2 + 1$ such that $\delta_{m-1}' < 1$ and $\delta_m' \geq 1$. Let $\delta_i = \delta_i'$ for $1 \leq i \leq m-1$ and $\delta_i = 1$ for $i \geq m$. Now note that

(i) $1 - \delta_i \delta_{i+1}^{-1} = \frac{\varepsilon}{2}(1 + \frac{\varepsilon}{2}) < \frac{\varepsilon}{2}$; and

(ii) $\delta_{i+1}\delta_i^{-1} - 1 = (1 + \frac{\varepsilon}{2}) - 1 \leq \frac{\varepsilon}{2}$,

for $1 \leq i \leq m-2$.

Also, $1 - \delta_{m-1}\delta_m^{-1} = 1 - \delta_{m-1}$. Now $\delta_{m-1} < 1$ but $\delta_{m-1}(1 + \frac{\varepsilon}{2}) \geq 1$ since $\delta_m' \geq 1$. Thus $\delta_{m-1} \geq 1 - \frac{\varepsilon}{2}\delta_{m-1} \geq 1 - \frac{\varepsilon}{2}$, implying that

$1-\delta_{m-1}\delta_m^{-1} \le \frac{\varepsilon}{2}$. Similarly, $\delta_m\delta_{m-1}^{-1} - 1 = \delta_{m-1}^{-1} - 1 \le (1-\frac{\varepsilon}{2})^{-1} - 1 = \varepsilon/2(1-\frac{\varepsilon}{2}) < \varepsilon$ as $\varepsilon < 1$. Finally note that

$$(1) \quad \|A_{i\ i+1} - (\delta_i I_i)A_{i\ i+1}(\delta_{i+1}I_{i+1})^{-1}\| = \|(1-\delta_i\delta_{i+1}^{-1})I_iA_{i\ i+1}\| \le |1-\delta_i\delta_{i+1}^{-1}|\,\|A_{i\ i+1}\| < \varepsilon\cdot 1 = \varepsilon; \text{ and}$$

$$(2) \quad \|A_{i+1\ i} - (\delta_{i+1}I_{i+1})A_{i+1\ i}(\delta_i I_i)^{-1}\| = \|(\delta_{i+1}\delta_i^{-1} - 1)I_{i+1}A_{i+1\ i}\| \le |\delta_{i+1}\delta_i^{-1} - 1|\,\|A_{i+1\ i}\| < \varepsilon\cdot 1 = \varepsilon$$

for $1 \le i \le m$.

The relevance of the above computations becomes clear when we define an operator $R \in \mathscr{B}(\mathscr{H}_A \oplus \mathscr{H}_B) = \mathscr{B}((\bigoplus_{n=1}^{\infty}\mathscr{H}_n)\oplus\mathscr{H}_B)$ by $R = (\bigoplus_{n=1}^{\infty}\delta_n I_n)\oplus I$. Since each H_n is finite dimensional and $\delta_n = 1$ except for $1 \le n < m$, $R \in (\mathscr{U}+\mathscr{K})(\mathscr{H}_A\oplus\mathscr{H}_B)$. Now consider RTR^{-1}: (in the matrix notation we shall abbreviate $\delta_n I_n$ to δ_n).

$$RTR^{-1} = \begin{bmatrix} \ddots & \ddots & & & & & \\ & A_{m+1\ m+1} & A_{m+1\ m} & & & & \\ \ddots & A_{m\ m+1} & A_{mm} & & & & \vdots \\ & & \ddots & \ddots & & & \delta_4 Z_4 \\ & & \ddots & A_{33} & \delta_3 A_{32}\delta_2^{-1} & & \delta_3 Z_3 \\ & & & \delta_2 A_{23}\delta_3^{-1} & A_{22} & \delta_2 A_{21}\delta_1^{-1} & \delta_2 Z_2 \\ & & & & \delta_1 A_{12}\delta_2^{-1} & A_{11} & \delta_1 Z_1 \\ & & & 0 & 0 & 0 & B \end{bmatrix}.$$

As such,

$$\begin{aligned} \|A\oplus B - RTR^{-1}\| &\le 2\max_{1\le i\le m}\{\|A_{i\ i+1} - (\delta_i I_i)A_{i\ i+1}(\delta_{i+1}I_{i+1})^{-1}\|, \\ &\qquad \|A_{i+1\ i} - (\delta_{i+1}I_{i+1})A_{i+1\ i}(\delta_i I_i)^{-1}\|\} \\ &\quad + \delta_1\|Z_1\| + \left\| \begin{bmatrix} \delta_m Z_m \\ \vdots \\ \delta_3 Z_3 \\ \delta_2 Z_2 \end{bmatrix} \right\| + \left\| \begin{bmatrix} \vdots \\ Z_{m+2} \\ Z_{m+1} \end{bmatrix} \right\| \\ &\le 2\varepsilon + \frac{\varepsilon}{2}(1) + \left(\sum_{n=2}^{m}\delta_n\|Z_m\|\right) + \varepsilon^3 \\ &\le \frac{5}{2}\varepsilon + m\cdot 1\cdot\varepsilon^3 + \varepsilon^3 \le \frac{5}{2}\varepsilon + \left(\frac{5}{\varepsilon^2}\right)\varepsilon^3 + \varepsilon^3 \\ &\le \frac{15}{2}\varepsilon + \varepsilon^3 < 8\varepsilon \text{ as } \varepsilon^2 < \frac{1}{2}. \end{aligned}$$

Letting ε tend to 0 yields the desired conclusion. □

2.8. REMARK. As was the case in the previous Corollary 2.6, we can extend this result to an $n \times n$ operator matrix with compact strictly upper triangular part by induction.

For $A \in \mathscr{B}(\mathscr{H})$, we denote by $\sigma_{\text{iso}}(A)$ the isolated points of $\sigma(A)$. Then $\sigma_{\text{iso}}(A)$ is a countable set which contains $\sigma_0(A)$ and which has no accumulation points outside of $\sigma_{\text{e}}(A)$. Note that $\sigma_{\text{iso}}(A)$ need not be closed in $\mathbb{C}$. We also define $\sigma_{\text{acc}}(A)$ to be the set of accumulation points of $\sigma(A)$. Then $\sigma_{\text{acc}}(A)$ is closed. If A is a semi-Fredholm operator (i.e. if $0 \in \rho_{\text{sF}}(A)$), then we define the minimal index of A, denoted min. ind.(A) to be the minimum of $\operatorname{nul} A$ and $\operatorname{nul} A^*$.

By $\sigma_{\text{p}}(A)$ we shall denote the point spectrum (i.e., eigenvalues) of A and for $\Delta \subseteq \mathbb{C}$, $\Delta^* = \{\overline{\lambda} : \lambda \in \Delta\}$. Following Apostol, we may define the *regular* points of $\rho_{\text{sF}}(A)$ as

$$\rho^r_{\text{sF}} = \{\lambda \in \rho_{\text{sF}}(A) : \operatorname{nul}(A-\mu) \text{ and } \operatorname{nul}(A-\mu)^* \text{ are continuous on some neighbourhood of } \lambda\}$$

as well as the *singular* points of $\rho_{\text{sF}}(A)$ as

$$\rho^{\text{s}}_{\text{sF}} = \rho_{\text{sF}}(A) \backslash \rho^{\text{r}}_{\text{sF}}(A).$$

The set $\rho^{\text{s}}_{\text{sF}}(A)$ consists of a countable sequence with no accumulation points in $\rho_{\text{sF}}(A)$. The reader is referred to **[Her 1]** for more information regarding these parts of the semi-Fredholm domain.

2.9. THEOREM. *Let $\mathscr{H}$ be a Hilbert space and $N \in \mathscr{B}(\mathscr{H})$ be normal. Suppose $T \in \overline{(\mathscr{U}+\mathscr{K})(N)}$. Then*

(i) $T \in (\mathscr{N}+\mathscr{K})(\mathscr{H})$;
(ii) $\sigma_{\text{e}}(T) = \sigma_{\text{e}}(N)$, $\sigma(N) \subseteq \sigma(T)$;
(iii) $\operatorname{nul}(T-\lambda) \geq \operatorname{nul}(N-\lambda)$ *for all* $\lambda \in \rho_{\text{sF}}(T) = \rho_{\text{F}}(T)$;
(iv) *if* $\{\lambda\} \in \sigma_{\text{iso}}(T) \cap \sigma_{\text{e}}(T)$, *then* $(\lambda - T)|\mathscr{H}(\lambda; T) = 0$;
(v) *if* $\{\lambda\} \in \sigma_0(T)$, *then* $\operatorname{rank} E(\lambda; T) = \operatorname{rank} E(\lambda; N)$.

REMARK. One can also combine conditions (iv) and (v) above to obtain the equivalent condition
(iv)′ if $\lambda \in \sigma_{\text{iso}}(T)$, then $(\lambda - T)|\mathscr{H}(\lambda; T) = 0$, and $\dim \mathscr{H}(\lambda; T) = \dim \mathscr{H}(\lambda; N)$.

Proof. The necessity of conditions (i) and (ii) is easily verified. As for conditions (iii), (iv) and (v), we turn to the Similarity Theorem of **[AFHV]**.

Let $R \in \mathscr{B}(\mathscr{H})$ and let $\rho: \mathscr{A}(\mathscr{H}) \to \mathscr{B}(\mathscr{H}_\rho)$ be a faithful unital $*$-representation. If μ is an isolated point of $\sigma_e(R)$, then the Riesz Decomposition Theorem implies that $\rho(\pi(R))$ is similar to $\mu I + Q_\mu \oplus W_\mu$, where Q_μ is quasinilpotent and $\mu \notin \sigma(W_\mu)$. In [**AFHV**, p. 3] is defined the function $k(\lambda, \pi(R))$ with domain $\mathbb{C}$ as follows:

$$k(\lambda; \pi(R)) = \begin{cases} 0 & \text{if } \lambda \notin \sigma_e(R), \\ n & \text{if } \lambda \in \sigma_{\mathrm{iso}}(R) \cap \sigma_e(R) \text{ and } Q_\lambda \\ & \text{is a nilpotent of order } n, \\ \infty & \text{otherwise.} \end{cases}$$

Also defined is $\sigma_{\mathrm{ne}}(R) = \{\lambda \in \mathbb{C}: 1 \le k(\lambda; \pi(R)) < \infty\}$.

The Similarity Theorem [**AFHV**, Thm. 9.2] shows that for $X \in \mathscr{B}(\mathscr{H})$ to be in $\overline{\mathscr{S}(R)}$, it is necessary (though not sufficient) that

(a) $\text{min. ind.}(X - \lambda)^k \ge \text{min. ind.}(R - \lambda)^k$ for all $k \ge 1$ and for all $\lambda \in \rho_{\mathrm{sF}}(R)$;

(b) if $\lambda \in \sigma_0(X)$, then $\operatorname{rank} E(\lambda; X) = \operatorname{rank} E(\lambda; R)$;

(c) if $\lambda \in \sigma_{\mathrm{ne}}(X) \cap \sigma_{\mathrm{iso}}(X)$, then

$$\operatorname{rank}[(\lambda - X)^k | \mathscr{H}(\lambda; X)] \le \operatorname{rank}[(\lambda - R)^k | \mathscr{H}(\lambda; R)]$$

for all $k \ge k(\lambda; \pi(X))$.

Now condition (b) is exactly condition (v). Meanwhile, since in our case $\pi(T) \cong \pi(N)$, $k(\lambda; \pi(T)) = k(\lambda; \pi(N))$ for all $\lambda \in \mathbb{C}$ and so it is easily seen that $k(\lambda; \pi(T)) = 1$ for all $\lambda \in \sigma_{\mathrm{iso}}(T) \cap \sigma_e(T)$. But then

$$\begin{aligned} \operatorname{rank}[(\lambda - T)^1 | \mathscr{H}(\lambda; T)] &\le \operatorname{rank}[(\lambda - N)^1 | \mathscr{H}(\lambda; N)] \\ &= 0 \quad \text{for } \lambda \in \sigma_{\mathrm{iso}}(T) \cap \sigma_e(T). \end{aligned}$$

Thus $(\lambda - T)|\mathscr{H}(\lambda; T) = 0$, which is condition (iv).

It remains only to show that condition (iii) is necessary. The fact that $T \in (\mathscr{N} + \mathscr{K})(\mathscr{H})$ implies that $\operatorname{ind}(T - \lambda)^k = 0$ for all $\lambda \in \rho_{\mathrm{sF}}(T) = \rho_{\mathrm{F}}(T) = \rho_{\mathrm{F}}(N) = \rho_{\mathrm{sF}}(N)$, and for all $k \ge 1$. The same holds true for N. Thus condition (a) above implies (in our case) that

$$\operatorname{nul}(T - \lambda)^k \ge \operatorname{nul}(N - \lambda)^k \quad \text{for all } k \ge 1 \text{ and for all } \lambda \in \rho_{\mathrm{F}}(T).$$

But $\operatorname{nul}(N - \lambda)^k = \operatorname{nul}(N - \lambda)$ for all $k \ge 1$ and $\lambda \in \rho_{\mathrm{F}}(T)$, and so we may in fact conclude that

$$\operatorname{nul}(T-\lambda) \geq \operatorname{nul}(N-\lambda) \quad \text{for all } \lambda \in \rho_{\mathrm{F}}(T),$$

which is condition (iii). This completes the proof. □

2.10. The main theorem—Theorem 2.14—below asserts that indeed, the five conditions above are also sufficient for membership in $\overline{(\mathscr{U}+\mathscr{K})(N)}$ and hence characterize this set. The difference between this theorem and Theorem 2.3 is, of course, that we now allow normal operators N which have isolated eigenvalues. As it turns out, those eigenvalues which are also isolated eigenvalues of T can be (relatively) easily handled using conditions (iii), (iv) and (v) above in combination with Corollary 2.6. The trouble begins when $\lambda \in \sigma_{\mathrm{iso}}(N)$ but $\lambda \notin \sigma_{\mathrm{iso}}(T)$, and this is perhaps best illustrated by the following example.

2.11. EXAMPLE. Let S denote the forward unilateral shift and B denote the bilateral shift. Let O_1 denote the O operator acting upon a 1-dimensional Hilbert space and Q denote an arbitrary—but fixed—compact, quasinilpotent operator acting on an infinite dimensional separable Hilbert space.

Consider the normal operator $N = B \oplus (O_1)^{(\infty)}$, and the operator $T = B \oplus Q$. It follows immediately from condition (iv) of Theorem 2.9 that $T \notin \overline{(\mathscr{U}+\mathscr{K})(N)}$. If we let $R = S \oplus S^* \oplus Q$, then $\sigma(R) = \{\lambda \in \mathbb{C}\colon |\lambda| \leq 1\}$. In other words, we have "filled in the hole" of $\sigma(N)$. It is not difficult to see that a simple application of the upper semicontinuity of the spectrum allows us to approximate (to within arbitrary $\varepsilon > 0$) Q by a finite rank nilpotent $F = F(\varepsilon)$ so that $F = F' \oplus O_1^{(\infty)}$, F' acting on a finite dimensional space, $\sigma(F') = \{0\}$. By Theorem 2.3, $S \oplus S^* \oplus F' \in \overline{(\mathscr{U}+\mathscr{K})(B)}$, while obviously $(O_1)^{(\infty)} \in \overline{(\mathscr{U}+\mathscr{K})((O_1)^{(\infty)})}$, and so $R_\varepsilon = S \oplus S^* \oplus F' \oplus (O_1)^{(\infty)} \in \overline{(\mathscr{U}+\mathscr{K})(N)}$. Since $\|R - R_\varepsilon\| < \varepsilon$ and $\varepsilon > 0$ is arbitrary, $R \in \overline{(\mathscr{U}+\mathscr{K})(N)}$.

We have used two main ideas here. First, we were able to "break up" Q into a finite dimensional piece whose spectrum lay in the hole of $\sigma(N)$, and an infinite dimensional direct summand of R corresponding to the isolated point of $\sigma_{\mathrm{e}}(R)$ which is not isolated in $\sigma(R)$.

Secondly, we were able to "glue" that finite dimensional piece onto an essentially normal operator with *non-zero* minimum index inside the hole. While this may generate singular points in the semi-Fredholm domain of R, Theorem 2.3 was nonetheless capable of handling these.

It will hopefully prove useful to keep the example of R and N in mind when reading Theorem 2.14. The general case is also complicated by the presence of elements of $\sigma_0(N)$ which are not isolated in $\sigma(T)$.

2.12. A key step is allowing us to deal with the "holes" of $\sigma(N)$ which are "filled in" in $\sigma(T)$ is the Lemma 2.13 below. This lemma is an adaptation to suit our specific needs of Lemma 5.1 of **[HTW]**. Two of the main ingredients in the proof are Apostol's triangular representation and the decomposition of certain multiplication operators on L^2-spaces. **[Apo 2]** is a good reference for the former, while **[HTW**, §3**]** is a good reference for the latter. What follows is a (very) brief synopsis of the salient features involved.

In **[Apo 2]**, C. Apostol showed that every $T \in \mathscr{B}(\mathscr{H})$ admits the representation

$$T \cong \begin{bmatrix} T_r & * & * \\ & T_0 & * \\ & & T_l \end{bmatrix} \begin{matrix} \mathscr{H}_r \\ \mathscr{H}_0 \\ \mathscr{H}_l \end{matrix}$$

where

$$\mathscr{H}_r = \overline{\operatorname{span}}\{\ker(\lambda - T) \colon \lambda \in \rho_{\mathrm{sF}}^{\mathrm{r}}(T)\},$$

$$\mathscr{H}_l = \overline{\operatorname{span}}\{\ker(\lambda - T)^* \colon \lambda \in \rho_{\mathrm{sF}}^{\mathrm{r}}(T)\} \quad \text{and} \quad \mathscr{H}_0 = \mathscr{H} \ominus (\mathscr{H}_r \oplus \mathscr{H}_l).$$

Under this representation, T_r (resp. T_l^*) is a triangular operator and all the components of its spectrum intersect the interior of $\rho_{\mathrm{sF}}(T) \cap \sigma_{\mathrm{p}}(T)$ (resp. of $\rho_{\mathrm{sF}}(T) \cap \sigma_{\mathrm{p}}(T^*)^*$). Also, $\sigma_{\mathrm{p}}(T_r^*) = \sigma_{\mathrm{p}}(T_l) = \varnothing$, $\rho_{\mathrm{sF}}(T) \subseteq \rho_{\mathrm{sF}}(T_r) \cap \rho_{\mathrm{sF}}(T_l) \cap (\mathbb{C} \backslash \sigma_0(T_0))$ and $\rho_{\mathrm{sF}}^{\mathrm{s}}(T) \subseteq \sigma_0(T_0)$.

As for the multiplication operators, let Ω be a non-empty bounded open subset of $\mathbb{C}$ and let $\mathscr{R}(\overline{\Omega})$ denote the uniform closure of the rational functions with poles outside $\overline{\Omega}$. As mentioned in **[HTW]**, one can find an appropriate measure μ on $\partial(\Omega)$ so that the operator $M(\partial\Omega) \in \mathscr{B}(L^2(\partial\Omega, d\mu))$ of "multiplication by λ" admits the following decomposition:

$$M(\partial\Omega) = \begin{bmatrix} M_+(\partial\Omega) & Z(\partial\Omega) \\ 0 & M_-(\partial\Omega) \end{bmatrix} \begin{matrix} H^2(\partial\Omega) \\ L^2(\partial\Omega) \ominus H^2(\partial\Omega) \end{matrix},$$

where $H^2(\partial\Omega)$ is the L^2-closure of $\mathscr{R}(\overline{\Omega})$. This representation has a multitude of special features:

(i) $H^2(\partial\Omega)$ is a rationally cyclic invariant subspace of $L^2(\partial\Omega)$, with $e_0(\lambda)\equiv 1$ being a rationally cyclic vector for $M_+(\partial\Omega)$;

(ii) $M(\partial\Omega)$ is normal, $\sigma_p(M(\partial\Omega))=\varnothing$;

(iii) $M_+(\partial\Omega)$ and $M_-(\partial\Omega)^*$ are pure subnormal operators such that $\sigma(M_+(\partial\Omega))=\sigma(M_-(\partial\Omega))=\overline{\Omega}$, $\sigma_e(M_+(\partial\Omega))=\sigma_e(M_-(\partial\Omega))=\sigma_e(M(\partial\Omega))=\sigma(M(\partial\Omega))=\partial(\overline{\Omega})$;

(iv) $\overline{\Omega}\backslash\partial(\overline{\Omega})=\sigma_p(M_-(\partial\Omega))=\sigma_p(M_+(\partial\Omega)^*)^*$;

(v) $\operatorname{ind}(\lambda-M_-(\partial\Omega))=\operatorname{nul}(\lambda-M_-(\partial\Omega))=\operatorname{nul}(\lambda-M_+(\partial\Omega))^*=-\operatorname{ind}(\lambda-M_+(\partial\Omega))$ for all $\lambda\in\overline{\Omega}\backslash\partial(\overline{\Omega})$; and

(vi) $Z(\partial\Omega)$ is compact.

2.13. LEMMA. *Let $T\in\mathscr{B}(\mathscr{H})$ be an essentially normal operator, and let τ be a component of $\rho_{sF}(T)\cap\operatorname{interior}[\sigma_p(T)\cap\sigma_p(T^*)^*]$. Then, given $\varepsilon>0$, there exists $K_\varepsilon\in\mathscr{K}(\mathscr{H})$ with $\|K_\varepsilon\|<\varepsilon$ such that*

$$T-K_\varepsilon\cong\begin{bmatrix}N(\tau) & W\\ 0 & T_\varepsilon\end{bmatrix},$$

where

(i) *$N(\tau)$ is a compact perturbation of a normal operator;*

(ii) $\sigma(N(\tau))=\overline{\tau}$, $\sigma_e(N(\tau))=\partial(\overline{\tau})$;

(iii) $\sigma_p(N(\tau))=\sigma_p(N(\tau)^*)^*=\overline{\tau}\backslash\partial(\overline{\tau})$, $\operatorname{nul}(N(\tau)-\lambda)=\operatorname{nul}(N(\tau)-\lambda)^*=1$ *for all* $\lambda\in\overline{\tau}\backslash\partial(\overline{\tau})$;

(iv) $\sigma_e(T_\varepsilon)=\sigma_e(T)$, $\rho_{sF}(T_\varepsilon)=\rho_{sF}(T)$;

(v) $\operatorname{ind}(T_\varepsilon-\lambda)=\operatorname{ind}(T-\lambda)$ *for all* $\lambda\in\rho_{sF}(T)$;

(vi)

$$\operatorname{nul}(T_\varepsilon-\lambda)=\begin{cases}\operatorname{nul}(T-\lambda)-1 & \text{if } \lambda\in\tau,\\ \operatorname{nul}(T-\lambda) & \text{if } \lambda\in\rho_{sF}(T)\backslash\tau\end{cases}$$

(vii)

$$\operatorname{nul}(T_\varepsilon-\lambda)^*=\begin{cases}\operatorname{nul}(T-\lambda)^*-1 & \text{if } \lambda\in\tau,\\ \operatorname{nul}(T-\lambda)^* & \text{if } \lambda\in\rho_{sF}(T)\backslash\tau.\end{cases}$$

Proof. If we let $\Omega=\overline{\tau}\backslash\partial(\overline{\tau})$, then we can consider the normal operator $M=M(\partial\Omega)$ as above. Note that $\sigma(M)=\sigma_e(M)=\partial\Omega=\partial(\overline{\tau})$, and that $\overline{\Omega}=\overline{\tau}$. To save on notation, we shall write

$$M=\begin{bmatrix}M_+ & Z\\ 0 & M_-\end{bmatrix}\begin{matrix}H^2(\partial\Omega)\\ L^2(\partial\Omega)\ominus H^2(\partial\Omega).\end{matrix}$$

Now $\partial(\overline{\tau}) \subseteq \sigma_{\mathrm{e}}(T)$ and therefore, since T is essentially normal, we can find $K_0 \in \mathscr{K}(\mathscr{H})$, $\|K_0\| < \frac{\varepsilon}{2}$ such that $T - K_0 \cong T \oplus M \oplus M$ (**[Sal]**). By furthermore applying Apostol's triangular representation of T, we obtain

$$T - K_0 \cong \begin{bmatrix} M_+ & 0 & 0 & 0 & 0 & 0 & Z \\ & M_- & 0 & 0 & 0 & 0 & 0 \\ & Z & M_+ & 0 & 0 & 0 & 0 \\ & & & T_r & X_1 & X_2 & 0 \\ & & & & T_0 & X_3 & 0 \\ & & & & & T_l & 0 \\ & & & & & & M_- \end{bmatrix}.$$

The operator T_r is triangular. Let Δ be that component of $\sigma(T_r)$ which intersects τ non-trivially (note: $\tau \subseteq \sigma(T_r)$ and thus the component is unique). By the Riesz Decomposition Theorem,

$$T_r \cong \begin{bmatrix} T_r(\Delta) & * \\ 0 & T_r' \end{bmatrix} \quad \begin{matrix} \mathscr{H}(\Delta;\, T_r) \\ \mathscr{H}_r \ominus \mathscr{H}(\Delta;\, T_r) \end{matrix},$$

and $\mathscr{H}(\Delta;\, T_r) = \overline{\operatorname{span}}\{\ker(\lambda - T_r)^k;\ \lambda \in \Delta \cap \sigma_p(T_r),\ k \geq 1\}$. Thus $T_r(\Delta)$ is triangular (cf. **[Her 1**, p. 73**]**). Moreover, since $\sigma(T_r(\Delta)) \cap \sigma(T_r') = \varnothing$, $\operatorname{ind}(T_r - \lambda) = \operatorname{ind}(T_r(\Delta) - \lambda) > 0$ for all $\lambda \in \rho_{\mathrm{sF}}(T_r) \cap \Delta$, while $\operatorname{ind}(T_r(\Delta) - \lambda) = 0$ for all $\lambda \notin \Delta$, since $\sigma(T_r(\Delta)) \subseteq \Delta$. Thus we may write

$$T - K_0 \cong \begin{bmatrix} M_+ & & & & & & & Z \\ & M_- & & & & & & \\ & Z & M_+ & & & & & \\ & & & T_r(\Delta) & * & * & * & \\ & & & & T_r' & * & * & \\ & & & & & T_0 & * & \\ & & & & & & T_l & \\ & & & & & & & M_- \end{bmatrix}.$$

The operator $M_+ \oplus T_r(\Delta)$ is quasitriangular (that is,

$$\operatorname{ind}((M_+ \oplus T_r(\Delta)) - \lambda) \geq 0 \quad \text{for all } \lambda \in \rho_{\mathrm{sF}}(M_+ \oplus T_r(\Delta));$$

see **[Her 1**; Thm. 6.4**]**). Since the spectrum of this operator has no isolated points, we are now in a position to apply the results of **[Her 4]** to obtain a compact operator K_1, $\|K_1\| < \frac{\varepsilon}{6}$ such that $A_r = (M_+ \oplus T_r(\Delta)) - K_1$ is triangular with diagonal entries $\{\lambda_j\}_{j=1}^{\infty}$ with respect to some orthonormal basis $\{e_j\}_{j=1}^{\infty}$, and so that $\lambda_j \in \partial(\overline{\tau}) \subseteq$

$\sigma_{\text{lre}}(M_+ \oplus T_r(\Delta))$, for all $j \geq 1$ (cf. [**Her 4**, Cor. 2.4]). This forces $\sigma_{\text{p}}(A_r^*)^* \subseteq \sigma_{\text{lre}}(A_r)$; moreover,

$$\operatorname{ind}(A_r - \lambda) = \begin{cases} \operatorname{ind}(T_r(\Delta) - 1) & \text{if } \lambda \in \tau, \\ \operatorname{ind}(T_r(\Delta)) & \text{if } \lambda \in \rho_{\text{sF}}(T_r(\Delta)) \backslash \overline{\tau}. \end{cases}$$

Now let P_n be the orthogonal projection onto $\operatorname{span}\{e_j\}_{j=1}^n$. Since $Z' = \left[\begin{smallmatrix} Z \\ 0 \end{smallmatrix}\right]$ is compact, $\lim_{n\to\infty} \|Z' - P_n Z'\| = 0$. Thus there exists K_2 a compact operator, $\|K_2\| < \varepsilon/6$, such that

$$\begin{bmatrix} M_- & 0 \\ Z' & A_r \end{bmatrix} - K_2 = \begin{bmatrix} M_- & 0 & 0 \\ P_n Z' & F_{r,n} & * \\ 0 & 0 & A_{r,n} \end{bmatrix}$$

(for some sufficiently large n), where

$$F_{r,n} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & * & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{matrix}$$

($F_{r,n} = A_r | \operatorname{ran} P_n$), and

$$A_{r,n} = \begin{bmatrix} \lambda_{n+1} & & \\ & \lambda_{n+2} & * \\ & & \ddots \end{bmatrix} \begin{matrix} e_{n+1} \\ e_{n+2} \\ \vdots \end{matrix}$$

($A_{r,n} = A_r | \ker P_n$) has the same characteristics as A_r (cf. [**Her 4**] or [**HTW**, Lemma 5.1]).

Observe that the spectrum of

$$R_n = \begin{bmatrix} M_- & 0 \\ P_n Z' & F_{r,n} \end{bmatrix}$$

is equal to $\sigma(M_-) \cup \sigma(F_{r,n}) = \overline{\tau}$ and $\rho_{\text{sF}}(R_n) \cap \sigma(R_n) = \rho_{\text{sF}}^+(R_n) = \overline{\tau} \backslash \partial(\overline{\tau})$. Furthermore, R_n is essentially normal and $\operatorname{ind}(R_n - \lambda) = 1$ for all $\lambda \in \overline{\tau} \backslash \partial(\overline{\tau})$.

By Proposition 3.4 of [**HTW**], there exists K_3 compact, $\|K_3\| < \varepsilon/6$ such that $R = R_n - K_3$ is essentially normal, $\sigma_{\text{p}}(R) = \rho_{\text{sF}}^+(R_n) = \overline{\tau} \backslash \partial(\overline{\tau})$, $\sigma_{\text{p}}(R^*)^* = \rho_{\text{sF}}^-(R_n) = \varnothing$, $\sigma_{\text{e}}(R) = \partial(\overline{\tau})$ and $\operatorname{nul}(R - \lambda) = \operatorname{ind}(R - \lambda) = 1$ for all $\lambda \in \overline{\tau} \backslash \partial(\overline{\tau})$.

Moreover,

$$\begin{aligned}\operatorname{nul}\left(\begin{bmatrix} R & * \\ 0 & A_{r,n}\end{bmatrix} - \lambda\right) &= \operatorname{ind}\left(\begin{bmatrix} R & * \\ 0 & A_{r,n}\end{bmatrix} - \lambda\right) \\ &= \operatorname{ind}(R-\lambda) + \operatorname{ind}(A_{r,n}-\lambda) \\ &= \operatorname{nul}(R-\lambda) + \operatorname{nul}(A_{r,n}-\lambda) \quad \text{for all } \lambda \in \rho_{\mathrm{sF}}(T_r(\Delta)).\end{aligned}$$

Summing up, there exists K_0' compact, $\|K_0'\| \leq \|K_1\| + \|K_2\| + \|K_3\| < \varepsilon/2$ such that

$$\begin{bmatrix} M_- & 0 & 0 \\ Z & M_+ & 0 \\ 0 & 0 & T_r(\Delta)\end{bmatrix} - K_0' \cong \begin{bmatrix} R & * \\ 0 & B_r\end{bmatrix}$$

where $B_r = A_{r,n}$. Let $N(\tau) = M_+ \oplus R$. Clearly conditions (i), (ii) and (iii) are met.

In a similar vein, T_l^* is triangular. Letting Δ' be that component of $\sigma(T_l^*)^*$ which intersects τ non-trivially, we can again use the Riesz decomposition to write

$$T_l \cong \begin{bmatrix} T_l' & * \\ 0 & T_l(\Delta')\end{bmatrix}$$

where $\sigma(T_l(\Delta')) = \Delta'$ and $T_l(\Delta')^*$ is triangular. Also, $\operatorname{ind}(T_l(\Delta')-\lambda) = \operatorname{ind}(T_l-\lambda) < 0$ for all $\lambda \in \rho_{\mathrm{sF}}(T_l) \cap \Delta'$.

Since $\sigma(T_l(\Delta') \oplus M_-) = \sigma(T_l(\Delta')) \cup \overline{\tau}$ has no isolated points, and $\operatorname{ind}((T_l(\Delta') \oplus M_-)-\lambda) \leq 0$ for all $\lambda \in \rho_{\mathrm{sF}}(T_l \oplus M_-)$, by using the results of **[Her 4]** we can find a compact operator K_0'', $\|K_0''\| < \varepsilon/2$ such that $B_l = (T_l(\Delta') \oplus M_-) - K_0''$ is the adjoint of a triangular operator whose diagonal entries belong to $\delta(\overline{\tau}) \subseteq \sigma_{\mathrm{lre}}(T_l(\Delta') \oplus M_-)$. It follows that $\sigma_{\mathrm{p}}(B_l) \subseteq \sigma_{\mathrm{lre}}(T_l(\Delta') \oplus M_-) \subseteq \sigma_{\mathrm{lre}}(T)$, and

$$\operatorname{ind}(B_l - \lambda) = \begin{cases} \operatorname{ind}(T_l(\Delta') - \lambda) + 1 & \text{if } \lambda \in \tau, \\ \operatorname{ind}(T_l(\Delta')) & \text{if } \lambda \in \rho_{\mathrm{sF}}(T_l(\Delta'))\backslash\overline{\tau}.\end{cases}$$

We conclude that there exists a compact operator $K_\varepsilon = K_0 + (K_0' \oplus 0 \oplus K_0'')$, $\|K_\varepsilon\| < \varepsilon$ such that

$$T - K_\varepsilon \cong \begin{bmatrix} N(\tau) & * & 0 & 0 & 0 & * \\ & B_r & * & * & * & * \\ & & T_r' & * & * & * \\ & & & T_0 & * & * \\ & & & & T_l' & * \\ & & & & & B_l \end{bmatrix}.$$

Finally, let

$$T_\varepsilon = \begin{bmatrix} B_r & * & * & * & * \\ & T_r' & * & * & * \\ & & T_0 & * & * \\ & & & T_l' & * \\ & & & & B_l \end{bmatrix}.$$

As in **[HTW**, Lemma 5.1], with the help of **[Her 1**, Chapter 3], one can verify that T_ε has the desired characteristics. □

The idea behind this lemma is perhaps obscured by its technical details. One may think of it as follows: if an operator $T \in (\mathscr{N}+\mathscr{K})(\mathscr{H})$ has the same spectrum, essential spectrum, index and minimum index as our operator $R = S \oplus S^* \oplus Q$ of Example 2.11, then we can in fact find an invariant subspace for a small compact perturbation of T where this perturbation of T behaves like $S \oplus S^*$. The corresponding T_ε then behaves like $B \oplus Q$. Since $T \in (\mathscr{N}+\mathscr{K})(\mathscr{H})$, so is T_ε, and W is compact. Another small perturbation allows us to "pull out" the 0-direct summand and proceed as before.

2.14. THEOREM. *Let $\mathscr{H}$ be a Hilbert space and $N \in \mathscr{B}(\mathscr{H})$ be normal. Then*

$$\overline{(\mathscr{U}+\mathscr{K})(N)} = \{T \in \mathscr{B}(\mathscr{H}) \colon T \textit{ satisfies conditions } \text{(i), (ii), (iii), (iv)} \textit{ and } \text{(v)} \textit{ of Theorem } 2.9\}.$$

In particular, the only difference in the spectra of N and of T is that $\sigma(T)$ may have fewer holes, while the index of $(T-\lambda)$ must equal 0 for all $\lambda \in \rho_{\mathrm{sF}}(T) = \rho_{\mathrm{F}}(T)$ in these holes. Moreover, if $\{\lambda\} \in \sigma_{\mathrm{iso}}(T)$, then the compression of T to the corresponding eigenspace $\mathscr{H}(\lambda; T)$ is a scalar, and $\dim \mathscr{H}(\lambda; T) = \dim \mathscr{H}(\lambda; N)$.

Proof. The necessity of conditions (i), (ii), ..., (v) for membership in $\overline{(\mathscr{U}+\mathscr{K})(N)}$ is precisely Theorem 2.9. We content ourselves now with showing their sufficiency. First we may assume without loss of generality as in Theorem 2.3 that N is a diagonal operator with all eigenvalues in $\sigma_{\mathrm{e}}(N)$ repeated with infinite multiplicity (**[Brg]**, **[Sik]**).

Step One: *The isolated points of $\sigma(T)$*. Let $\{\lambda_i\}_{i=1}^{\mu}$ $(0 \le \mu \le \infty)$ denote the countable set $\sigma_{\mathrm{iso}}(T)$, in decreasing order of distance to $\sigma_{\mathrm{acc}}(T)$. (Note that if $\sigma_{\mathrm{iso}}(T)$ has infinite cardinality, then in fact $\{\lambda_i\}_{i=1}^{\mu}$ tend to $\partial(\sigma_{\mathrm{e}}(T))$.) Now if $\lambda \in \sigma_0(T)$, then by (iii) and (v), we find that $\dim \mathscr{H}(\lambda; T) = \mathrm{nul}(N-\lambda)$. Morever, by (v), $\mathscr{H}(\lambda; T) = \ker(T-\lambda)$. Similarly, by (iv), if $\{\lambda\} \subseteq \sigma_{\mathrm{iso}}(T) \cap \sigma_{\mathrm{e}}(T)$, then we find that $\{\lambda\} \subseteq \sigma_{\mathrm{e}}(N)$ and $\mathscr{H}(\lambda; T) = \ker(T-\lambda)$.

Because of the countability of $\sigma_{\mathrm{iso}}(T)$, we can choose $\varepsilon > 0$ arbitrarily small yet subject to the condition that $\partial((\sigma_{\mathrm{acc}}(T))_\varepsilon) \cap \sigma_{\mathrm{iso}}(T) = \varnothing$. Having chosen such an $\varepsilon > 0$, let $\Omega_\varepsilon = (\sigma_{\mathrm{acc}}(T))_\varepsilon$ and let $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be an enumeration of $\sigma(T)\backslash\Omega_\varepsilon$. If $\mu < \infty$, we choose $\varepsilon > 0$ small enough so that $\sigma_{\mathrm{iso}}(T) \cap \Omega_\varepsilon = \varnothing$. We then note that T admits the representation

$$T \cong \begin{bmatrix} T_1 & & & & T_{10} \\ & T_2 & & T_{ij} & T_{20} \\ & & \ddots & & \vdots \\ & & & T_n & T_{n0} \\ & & & & T_0 \end{bmatrix} \begin{matrix} \mathscr{H}_1 \\ \mathscr{H}_2 \\ \vdots \\ \mathscr{H}_n \\ \mathscr{H}_0 \end{matrix}$$

where $\mathscr{H}_1, \mathscr{H}_2, \dots, \mathscr{H}_n, \mathscr{H}_0$ are so defined that $\bigoplus_{j=1}^k \mathscr{H}_j$ coincides with $\bigoplus_{j=1}^k \mathscr{H}(\lambda_j; T)$ for $1 \le k \le n$ and $\bigoplus_{j=0}^n \mathscr{H}_j = \mathscr{H}$. Then $T_i \cong \lambda_i I_i$ for $1 \le i \le n$, where I_i is the identity operator acting on $\mathscr{H}_i$. Since $T \in (\mathscr{N} + \mathscr{K})(\mathscr{H})$, an easy computation now shows that each T_{ij}, $1 \le i \le n$, $0 \le j \le n$, above is compact. Moreover, by simple index considerations, we get back that not only is T_0 essentially normal, but in fact, $T_0 \in (\mathscr{N} + \mathscr{K})(\mathscr{H}_0)$. Note in particular that $\dim \mathscr{H}_i = \mathrm{nul}(N-\lambda_i)$, $1 \le i \le n$ and that $\lambda_i \notin \sigma(T_0)$, $1 \le i \le n$.

Now N is normal and $\{\lambda_i\}_{i=1}^n \subseteq \sigma_{\mathrm{iso}}(N)$. Also, $\mathrm{nul}(N-\lambda_i) = \dim \mathscr{H}_i$ allows us to write $N \cong (\bigoplus_{i=1}^n \lambda_i I_i) \oplus N_0$ with respect to the decomposition $\mathscr{H} = (\bigoplus_{i=1}^n \mathscr{H}_i) \oplus \mathscr{H}_0$, where $N_0 \cong N|(\bigoplus_{i=1}^n \mathscr{H}(\lambda_i; N))^\perp$. Note that N_0 is normal with $\sigma_{\mathrm{acc}}(N_0) = \sigma_{\mathrm{acc}}(N)$, $\sigma(N_0) = \sigma(N)\backslash\{\lambda_i\}_{i=1}^n = \sigma(N) \cap \Omega_\varepsilon$. Obviously the multiplicities are preserved.

Suppose, temporarily, that we can show that there exists $V_0 \in (\mathscr{U} + \mathscr{K})(\mathscr{H}_0)$ such that $\|V_0^{-1} N_0 V_0 - T_0\| < 7\varepsilon$. Then

$$N \cong \left(\bigoplus_{i=1}^n \lambda_i I_i\right) \oplus N_0 \cong_{u+k} N' = \begin{bmatrix} \lambda_1 I_1 & & & & T_{10} V_0^{-1} \\ & \lambda_2 I_2 & & T_{ij} & T_{20} V_0^{-1} \\ & & \ddots & & \vdots \\ & & & \lambda_n I_n & T_{n0} V_0^{-1} \\ & & & & N_0 \end{bmatrix}$$

by Corollary 2.6.

Thus

$$N \cong_{u+k} \left(\left(\bigoplus_{i=1}^{n} I_i\right) \oplus V_0^{-1}\right) N' \left(\left(\bigoplus_{i=1}^{n} I_i\right) \oplus V_0\right)$$

$$= N'' = \begin{bmatrix} \lambda_1 I_1 & & & & T_{10} \\ & \lambda_2 I_2 & & T_{ij} & T_{20} \\ & & \ddots & & \vdots \\ & & & \lambda_n I_n & T_{n0} \\ & & & & V_0^{-1} N_0 V_0 \end{bmatrix}.$$

But then $\|T - N''\| = \|T_0 - V_0^{-1} N_0 V_0\| < 7\varepsilon$. Since $\varepsilon > 0$ can be chosen arbitrarily small, we see that $T \in \overline{(\mathscr{U}+\mathscr{K})(N)}$. Thus it suffices to show that $\operatorname{dist}(T_0, (\mathscr{U}+\mathscr{K})(N_0)) < 7\varepsilon$. We therefore proceed in this direction.

Step Two: $\sigma_0(N)\backslash\sigma_{\text{iso}}(T)$. Essentially we have reduced the original problem to the case where $\lambda \in \sigma(T_0)$ implies $\operatorname{dist}(\lambda, \sigma_{\text{acc}}(T_0)) < \varepsilon$. The spectrum of T_0 looks like that of N_0, except that some of the holes (i.e., bounded components of $\rho(N_0)$) of $\sigma(N_0)$ may be filled in. In much the same way that we dealt with $\sigma_{\text{iso}}(T)$, we shall now deal with the points $\beta \in \sigma_0(N)$ which lie in a hole of $\sigma(N)$, but which are not isolated in $\sigma(T)$.

Let $\{\beta_i\}_{i=1}^{\nu}$ $(0 \leq \nu \leq \infty)$ denote the countable set $\sigma_0(N)\backslash\sigma_{\text{iso}}(T) = \sigma_0(N_0)\backslash\sigma_{\text{iso}}(T_0)$ in decreasing order of distance to $\sigma_{\text{acc}}(N_0) \subseteq \sigma_{\text{e}}(N_0)$. Let $b_i = \operatorname{nul}(N_0 - \beta_i)$ for $i \geq 1$. Then $b_i \leq \operatorname{nul}(T_0 - \beta_i)$ by condition (iii), for $i \geq 1$ (it is not hard to see that we may indeed use T_0 and N_0 here instead of T and N).

As before, because of the countability of $\sigma_0(N_0)\backslash\sigma_{\text{iso}}(T_0)$, we can choose $0 < \varepsilon_1 < \varepsilon$ such that $\partial((\sigma_{\text{acc}}(N_0))_{\varepsilon_1}) \cap \{\beta_i\}_{i=1}^{\nu} = \varnothing$. Having done so, let $\{\beta_i\}_{i=1}^{p}$ denote those elements of $\{\beta_i\}_{i=1}^{\nu}$ which do not lie in $(\sigma_{\text{acc}}(N_0))_{\varepsilon_1}$. (Again, if $\nu < \infty$, choose $\varepsilon_1 > 0$ small enough so that $p = \nu$.) Then $b_i \leq \operatorname{nul}(T_0 - \beta_i)$, $1 \leq i \leq p$, implies that T_0 admits the representation

$$T_0 \cong \begin{bmatrix} \beta_1 I_1' & & & & \beta_{10} \\ & \beta_2 I_2' & & \beta_{ij} & \beta_{20} \\ & & \ddots & & \vdots \\ & & & \beta_p I_p' & \beta_{p0} \\ & & & & T_1 \end{bmatrix} \begin{matrix} \mathscr{M}_1 \\ \mathscr{M}_2 \\ \vdots \\ \mathscr{M}_p \\ \mathscr{M}_0 \end{matrix}$$

where I_i' is the identity operator acting on a space $\mathscr{M}_i$ of dimension b_i, $1 \le i \le p$. Again, using the fact that $T_0 \in (\mathscr{N} + \mathscr{K})(\mathscr{H}_0)$, a simple matrix computation shows that T_1 is essentially normal, while index considerations and **[BDF]** imply that $T_1 \in (\mathscr{N} + \mathscr{K})(\mathscr{M}_0)$. In fact, a simple computation shows that $\sigma(T_1) = \sigma(T_0)$, $\sigma_e(T_1) = \sigma_e(T_0)$, and $\text{nul}(T_1 - \beta) = \text{nul}(T_0 - \beta)$ if $\beta \in \rho_{sF}(T_1) \backslash \{\beta_i\}_{i=1}^p$.

Since N_0 is normal and $\beta_i \in \sigma_0(N_0)$ for each $1 \le i \le p$, we can also decompose N_0 as $N_0 \cong (\bigoplus_{i=1}^p \beta_i I_i') \oplus N_1$ with respect to the same decomposition $\mathscr{H}_0 \cong (\bigoplus_{i=1}^p \mathscr{M}_i) \oplus \mathscr{M}_0$. The key reason for doing this is that $\beta_i \notin \sigma(N_1)$ for $1 \le i \le p$. In particular, if $\beta \in \sigma(N_1)$, then either $\text{dist}(\beta, \sigma_{\text{acc}}(N_1)) = \sigma_{\text{acc}}(N) < \varepsilon$ or $\beta \in \sigma_e(N_1) = \sigma_e(N_0)$ and β is in some hole of $\sigma(N_1)$, but β is not isolated in $\sigma(T_1)$. (This is the situation illustrated by Example 2.11.)

Step Three: Emptying the "Big Holes" of $\sigma(T_1)$. Since $\sigma(T_1) = \sigma(T_0)$, it also looks like $\sigma(N_1)$ with some of the holes filled in. Now $\sigma(N_1)$ is compact, and as such it can have at most countably many holes. By $\{\tau_j\}_{j=1}^\eta$ $(1 \le \eta \le \infty)$ we shall denote the holes of $\sigma(N_1)$ which lie in $\sigma(T_1)$. Again using the compactness of $\sigma(N_1)$, the sequence $\{\tau_j\}_{j=1}^\eta$ must be decreasing in the sense that given $\varepsilon_2 > 0$, there exists $N = N(\varepsilon_2) > 0$ such that $\{\tau_j\}_{j=N+1}^\eta \subseteq (\sigma_{\text{acc}}(N_1))_{\varepsilon_2}$. Let us therefore fix $0 < \varepsilon_2 < \varepsilon/2$ and find the appropriate $N = N(\varepsilon_2)$. The holes $\{\tau_j\}_{j=1}^N$ we shall call "big", while $\{\tau_j\}_{j=N+1}^\eta$ we shall think of as "small". (As usual, if $\eta < \infty$, we choose ε_2 small enough so that $N = \eta$, i.e., all holes are "big".)

Suppose $\text{min. ind.}(T_1 - \alpha) = \kappa_j$ for all $\lambda \in \tau_j \cap \rho_{sF}^r(T_1)$, $1 \le j \le N$. Let $\kappa = \sum_{j=1}^N \kappa_j$, and let $0 < \varepsilon_3 < \varepsilon_2/\kappa$.

We can now apply Lemma 2.13 to T_1, first setting τ (there) equal to τ_1 and ε (there) equal to ε_3. We obtain a compact operator K_1 such that $\|K_1\| < \varepsilon_3$ and

$$T_1 - K_1 = \begin{bmatrix} N_1(\tau_1) & W \\ 0 & T_1' \end{bmatrix}$$

with $N_1(\tau_1)$ (resp. T_1') playing the role of $N(\tau)$ (resp. T_ε). The point is that the nullity of $(T_1' - \alpha)$ is one less than the nullity of $(T_1 - \alpha)$ for all $\lambda \in \tau_1 \cap \rho_{sF}(T_1)$.

If we reiterate this process on T_1' $(\kappa_1 - 1)$ more times using τ_1, and then (κ_j) more times using τ_j, $2 \le j \le N$, the result is a compact operator K_2, with $\|K_2\| < (\sum_{j=1}^N \kappa_j)\varepsilon_3 < \varepsilon_2$, such that

$$T_1 - K_2 \cong \begin{bmatrix} N_1(\tau_1) & & & & & & W_{10} \\ & \ddots & & & W_{ij} & & W_{20} \\ & & N_{\mathscr{K}_1}(\tau_1) & & & & \\ & & & N_1(\tau_2) & & & \vdots \\ & & & & \ddots & & \\ & & & & & N_{\kappa_N}(\tau_N) & W_{k0} \\ & & & & & & T_2 \end{bmatrix} \begin{matrix} \mathscr{J}_1 \\ \mathscr{J}_2 \\ \vdots \\ \mathscr{J}_k \\ \mathscr{J}_0 \end{matrix}$$

where

(i) each $N_i(\tau_j)$ is a compact perturbation of a normal operator;

(ii) $\sigma(N_i(\tau_j)) = \overline{\tau}_j$, $\sigma_e(N_i(\tau_j)) = \partial(\overline{\tau}_j)$;

(iii) $\sigma_p(N_i(\tau_j)) = \sigma_p(N_i(\tau_j))^*)^* = \overline{\tau}_j \backslash \partial(\overline{\tau}_j)$, $\mathrm{nul}(N_i(\tau_j) - \lambda) = \mathrm{nul}(N_i(\tau_j) - \lambda)^* = 1$ for all $\lambda \in \overline{\tau}_j \backslash \partial(\overline{\tau}_j)$;

(iv) $\sigma_e(T_2) = \sigma_e(T_1)$, $\rho_{sF}(T_2) = \rho_{sF}(T_1)$;

(v) $\mathrm{ind}(T_2 - \lambda) = \mathrm{ind}(T_1 - \lambda) = 0$ for all $\lambda \in \rho_{sF}(T_1)$;

(vi)

$$\mathrm{nul}(T_2 - \lambda) = \begin{cases} \mathrm{nul}(T_1 - \lambda) - \kappa_j & \text{if } \lambda \in \tau_j,\ 1 \le j \le N, \\ \mathrm{nul}(T_1 - \lambda) & \text{if } \lambda \notin \bigcup_{j=1}^N \tau_j; \end{cases}$$

(vii)

$$\mathrm{nul}(T_2 - \lambda)^* = \begin{cases} \mathrm{nul}(T_1 - \lambda)^* - \kappa_j & \text{if } \lambda \in \tau_j,\ 1 \le j \le N, \\ \mathrm{nul}(T_1 - \lambda) & \text{if } \lambda \notin \bigcup_{j=1}^N \tau_j. \end{cases}$$

Moreover, as $T_1 \in (\mathscr{N}+\mathscr{K})(\mathscr{M}_0)$, using (i) and still another matrix calculation of the same type as above, we find that each W_{ij}, $1 \le i \le k$, $0 \le j \le k$, is compact and that T_2 is essentially normal. As always, index considerations show that $T_2 \in (\mathscr{N}+\mathscr{K})(\mathscr{J}_0)$.

Conditions (iv), (v), (vi) and (vii) together show that $\sigma_0(T_2) = \sigma_0(T_1) \cup (\rho_{sF}^s(T_1) \cap (\bigcup_{j=1}^N \tau_j))$. Moreover,

$$\sigma(T_2) = \left(\sigma(T_1) \backslash \left(\bigcup_{j=1}^N \tau_j\right)\right) \cup \left(\rho_{sF}^s(T_0) \cap \left(\bigcup_{j=1}^N \tau_j\right)\right).$$

Step Four: *Factoring out the singular points of* T_2. Let us now consider T_2. We know $\sigma_e(T_2) = \sigma_e(T_1) = \sigma_e(T_0) = \sigma_e(N_0) = \sigma_e(N_1)$. Since both T_2 and N_1 are of the form normal plus compact, it follows from **[BDF]** that there exists a compact operator $L_1 \in \mathscr{K}(\mathscr{J}_0)$ and a unitary operator $U: \mathscr{M}_0 \to \mathscr{J}_0$ such that

$$T_2 \cong UN_1U^* + L_1 = U(N_1 + L)U^*,$$

where $L = U^*L_1U \in \mathscr{K}(\mathscr{M}_0)$. Now we adopt an approach similar to that of Theorem 2.3. Namely, we let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for $\mathscr{M}_0$ with respect to which N_1 is diagonal (recall that N_1 is a direct summand of N, which we assumed was a diagonal normal operator). Let P_m be the orthogonal projection onto $\operatorname{span}\{e_i\}_{i=1}^{m}$, $m \geq 1$. Then $\{F_i = P_iLP_i\}_{i=1}^{\infty}$ is a sequence of finite rank operators satisfying $\lim_{i\to\infty} \|L - F_i\| = 0$.

Now

$$N_1 + F_m \cong \begin{bmatrix} \gamma_1 & & & \\ & \gamma_2 & \gamma_{ij} & \\ & & \ddots & \\ & & & \gamma_m \end{bmatrix} \oplus \operatorname{diag}\{\gamma_i\}_{i>m},$$

and the upper semicontinuity of the spectrum ensures us that by choosing m large enough, we get

(i) $\|(N_1 + F_m) - (N_1 + L)\| < \varepsilon$; and
(ii) $\sigma(N_1 + F_m) \subseteq (\sigma(N_1 + L))_\varepsilon = (\sigma(T_2))_\varepsilon$.

Clearly $\sigma_e(N_1 + F_m) = \sigma_e(N_1 + L)$.

Now $\sigma_0(N_1+F_m) \subseteq \{\gamma_i\}_{i=1}^{\infty}$ and since $\sigma_0(N_1+F_m)$ forms a sequence with no accumulation points in $\rho_{sF}(N_1 + F_m) = \rho_{sF}(N_1)$, we can find $m_0 > m$ such that $\operatorname{dist}(\gamma_i, \sigma_e(N_1)) < \varepsilon$ for $i > m_0$. Consider

$$G_m = \begin{bmatrix} \gamma_1 & & \gamma_{ij} \\ & \ddots & \\ & & \gamma_m \end{bmatrix} \oplus \operatorname{diag}\{\gamma_i\}_{i=m+1}^{m_0} \quad \text{and} \quad M = \operatorname{diag}\{\gamma_i\}_{i>m_0},$$

so that $N_1 + F_m \cong G_m \oplus M$. If $\gamma_i \in \sigma(M)$, then $\operatorname{dist}(\gamma_i, \sigma_e(M) = \sigma_e(N_1)) < \varepsilon$, and so clearly we can choose $f(i) > m_0$ so that $\gamma_i' = \gamma_{f(i)} \in \{\gamma_j\}_{j>m_0}$ so that $\gamma_i' = \gamma_{f(i)} \in \{\gamma_j\}_{j>m_0} \cap \sigma_e(N_1)$ satisfies $|\gamma_i - \gamma_i'| < \varepsilon$. Let $M_1 = \operatorname{diag}\{\gamma_i'\}_{i>m_0}$. Then $\|M_1 - M\| < \varepsilon$ and $\sigma(M_1) = \sigma_e(M_1) = \sigma_e(N_1)$. Note, therefore, that if $i > m_0$ and $\gamma_i' \notin \bigcup_{j=1}^{N} \tau_j$, then $\operatorname{dist}(\gamma_i', \sigma_{acc}(N_1)) < \varepsilon_2 < \varepsilon$.

As for G_m, $\sigma(G_m) \subseteq \sigma(N_1 + F_m) \subseteq (\sigma(T_2))_\varepsilon$, and so for $1 \leq i \leq m_0$, either $\gamma_i \in \bigcup_{j=1}^{N} \tau_j$, or $\operatorname{dist}(\gamma_i, \sigma_{acc}(N_1)) < 2\varepsilon$. If $\gamma_i \notin \bigcup_{j=1}^{N} \tau_j$ for some $1 \leq i \leq m_0$, then choose $\gamma_i' \in \sigma_{acc}(N_1)$ such that $|\gamma_i - \gamma_i'| < 2\varepsilon$. Otherwise, let $\gamma_i' = \gamma_i$.

Letting

$$G_m' = \begin{bmatrix} \gamma_1' & & \gamma_{ij} \\ & \ddots & \\ & & \gamma_m' \end{bmatrix} \oplus \operatorname{diag}\{\gamma_i'\}_{i=m+1}^{m_0},$$

$$\|G'_m - G_m\| = \max_{1\le i\le m_0} |\gamma_i - \gamma'_i| < 2\varepsilon \quad \text{and}$$

$$\sigma(G'_m) \subseteq \left(\bigcup_{j=1}^{N} \tau_j\right) \cup (\sigma_{\mathrm{acc}}(N_1)).$$

Now $\|(N_1+F_m)-(G'_m\oplus M_1)\| < 2\varepsilon$. Hence $\|(N_1+L)-(G'_m\oplus M_1)\| < 3\varepsilon$, and so $\|T_2 - U(G'_m \oplus M_1)U^*\| < 4\varepsilon$.

Let $K'_2 \in \mathscr{K}(\mathscr{H}_0)$, $K'_2 = 0 \oplus K_2$ with respect to the decomposition $\mathscr{H}_0 = (\mathscr{M}_0)^\perp \oplus \mathscr{M}_0$. Then $\|K'_2\| = \|K_2\| < \varepsilon_2$ and

$$T_0 - K'_2 \cong \begin{bmatrix} \beta_1 I'_1 & & & \beta_{10} \\ & \beta_2 I'_2 & \beta_{ij} & \beta_{20} \\ & & \ddots & \vdots \\ & & \beta_p I'_p & \beta_{p0} \\ & & & (T_1 - K_2) \end{bmatrix} \begin{matrix} \mathscr{M}_1 \\ \mathscr{M}_2 \\ \vdots \\ \mathscr{M}_p \\ \mathscr{M}_0 \end{matrix}$$

$$\cong \begin{bmatrix} \beta_1 I'_1 & & & C_{11} & \cdots & C_{1\kappa} & C_{10} \\ & \ddots & \beta_{ij} & \vdots & & \vdots & \vdots \\ & & \beta_p I'_p & C_{p1} & \cdots & C_{p\kappa} & C_{p0} \\ & & & N_1(\tau_1) & & & W_{10} \\ & & & & \ddots & W_{ij} & \vdots \\ & & & & & N_{k_N}(\tau_N) & W_{\kappa 0} \\ & & & & & & T_2 \end{bmatrix}.$$

If we now conjugate this by $(I \oplus U)$ with respect to the decomposition $\mathscr{H}_0 \cong (J_0)^\perp \oplus J_0$, and then replace U^*T_2U by $G'_m \oplus M_1$, we see that

$$T_0 \cong_\varepsilon T_0 - K'_2 \cong_{4\varepsilon} \begin{bmatrix} \beta_1 I'_1 & & & C_{11} & \cdots & C_{1\kappa} & C_{10}U \\ & \ddots & \beta_{ij} & \vdots & & \vdots & \vdots \\ & & \beta_p I'_p & C_{p1} & \cdots & C_{p\kappa} & C_{p0}U \\ & & & N_1(\tau_1) & & & W_{10}U \\ & & & & \ddots \; W_{ij} & & \vdots \\ & & & & & N_{k_N}(\tau_N) & W_{\kappa 0}U \\ & & & & & & G'_m \oplus M_1 \end{bmatrix} \begin{matrix} \mathscr{M}_1 \\ \vdots \\ \mathscr{M}_p \\ \mathscr{J}_1 \\ \vdots \\ \mathscr{J}_\kappa \\ \mathscr{J}_0 \end{matrix}$$

Now, since each $C_{i0}U$, $1 \le i \le p$, and $W_{i0}U$, $1 \le i \le \kappa$, is compact, we can approximate them individually to within $\varepsilon/(\kappa + p)$ by finite rank operators $C'_{i0} = (C_{i0}U)P_r$ and $W'_{i0} = (W_{i0}U)P_r$ for some $r > m_0$ sufficiently large, r independent of i in each case. Let $D_{r+1} = \mathrm{diag}\{\gamma'_i\}_{i>r}$. Consider $G'_m \oplus \mathrm{diag}\{\gamma'_i\}_{i=m_0+1}^r$. Let $\{\omega_i\}_{i=1}^t$ denote the subset (including multiplicity) of $\sigma(G'_m \oplus \mathrm{diag}\{\gamma'_i\}_{i=m_0+1}^r)$ which lies in $\overline{(\bigcup_{j=1}^{N} \tau_j)}$, and let $\{\omega_i\}_{i=t+1}^{r}$ denote the remaining eigenvalues. Clearly we can find a new orthonormal basis for $G'_m \oplus \operatorname{diag}\{\gamma'_i\}_{i=m_0+1}^{r}$ such that

$$T_0 \cong_{6\varepsilon} \begin{bmatrix} \beta_1 I'_1 & \beta_{ij} & C_{11} & \cdots & C_{1\kappa} & & C''_{10} & & & C'''_{10} & \\ & \ddots & \vdots & & & & \vdots & & & \vdots & \\ & \beta_p I'_p & C_{p1} & \cdots & C_{p\kappa} & & C''_{p0} & & & C'''_{p0} & \\ & & N_1(\tau_1) & & & & W''_{10} & & & W'''_{10} & \\ & & & \ddots & & & \vdots & & & \vdots & \\ & & & & N_{k_N}(\tau_N) & & W''_{\kappa 0} & & & W'''_{\kappa 0} & \\ & & & & & \omega_1 & & \omega_{ij} & & & \\ & & & & & & \ddots & & & \omega_{ij} & \\ & & & & & & & \omega_t & & & \\ & & & & & & & & \omega_{t+1} & & \omega_{ij} \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & \omega_r \end{bmatrix}$$

$\oplus D_{r+1}$.

Step Five: *Rebuilding* T_0 *from* N_0. We are now (finally!) in a position to show that T_0 is close to $(\mathscr{U} + \mathscr{K})(N_0)$.

For $1 \le j \le N$, $\overline{\tau_j} \subseteq \sigma_e(N_1)$. Thus $N_1 \cong_a (\bigoplus_{j=1}^{N} N'(\tau_j)) \oplus N_1$, where $N'(\tau_j)$ is a normal operator whose spectrum is the *perfect* set $\overline{\tau_j}$.

Moreover, since $\omega_i \in (\sigma_{\mathrm{acc}}(N_1))_{\varepsilon_2}$ for $t+1 \le i \le r$, we can find $\{d_i\}_{i=1}^{r-t} \in \sigma_{\mathrm{acc}}(N_0)$ satisfying

(i) $d_i \ne d_j$, $1 \le i \ne j \le r-t$;
(ii) $|d_i - \omega_{t+i}| < 2\varepsilon_2 < \varepsilon$, $1 \le i \le r-t$;
(iii) $d_i \notin \{\beta_j\}_{j=1}^{p}$, $1 \le i \le r-t$; and
(iv) $d_i \notin \overline{(\bigcup_{j=1}^{N} \tau_j)}$ $1 \le i \le r-t$.

Since $d_i \in \sigma_{\mathrm{acc}}(N_1) \subseteq \sigma_e(N_1)$ for $1 \le i \le r-t$, $N_1 \cong_a \operatorname{diag}\{d_i\}_{i=1}^{r-t} \oplus N_1$. Thus

$$N_0 \cong_a \left(\bigoplus_{i=1}^{p} \beta_i I'_i\right) \oplus \left(\bigoplus_{j=1}^{N} N'(\tau_j)\right) \oplus \operatorname{diag}\{d_i\}_{i=1}^{r-t} \oplus N_1.$$

Choose $R_1 \in (\mathscr{U} + \mathscr{K})(\mathscr{M}_1 \oplus \cdots \oplus \mathscr{M}_p)$ such that

$$R_1 \left(\bigoplus_{i=1}^{p} \beta_i I'_i\right) R_1^{-1} = \begin{bmatrix} \beta_1 I'_1 & & \beta_{ij} \\ & \ddots & \\ & & \beta_p I'_p \end{bmatrix}.$$

This is a simple application of Corollary 2.6, as $\dim \mathscr{M}_i < \infty$, $1 \le i \le p$.

Choose $R_2 \in (\mathscr{U}+\mathscr{K})(\mathscr{J}_1 \oplus \cdots \oplus \mathscr{J}_k \oplus \operatorname{span}\{e_i\}_{i=1}^t)$ such that

$$\left\| R_2 \left(\bigoplus_{j=1}^{N} N'(\tau_j) \right) R_2^{-1} - \begin{bmatrix} N_1(\tau_1) & & & W''_{10} & \\ & \ddots & & \vdots & \\ & & N_{k_N}(\tau_N) & W''_{\kappa 0} & \\ & & & \omega_1 & & \omega_{ij} \\ & & & & \ddots & \\ & & & & & \omega_t \end{bmatrix} \right\| < \varepsilon.$$

This can be done since the second operator in this difference satisfies the conditions of Theorem 2.3 with respect to the normal operator $\bigoplus_{j=1}^{N} N'(\tau_j)$. Choose $R_3 \in (\mathscr{U}+\mathscr{K})(\operatorname{span}\{e_i\}_{i=t+1}^r)$ such that

$$R_3(\operatorname{diag}\{d_i\}_{i=1}^{r-t})R_3^{-1} = \begin{bmatrix} d_1 & & & \\ & d_2 & \omega_{t+i\,t+j} & \\ & & \ddots & \\ & & & d_{t-r} \end{bmatrix}.$$

This is again Corollary 2.6. Then

$$\left\| R_3(\operatorname{diag}\{d_i\}_{i=1}^{r-t})R_3^{-1} - \begin{bmatrix} \omega_{t+1} & \omega_{t+1\,t+2} & \cdots & \omega_{t+1\,r} \\ & \omega_{t+2} & & \\ & & \ddots & \vdots \\ & & & \omega_r \end{bmatrix} \right\|$$

$$= \max_{1 \le i \le t-r} |d_i - \omega_{t+i}| < \varepsilon.$$

Now

$$N_0 \cong_{u+k} \begin{bmatrix} R_1 \left(\bigoplus_{i=1}^{p} \beta_i I'_i \right) R_1^{-1} & & \\ & R_2 \left(\bigoplus_{j=1}^{N} N(\tau_j) \right) R_2^{-1} & \\ & & R_3(\operatorname{diag}\{d_i\}_{i=1}^{r-t})R_3^{-1} \end{bmatrix} \oplus N_1$$

and since the spectra of the diagonal elements of the 3×3 operator matrix are disjoint, we conclude by Corollary 2.6 that

$$N_0 \cong_{u+k} \begin{bmatrix} R_1\left(\bigoplus_{i=1}^{p} \beta_i I_i'\right) R_1^{-1} & C_{11}\cdots C_{1\kappa} & C_{10}'' & C_{10}''' \\ & \vdots & & \vdots \\ & C_{p1}\cdots C_{p\kappa} & C_{p0}'' & C_{p0}''' \\ & & & W_{10}''' \\ & & & \vdots \\ & R_2\left(\bigoplus_{j=1}^{N} N'(\tau_j)\right) R_2^{-1} & & W_{\kappa 0}''' \\ & & & \omega_{ij} \\ & & & R_3(\operatorname{diag}\{d_i\}_{i=1}^{r-t})R_3^{-1} \end{bmatrix} \oplus N_1 .$$

But finally

$$\sigma(N_1) = \sigma(N)\backslash(\{\lambda_i\}_{i=1}^{n} \cup \{\beta_i\}_{i=1}^{p}),$$

while

$$\sigma(D_{r+1}) = \sigma(M_1) = \sigma_{\mathrm{e}}(M_1) = \sigma_{\mathrm{e}}(N_1).$$

Since the Hausdorff distance $d_H(\sigma(N_1), \sigma_{\mathrm{e}}(N_1)) < \varepsilon$, it follows from **[Dav 2]** that $\operatorname{dist}(D_{r+1}, \mathscr{U}(N_1)) < \varepsilon$. Choose R_4 unitary such that $\|D_{r+1} - R_4 N_1 R_4^*\| < \varepsilon$. Then

$$N_0 \cong_{U+K} N_0' = \begin{bmatrix} R_1\left(\bigoplus_{i=1}^{p} \beta_i I_i'\right) R_1^{-1} & C_{11}\cdots C_{1\kappa} & C_{10}'' & C_{10}''' \\ & \vdots & & \vdots \\ & C_{p1}\cdots C_{p\kappa} & C_{p0}'' & C_{p0}''' \\ & & & W_{10}''' \\ & & & \vdots \\ & R_2\left(\bigoplus_{j=1}^{N} N'(\tau_j)\right) R_2^{-1} & & W_{\kappa 0}''' \\ & & & \omega_{ij} \\ & & & R_3(\operatorname{diag}\{d_i\}_{i=1}^{r-t})R_3^{-1} \end{bmatrix}$$
$$\oplus R_4 N_1 R_4^*,$$

and $\|T_0 - N_0'\| < 7\varepsilon$.

But as we saw at the end of Step One, this is indeed sufficient to prove our theorem. □

2.15. Let $N \in \mathscr{B}(\mathscr{H})$ be normal. Let $\mathscr{U}(\pi(N))$ be the (necessarily closed) unitary orbit of $\pi(N)$ in $\mathscr{A}(\mathscr{H})$ (cf. **[BDF]**). Let $\pi^{-1}(\mathscr{U}(\pi(N))) = \{T \in \mathscr{B}(\mathscr{H}) \colon \pi(T) \in \mathscr{U}(\pi(N))\}$ be the lifting to

$\mathscr{B}(\mathscr{H})$ of $\mathscr{U}(\pi(N))$. Since $T \in \overline{(\mathscr{U}+\mathscr{K})(N)}$ implies $T \in \overline{\mathscr{S}(N)} \cap \pi^{-1}(\mathscr{U}(\pi(N)))$, it is natural to ask whether or not $\overline{(\mathscr{U}+\mathscr{K})(N)} = \overline{\mathscr{S}(N)} \cap \pi^{-1}(\mathscr{U}(\pi(N)))$. The answer is yes.

COROLLARY. *Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator. Then*

$$\overline{(\mathscr{U}+\mathscr{K})(N)} = \overline{\mathscr{S}(N)} \cap \pi^{-1}(\mathscr{U}(\pi(N))).$$

Proof. Again, $\overline{(\mathscr{U}+\mathscr{K})(N)} \subseteq \overline{\mathscr{S}(N)} \cap \pi^{-1}(\mathscr{U}(\pi(N)))$ is easily seen. As in Theorem 2.9, if $T \in \overline{\mathscr{S}(N)}$, then T must satisfy conditions (iii), (iv) and (v) of that theorem, and moreover, $\sigma(N) \subseteq \sigma(T)$. If $T \in \pi^{-1}(\mathscr{U}(\pi(N)))$, then we also have that $\sigma_e(T) = \sigma_e(N)$, so that T satisfies (ii), and from **[BDF]**, we can also deduce that $T \in \overline{(\mathscr{N}+\mathscr{K})(\mathscr{H})}$, so that T satisfies (i). Thus $T \in \overline{(\mathscr{U}+\mathscr{K})(N)}$, completing the proof. □

2.16. *Question*. In general, for $A \in \mathscr{B}(\mathscr{H})$, $\mathscr{U}(\pi(A))$ need not be closed **[Dav]**. Nevertheless, we can define $\pi^{-1}(\mathscr{U}(\pi(A)))$ as above. Is it true in general that

$$\overline{(\mathscr{U}+\mathscr{K})(A)} = \overline{\mathscr{S}(A)} \cap \overline{\pi^{-1}(\mathscr{U}(\pi(A)))}?$$

2.17. COROLLARY. *Let $N \in \mathscr{B}(\mathscr{H})$ be a normal operator. Then $\overline{(\mathscr{U}+\mathscr{K})(N)}+\mathscr{K}(\mathscr{H}) = \mathscr{U}(N)+\mathscr{K}(\mathscr{H})$. In particular, $\overline{(\mathscr{U}+\mathscr{K})(N)}+\mathscr{K}(\mathscr{H})$ is closed.*

Proof. Clearly $\mathscr{U}(N)+\mathscr{K}(\mathscr{H}) \subseteq \overline{(\mathscr{U}+\mathscr{K})(N)}+\mathscr{K}(\mathscr{H})$. But if $T \in \overline{(\mathscr{U}+\mathscr{K})(N)}$, then $\pi(T) \in \mathscr{U}(\pi(N))$, and so by **[BDF]**, $T = U^*NU+K$ for some unitary U and some compact operator K. That is, $T \in \mathscr{U}(N)+\mathscr{K}(\mathscr{H})$. Now $\mathscr{U}(N)+\mathscr{K}(\mathscr{H}) = \pi^{-1}(\mathscr{U}(\pi(N)))$, again by **[BDF]**, and since $\mathscr{U}(\pi(N))$ is closed, so is $\pi^{-1}(\mathscr{U}(\pi(N)))$, as was to be shown. □

2.18. COROLLARY. $(\mathscr{N}+\mathscr{K})(\mathscr{H}) = \overline{(\mathscr{U}+\mathscr{K})(\mathscr{N}(\mathscr{H}))}$, *where $\mathscr{N}(\mathscr{H})$ is the set of normal operators on $\mathscr{H}$, and $(\mathscr{U}+\mathscr{K})(\mathscr{N}(\mathscr{H})) = \bigcup\{(\mathscr{U}+\mathscr{K})(N) : N \in \mathscr{N}(\mathscr{H})\}$.*

Proof. Suppose $T = N+K$ where $N \in \mathscr{N}(\mathscr{H})$ and $K \in \mathscr{K}(\mathscr{H})$. As before, it suffices to consider the case where N is diagonal with respect to an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ for $\mathscr{H}$. Let P_m be the orthogonal projection onto $\text{span}\{e_i\}_{i=1}^{m}$. Let $\varepsilon > 0$.

Consider

$$T_n = N + P_n K P_n$$
$$\cong \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} & 0 \\ & \alpha_{22} & \cdots & \alpha_{2n} & 0 \\ & & \ddots & & \vdots \\ & & & \alpha_{nn} & 0 \\ 0 & 0 & \cdots & 0 & \operatorname{diag}\{d_i\}_{i>n} \end{bmatrix}.$$

Choose $M_n \in \mathcal{N}(\mathcal{H})$ such that $M_n = \operatorname{diag}\{d_i(n)\}_{i=1}^{\infty}$ satisfies
(a) $d_i(n) = d_i$, $i > n$;
(b) $|d_i(n) - \alpha_{ii}| < \varepsilon/n$; and
(c) $d_j(n) \neq d_k(n)$ if $i \leq j \neq k \leq n$. Then

$$M_n \cong_{u+k} R_n = \begin{bmatrix} d_1(n) & \alpha_{12} & & \alpha_{1n} & 0 \\ & d_2(n) & & \alpha_{2n} & 0 \\ & & \ddots & \vdots & \vdots \\ & & & d_n(n) & 0 \\ 0 & 0 & \cdots & 0 & \operatorname{diag}\{d_i\}_{i>n} \end{bmatrix}$$

and $\|R_n - T_n\| = \max_{1 \leq i \leq n} |d_i(n) - \alpha_{ii}| < \varepsilon/n$. Thus

$$\begin{aligned} \operatorname{dist}(T, (\mathcal{U} + \mathcal{K})(\mathcal{N}(\mathcal{H}))) &\leq \|T - T_n\| + \operatorname{dist}(T_n, (\mathcal{U} + \mathcal{K})(M_n)) \\ &\leq \|T - T_n\| + \|T_n - R_n\| \\ &\leq \|T - T_n\| + \varepsilon/n. \end{aligned}$$

Since $T = \lim_{n\to\infty} T_n$, letting n tend to ∞ does the trick. □

3. The compact case. In this section we consider the case of compact operators. Let $(I+\mathcal{K})(\mathcal{H}) = \{R \in \mathcal{B}(\mathcal{H}) | R$ is invertible and R is of the form identity plus compact$\}$ (note: $(I+\mathcal{K})(\mathcal{H}) \subset (\mathcal{U}+\mathcal{K})(\mathcal{H})$). For $T \in \mathcal{B}(\mathcal{H})$ let $(I+\mathcal{K})(T) = \{R^{-1}TR | R \in (I+\mathcal{K})(\mathcal{H})\}$. We show that for K a compact operator

$$\overline{(I+\mathcal{K})(K)} = \overline{\mathcal{S}(K)}$$

(and hence $\overline{(\mathcal{U}+\mathcal{K})(K)} = \overline{\mathcal{S}(K)}$).

After submitting this paper for publication, we learnt that Al-Musallam has independently obtained a characterization of $\overline{(\mathcal{U}+\mathcal{K})(K)}$ in the case of a compact operator K (cf. **[A1-M]**). The methods used and the characterization of $\overline{(\mathcal{U}+\mathcal{K})(K)}$ given there are substantially different from those below, and are more along the lines of our Theorem 2.14. The development here is indeed much shorter, and actually identifies $\overline{(\mathcal{U}+\mathcal{K})(K)}$ with both $\overline{(I+\mathcal{K})(K)}$ and $\overline{\mathcal{S}(K)}$.

3.1. LEMMA. *Any compact operator* $K \in \mathscr{K}(\mathscr{H})$ *is the norm limit of finite rank operators* F_n *which are invertible when restricted to the subspace* $\operatorname{supp}(F_n) = \operatorname{span}\{F_n\mathscr{H}, F_n^*\mathscr{H}\}$.

Proof. Let F_n' be a sequence of finite rank operators converging in norm to K. For each n one can find a μ_n, $0 < \mu_n < 2^{-n}$, such that $F_n' + \mu_n P_{\operatorname{supp}(F_n')}$, where $P_{\operatorname{supp}(F_n')}$ is the orthogonal projection onto $\operatorname{supp}(F_n')$, has the desired property. □

3.2. LEMMA. *The action of a similarity induced by* $S \in \mathscr{B}(\mathscr{H})$ *on a finite rank operator* F *of the form of the previous lemma can be induced by an operator* $S' \in (I+\mathscr{K})(\mathscr{H})$ *such that* $\|S'\|\,\|S'^{-1}\| \le \|S\|\,\|S^{-1}\|$.

Proof. Let $F' = S^{-1}FS$. We have $\dim\operatorname{supp}(F) = \dim\operatorname{supp}(F')$. Decompose $\mathscr{H}$ as $\mathscr{H}_1 \oplus \mathscr{H}_1^{\perp}$ where $\mathscr{H}_1 = \operatorname{span}\{\operatorname{supp}(F), \operatorname{supp}(F')\}$. We can find a unitary $U = U_1 \oplus \mathrm{Id}$ with respect to this decomposition such that if $F'' = U^*F'U$ then $\operatorname{supp}(F) = \operatorname{supp}(F'')$. Let $R = SU$.

Now consider the decomposition $\mathscr{H} = \operatorname{supp}(F) \oplus (\operatorname{supp}(F))^{\perp}$. With respect to this decomposition we have

$$RF'' = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} F_{11}'' & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} F_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = FR.$$

As F_{11}'' and F_{11} are invertible when restricted to $\operatorname{supp}(F)$ this implies that $R_{12} = R_{21} = 0$. Thus the operator $R' = R_{11} \oplus \mathrm{Id}$ implements the similarity of F and F''. Thus we have $F' = UR'^{-1}FR'U^*$. We also have $\|R'U^*\|\,\|UR'^{-1}\| \le \|S\|\,\|S^{-1}\|$ and $R'U^*$ is of the form the identity plus a compact. □

3.3. THEOREM. *If* $K \in \mathscr{K}(\mathscr{H})$ *then the closures of the* $(I+\mathscr{K})$ *orbit of* K *and the similarity orbit coincide.*

Proof. Let F_n be a sequence of finite rank operators of the type constructed in Lemma 3.1 converging in norm to K. Let $T \in \overline{\mathscr{S}(K)}$ and let $S_i \in \mathscr{B}(\mathscr{H})$ be a sequence of invertible operators such that $S_iKS_i^{-1} \to T$. Let $S_{i,n} \in \mathscr{B}(\mathscr{H})$ be the invertible operator in $(I+\mathscr{K})(\mathscr{H})$ constructed in the previous lemma such that $S_{i,n}F_nS_{i,n}^{-1} = S_iF_nS_i^{-1}$ and $\|S_{i,n}\|\,\|S_{i,n}^{-1}\| \le \|S_i\|\,\|S_i^{-1}\|$. We have

$$\begin{aligned} \|S_{i,n}F_nS_{i,n}^{-1} - S_iKS_i^{-1}\| &= \|S_i(F_n - K)S_i^{-1}\| \\ &\le \|S_i\|\,\|F_n - K\|\,\|S_i^{-1}\|. \end{aligned}$$

By passing to a subsequence $F_{n(i)}$ of the F_n we can force this to go to zero as i goes to infinity. Hence $S_{i,n(i)}F_{n(i)}S_{i,n(i)}^{-1}$ converges to T. The same subsequence gives

$$\|S_{i,n(i)}F_{n(i)}S_{i,n(i)}^{-1} - S_{i,n(i)}KS_{i,n(i)}^{-1}\| \\ \leq \|S_{i,n(i)}\| \, \|F_{n(i)} - K\| \, \|S_{i,n(i)}^{-1}\|.$$

As $\|S_{i,n(i)}\| \, \|S_{i,n(i)}^{-1}\| \leq \|S_i\| \, \|S_i^{-1}\|$, the right-hand side converges to zero. Hence $S_{i,n(i)}KS_{i,n(i)}^{-1}$ converges to T, completing the proof. □

3.4. COROLLARY. *If K is a compact quasinilpotent then* $0 \in \overline{(I+\mathcal{K})(K)}$.

Proof. By a result of Rota [**Rot**], $0 \in \overline{\mathcal{S}(K)}$. □

3.5. COROLLARY. *The closure of the $(I+K)$ orbit of a compact quasinilpotent which is not nilpotent consists of all compact quasinilpotents.*

Proof. Apostol [**Apo**] has shown that the result holds for the similarity orbit. □

3.6. REMARK. It is worth noting that for $K \in \mathcal{K}(\mathcal{H})$, the answer to Question 2.12 is again positive. In this case, $\overline{\mathcal{U}_L(\pi(K))} = \mathcal{K}(\mathcal{H})$, of course, yielding, $\overline{(\mathcal{U}+\mathcal{K})(K)} = \overline{\mathcal{S}(K)} \cap \mathcal{K}(\mathcal{H}) = \overline{\mathcal{S}(K)}$.

3.7. The coincidence of $\overline{\mathcal{S}(T)}$ and $\overline{(\mathcal{U}+\mathcal{K})(T)}$ for $T \in \mathcal{K}(\mathcal{H})$ is, understandably, a very special phenomenon. In fact we have the following

PROPOSITION. *For $T \in \mathcal{B}(\mathcal{H})$, $\overline{\mathcal{S}(T)} = \overline{(\mathcal{U}+\mathcal{K})(T)}$ if and only if T is of the form scalar plus compact.*

Proof. That $\overline{\mathcal{S}(T)} = \overline{(\mathcal{U}+\mathcal{K})(T)}$ when T is a scalar plus compact follows immediately from Theorem 3.3.

Let T be such that $\overline{\mathcal{S}(T)} = \overline{(\mathcal{U}+\mathcal{K})(T)}$. Note that if $A \in \overline{(\mathcal{U}+\mathcal{K})(T)}$, then $\|\pi(A)\| = \|\pi(T)\|$; the essential norm is preserved. Consider a decomposition of $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into two infinite dimensional subspaces. Let $R \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$ be an arbitrary operator from $\mathcal{H}_2$ to $\mathcal{H}_1$. With respect to this decomposition we have the following application of a similarity to T.

$$\begin{bmatrix} I & R \\ 0 & I \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} I & -R \\ 0 & I \end{bmatrix}$$
$$= \begin{bmatrix} T_{11}+RT_{21} & -(T_{11}R+RT_{21}R)+T_{12}+RT_{22} \\ T_{21} & T_{22}-T_{21}R \end{bmatrix}$$

In order for this latter operator to have the same essential norm as T for all such possible R, T_{21} must be compact (or else we could scale R as we wished to increase the essential norm of the bottom right-hand corner). A similar calculation with R in the lower left-hand corner shows that T_{12} must also be compact. The same argument also forces $RT_{22}-T_{11}R$ to be compact for all $R \in \mathscr{B}(\mathscr{H}_2, \mathscr{H}_1)$. Passing to the Calkin algebra we have

$$\pi(R)\pi(T_{22}) - \pi(T_{11})\pi(R) = 0 \quad \text{for all } R \in \mathscr{B}(\mathscr{H}_2, \mathscr{H}_1).$$

Putting $R = I$ we have $\pi(T_{22}) \cong \pi(T_{11})$. Thus $\pi(T_{11})$ is in the commutant of $\mathscr{A}(\mathscr{H}_1)$, so that $\pi(T_{11}) \cong \pi(T_{22}) \cong \lambda\pi(I)$ for some $\lambda \in \mathbb{C}$. Lifting back to $\mathscr{B}(\mathscr{H})$ we have T is of the form scalar plus compact. □

4. Further comments. Having described the $\mathscr{U}+\mathscr{K}$ orbit of a normal operator, one would like to obtain similar results for essentially normal operators. In this direction we have the following results which describe the $\mathscr{U}+\mathscr{K}$ orbit of the forward unilateral shift.

4.1. LEMMA. *Let S be the forward unilateral shift and let λ be a complex number such that $|\lambda| < 1$. Then $\lambda I' \oplus S \in \overline{(\mathscr{U}+\mathscr{K})(S)}$, where I' is the identity operator acting on a one-dimensional space.*

Proof. As $|\lambda| < 1$, $\bar{\lambda}$ is an eigenvalue of multiplicity one for S^*. Let x_0 be an associated eigenvector. Then S has a matrix representation of the form

$$S = \begin{bmatrix} \lambda I' & 0 \\ A & S' \end{bmatrix} \begin{matrix} \mathbb{C}x_0 \\ (\mathbb{C}x_0)^{\perp} \end{matrix}.$$

By Proposition 2.7, $\lambda I' \oplus S' \in \overline{(\mathscr{U}+\mathscr{K})(S)}$. Since the restriction of S to a cyclic invariant subspace is unitarily equivalent to S ([**RR**, Thm. 3.33]), it suffices to show that $\{x_0\}^{\perp}$ is a cyclic subspace. Direct computation shows that the orthogonal projection of the standard basis vector e_0 (with respect to S) onto the subspace $\{x_0\}^{\perp}$ is indeed a cyclic vector. □

4.2. COROLLARY. *Let* $\{\lambda_1, \dots, \lambda_n\}$ *be complex numbers of modulus less than one. When* S *is restricted to the invariant subspace formed by* $(\operatorname{span}\{\ker(S-\lambda_i I)^*\}_{i=1}^n)^\perp$, *the resulting operator is unitarily equivalent to* S.

Proof. Induction. □

4.3. COROLLARY. *Let* $\{\lambda_1, \dots, \lambda_n\}$ *be distinct complex numbers of modulus less than one. Then there exists an operator* C *such that*

$$S \cong_{u+k} \begin{bmatrix} F_d & 0 \\ C & S \end{bmatrix},$$

where F_d *is the* $n \times n$ *diagonal matrix* $F_d = \operatorname{diag}\{\lambda_i\}_{i=1}^n$.

Proof. Consider the decomposition

$$\mathscr{H} = (\operatorname{span}\{\ker(S-\lambda_i I)^*\}_{i=1}^n) \oplus (\operatorname{span}\{\ker(S-\lambda_i I)^*\}_{i=1}^n)^\perp$$

to get $S \cong \left[\begin{smallmatrix} F_0 & 0 \\ C_0 & S_0 \end{smallmatrix}\right]$, with $\sigma(F_0) = \{\lambda_i\}_{i=1}^n$. By the above Corollary, $S_0 \cong S$. Since F_0 has no repeated eigenvalues, it is similar to F_d via a matrix R. Apply the similarity transformation $\left[\begin{smallmatrix} R & 0 \\ 0 & I \end{smallmatrix}\right] \in (\mathscr{U}+\mathscr{K})(\mathscr{H})$ to S to obtain

$$S \cong_{u+k} \begin{bmatrix} F_d & 0 \\ C & S \end{bmatrix},$$

with $C = C_0 R$. □

4.4. COROLLARY. *If* A *is a shift of arbitrary multiplicity and* F *is an operator on a finite dimensional space whose spectrum lies inside the unit disk, then* $F \oplus A \in \overline{(\mathscr{U}+\mathscr{K})(A)}$.

Proof. Let $\varepsilon > 0$. We can clearly approximate F by an operator G such that $\|F - G\| < \varepsilon$, the eigenvalues of G still lie inside the disk, and the eigenvalues of G all have multiplicity one. From above, we know that

$$S \cong_{u+k} \begin{bmatrix} G_d & 0 \\ C & S \end{bmatrix}$$

for some operator C, where G_d is the diagonal matrix with the same eigenvalues as G. Now G_d is similar to G, say $G = R^{-1} G_d R$. Thus

$$\begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} G_d & 0 \\ C & S \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} G & 0 \\ CR & S \end{bmatrix} \in (\mathscr{U}+\mathscr{K})(S).$$

By Proposition 2.7, $\left[\begin{smallmatrix} G & 0 \\ 0 & S \end{smallmatrix}\right] \in \overline{(\mathscr{U}+\mathscr{K})(S)}$. Thus

$$\operatorname{dist}((F \oplus S), \overline{(\mathscr{U}+\mathscr{K})(S)}) \le \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, $F \oplus S \in \overline{(\mathscr{U}+\mathscr{K})(S)}$. □

4.5. LEMMA. *An operator of the form*

$$C = \begin{bmatrix} F_d & 0 \\ C_{21} & S \end{bmatrix},$$

where F_d *is a diagonal matrix is similar to* S *if and only if the diagonal entries* $\{\lambda_1, \ldots, \lambda_n\}$ *of* F_d *are distinct and have modulus less than one, and the* ith *column of* C_{21} *is not in* $\operatorname{ran}(S-\lambda_i I)$, $1 \le i \le n$. *Moreover, the similarity can be implemented by an operator in* $(\mathcal{U}+\mathcal{K})(\mathcal{H})$.

Proof. The necessity of the restrictions on the λ_i's is immediate from spectral considerations. To see the necessity that the ith column of C_{21} not be in $\operatorname{ran}(S-\lambda_i I)$, consider adjoints, that is,

$$C^* = \begin{bmatrix} F_d^* & C_{21}^* \\ 0 & S^* \end{bmatrix}$$

which is similar to S^*. We are now concerned with whether or not the ith row of C_{21}^* is perpendicular to $\ker(S^* - \overline{\lambda}_i I)$. Assume $i = 1$. Then

$$C^* - \overline{\lambda}_1 I = \begin{bmatrix} 0 & & & & \\ & \overline{\lambda}_2 - \overline{\lambda}_1 & & & \\ & & \ddots & & C_{21}^* \\ & & & \overline{\lambda}_n - \overline{\lambda}_1 & \\ & & 0 & & S^* - \overline{\lambda}_1 I \end{bmatrix}.$$

Obviously the first basis element is in the kernel. Let v be a non-zero vector in the kernel of $S^* - \overline{\lambda}_1 I$. Consider the action of $C^* - \overline{\lambda}_1 I$ on the vector

$$\begin{bmatrix} 0 \\ \alpha_2 \\ \vdots \\ \alpha_n \\ v \end{bmatrix}.$$

The resulting vector is (using $\overline{c}_i$ to denote the ith row of C_{21}^*):

$$\begin{bmatrix} \overline{c}_1 \cdot v \\ (\overline{\lambda}_2 - \overline{\lambda}_1)\alpha_2 + \overline{c}_2 \cdot v \\ \vdots \\ (\overline{\lambda}_n - \overline{\lambda}_1)\alpha_n + \overline{c}_n \cdot v \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

An appropriate choice of the α_i causes the entries below the first one to be zero. As C^* is similar to S^*, the kernel of $C^* - \overline{\lambda}_1 I$ must be one dimensional and hence is just the span of the first basis element. Thus $\overline{c}_1 \cdot v \neq 0$. Thus, as $\operatorname{ran}(S - \lambda_1 I) = \ker(S^* - \overline{\lambda}_1 I)^\perp$, c_1 is not in the range of $S - \lambda_1 I$, where c_1 is the ith column of C_{21}.

To demonstrate the sufficiency, first note that given $\lambda_1, \dots, \lambda_n$ satisfying the above conditions, Lemma 4.3 says that there is an operator of the form

$$B = \begin{bmatrix} F_d & 0 \\ B_{21} & S \end{bmatrix}$$

such that $B \cong_{u+k} S$. Next consider the equation

$$\begin{bmatrix} I & 0 \\ D & I \end{bmatrix} \begin{bmatrix} F_d & 0 \\ B_{21} & S \end{bmatrix} \begin{bmatrix} I & 0 \\ -D & I \end{bmatrix} = \begin{bmatrix} F_d & 0 \\ C_{21} & S \end{bmatrix},$$

which is equivalent to $SD - DF_d = B_{21} - C_{21}$. The ith column of the left-hand side is given by $(S - \lambda_i I_i) d_i$, where d_i is the ith column of D. Thus one can find a D to implement the similarity provided that the ith column of $B_{21} - C_{21}$ is in $\operatorname{ran}(S - \lambda_i I)$. Note that as $S - \lambda_i I$ is Fredholm, $\operatorname{ran}(S - \lambda_i I)$ is closed. By acting on B with similarities of the form $\left[\begin{smallmatrix} R_d & 0 \\ 0 & I \end{smallmatrix}\right]$, R_d a diagonal matrix, the columns of B_{21} can be scaled by arbitrary non-zero scalars. Note also that as F_d is diagonal these similarities do not change F_d. Since both the ith column of B_{21} and the ith column of C_{21} are not in $\operatorname{ran}(S - \lambda_i I)$, which is of codimension one, this suffices to get the ith column of $B_{21} - C_{21}$ into $\operatorname{ran}(S - \lambda_i I)$. Observe that the similarities used are all in $(\mathscr{U} + \mathscr{K})(\mathscr{H})$. □

4.6. Corollary. *If C is an operator of the form*

$$C = \begin{bmatrix} F_d & 0 \\ C_{21} & S \end{bmatrix}$$

where F_d is a diagonal matrix with distinct diagonal entries of modulus less than one, then $C \in \overline{(\mathscr{U} + \mathscr{K})(S)}$.

Proof. An arbitrarily small perturbation of C will get the ith column of C_{21} out of $\operatorname{range}(S - \lambda_i I)$. Then by the lemma this perturbed operator is in $(\mathscr{U} + \mathscr{K})(S)$. Hence $C \in \overline{(\mathscr{U} + \mathscr{K})(S)}$. □

4.7. THEOREM. *Let S be the (forward) unilateral shift on a Hilbert space $\mathscr{H}$. Then*

$$\overline{(\mathscr{U}+\mathscr{K})(S)} = \{T \in \mathscr{B}(\mathscr{H}):$$
$$\text{(i) } T \text{ is essentially normal},$$
$$\text{(ii) } \sigma(T) = \{z \in \mathbb{C} \mid |z| \leq 1\},$$
$$\text{(iii) } \sigma_{\mathrm{e}}(T) = \{z \in \mathbb{C}\colon |z| = 1\},$$
$$\text{(iv) } \operatorname{ind}(T-\lambda) = -1 \text{ for all } \lambda \in \{z \in \mathbb{C}\colon |z| < 1\}\}.$$

Alternatively, $\overline{(\mathscr{U}+\mathscr{K})(S)}$ consists of all essentially normal operators T having the same spectrum and essential spectrum as S, and satisfying $\operatorname{ind}(T-\lambda) = \operatorname{ind}(S-\lambda)$ *for all λ not in the essential spectrum.*

Proof. That these conditions are necessary is easily verified. We now consider their sufficiency.

By **[BDF]**, if T satisfies the above conditions, then $T = U^*SU + L = U^*(S+ULU^*)U$, where U is unitary and $K = ULU^*$ is compact. Thus it suffices to show that $S+K \in \overline{(\mathscr{U}+\mathscr{K})(S)}$. Let $\{e_n\}_{n=1}^{\infty}$ be the standard orthonormal basis for $\mathscr{H}$ with respect to which S is a shift, and let P_n be the orthogonal projection onto $\operatorname{span}\{e_i\}_{i=1}^{n}$. The sequence $\{S+P_nKP_n\}_{n=1}^{\infty}$ of operators converges to $S+K$. These operators are of the form

$$S+P_nKP_n \cong \begin{bmatrix} F_n & 0 \\ C_n & S \end{bmatrix} \quad \begin{matrix} (P_n\mathscr{H}) \\ (P_n\mathscr{H})^{\perp} \end{matrix}.$$

By passing to a subsequence (if necessary) and by using the upper semicontinuity of the spectrum, we may perturb F_n to get a new operator G_n such that

(i) $\|G_n - F_n\| < \frac{1}{n}$;
(ii) $\sigma(G_n) \subseteq \{z \in \mathbb{C}\colon |z| < 1\}$; and
(iii) G_n has no multiple eigenvalues.

Clearly the sequence $T_n = \left[\begin{smallmatrix} G_n & 0 \\ C_n & S \end{smallmatrix}\right]$ still converges to $S+K$. Now if $G_d(n)$ is the diagonal matrix whose eigenvalues are exactly those of G_n, then by Corollary 4.3 there exists an operator B_n such that $S \cong_{u+k} \left[\begin{smallmatrix} G_d(n) & 0 \\ B_n & S \end{smallmatrix}\right]$. Moreover, $G_n = R_n^{-1}G_d(n)R_n$ for some similarity R_n since all eigenvalues here are of multiplicity one. Thus, by Corollary 4.6,

$$\begin{bmatrix} G_d(n) & 0 \\ C_nR_n^{-1} & S \end{bmatrix} \in \overline{(\mathscr{U}+\mathscr{K})(S)},$$

implying that

$$T_n = \begin{bmatrix} G_n & 0 \\ C_n & S \end{bmatrix}$$
$$= \begin{bmatrix} R_n^{-1} & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G_d(n) & 0 \\ C_n R_n^{-1} & S \end{bmatrix} \begin{bmatrix} R_n & 0 \\ 0 & I \end{bmatrix} \in \overline{(\mathscr{U} + \mathscr{K})(S)}.$$

Since $T_n \in \overline{(\mathscr{U} + \mathscr{K})(S)}$ for all $n \geq 1$, $T = S + K = \lim_{n\to\infty} T_n \in \overline{(\mathscr{U} + \mathscr{K})(S)}$, completing the proof. □

4.8. It is also reasonable to ask about the strong and weak operator closures of the $(\mathscr{U} + \mathscr{K})$-orbits of bounded linear operators T on $\mathscr{H}$. In this context Hadwin, Nordgren, Radjavi and Rosenthal **[HNRR]** have shown that

(1) If $T \in \mathscr{B}(\mathscr{H})$ and T is not the sum of a scalar and a finite rank operator, then $S(T)$ is strongly (thus weakly) dense in $\mathscr{B}(\mathscr{H})$; and

(2) If $T \in \mathscr{B}(\mathscr{H})$ and rank $(T - \lambda I) = m < \infty$ for some scalar λ, then the strong (weak) closure of $S(T)$ is $\{\lambda I + F \colon \operatorname{rank} F \leq m\}$.

As might be expected, the same results hold true if $S(T)$ is replaced by $(\mathscr{U} + \mathscr{K})(T)$. The proofs are identical to theirs, noting only that the invertible operator A which appears in their proofs of Theorem 1 and Theorem 2 (**[HNRR]**) can be taken to be of the form unitary plus compact.

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