INEQUALITIES FOR QUASICONFORMAL MAPPINGS IN SPACE

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A new lower bound for the conformal capacity of the Grötzsch ring and sharp bounds for the radial distortion of a quasiconformal automorphism of the unit ball are obtained in n-space, $n \geq 2$.

1. Introduction. The conformal capacities of the Grötzsch and Teichmüller extremal rings in $\mathbb{R}^n$, $n \geq 2$ (see §2), are denoted by

$$(1.1) \qquad \gamma_n(s) = \operatorname{cap} R_{G,n}(s) \quad \text{and} \quad \tau_n(t) = \operatorname{cap} R_{T,n}(t),$$

respectively, where $s > 1$ and $t > 0$. The modulus $M_n(r)$ of the Grötzsch ring $R_{G,n}(1/r)$, $0 < r < 1$, is defined by

$$(1.2) \qquad \gamma_n(1/r) = \omega_{n-1} M_n(r)^{1-n},$$

where ω_{n-1} is the $(n-1)$-dimensional measure of the unit sphere S^{n-1} in $\mathbb{R}^n$. The capacities in (1.1) are related [G, §18] by

$$(1.3) \qquad \gamma_n(s) = 2^{n-1} \tau_n(s^2 - 1), \qquad s > 1.$$

For $K > 0$ define increasing homeomorphisms $\varphi_{K,n}$ and $\psi_{K,n}$ from $(0, 1)$ onto $(0, 1)$ by

$$(1.4) \qquad \begin{cases} \varphi_{K,n}(r) = 1/\gamma_n^{-1}(K\gamma_n(1/r)) = M_n^{-1}(\alpha M_n(r)), \\ \psi_{K,n}(r) = (1 - \varphi_{1/K,n}^2(r'))^{1/2}, \end{cases}$$

where $r' = \sqrt{1 - r^2}$ and $\alpha = K^{1/(1-n)}$. Given a domain D in $\mathbb{R}^n$, for $K \geq 1$ let $QC_K(D)$ and $QR_K(D)$ denote the class of all K-quasiconformal and K-quasiregular mappings, respectively, of D into itself [V1], [Vu2]. For $K \geq 1$, $0 < r < 1$, define [AVV2]

$$(1.5) \qquad \varphi_{K,n}^*(r) = \sup\{|f(x)|: |x| = r, f(0) = 0, f \in QC_K(B^n)\},$$

$$(1.6) \quad \varphi_{1/K,n}^*(r) = \inf\{|f(x)|: |x| = r, f(0) = 0,$$
$$f \in QC_K(B^n), f(B^n) = B^n\}.$$

We extend the functions in (1.4), (1.5), (1.6) to [0,1] by defining them to be 0 at 0 and 1 at 1. For $n \geq 2$, $K \geq 1$, $0 < r < 1$, these

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distortion functions are related by the inequalities [**Vu1**, 3.5, 5.20], [**AVV2**, (1.4), Theorem 2.24]

$$(1.7) \quad \begin{cases} \varphi^*_{K,n}(r) \le \min\{\varphi_{K,n}(r),\, \psi_{K,n}(r)\}, \\ \varphi^*_{1/K,n}(r) \ge \max\{\varphi_{1/K,n}(r),\, \psi_{1/K,n}(r)\}. \end{cases}$$

Each of these three functions is increasing from $[0, 1]$ onto $[0, 1]$ (see §3 below). For $n = 2$ the inequalities in (1.7) reduce to equalities.

It is well known that $M_n(r) + \log r$ is monotone decreasing on $(0, 1)$ [**T**, p. 632], [**G**]. The so-called Grötzsch ring constant λ_n defined by

$$(1.8) \quad \log \lambda_n = \lim_{r \to 0+} (M_n(r) + \log r)$$

satisfies $\lambda_2 = 4$ [**T**], [**LV**, (2.11), p. 62] and $\lambda_n \in (2e^{0.76(n-1)}, 2e^{n-1})$ for $n \ge 3$ [**G**], [**AVV4**, pp. 120–121], [**AF**].

The main purpose of this paper is to show how one can translate information about the special function $M_n(r)$ into information about geometric properties of quasiconformal mappings. An important tool is the following result, which improves the above-mentioned monotone property of $M_n(r) + \log r$.

1.9. THEOREM. *For each* $n \ge 2$, *the function*

$$f_n(r) \equiv M_n(r) + \log\left(\frac{r}{1+r'}\right)$$

is strictly decreasing from $(0, 1)$ *onto* $(0, \log(\lambda_n/2))$, *where* λ_n *is as in* (1.8) *and* $r' = (1 - r^2)^{1/2}$. *Moreover,* $f_2(r)$ *is strictly concave on* $(0, 1)$.

The next result is an immediate consequence of Theorem 1.9 (cf. [**G**, Lemma 8], [**AVV4**, Corollary 2.30]).

1.10. COROLLARY. *For each* $n \ge 2$, $0 < r < 1$, $r' = \sqrt{1 - r^2}$,

$$(1) \quad M_n(r) < \log\left(\frac{\lambda_n}{2} \frac{1+r'}{r}\right) < \log \frac{\lambda_n}{r},$$

$$(2) \quad \frac{M_n(r)}{r \log r} \le \frac{d}{dr} M_n(r) \le -\frac{1}{rr'}.$$

The usual method of obtaining lower bounds for the capacity of a ring is to use spherical symmetrization [**G**] together with the extremal property of the Teichmüller ring and the fundamental inequality $M_n(r) \le \log(\lambda_n/r)$. By virtue of Corollary 1.10(1), all such earlier bounds can now be improved. A graphical comparison of some bounds for $\gamma_3(1/r)$ appears in Figure 1 in §2.

From Theorem 1.9 and Corollary 1.10(1) we shall derive several inequalities for the distortion functions in (1.4), (1.5), and (1.6). These functions are interesting not only for studying the radial behavior of mappings in $QC_K(B^n)$, but also as means of expressing other special functions.

1.11. THEOREM. *For $n \geq 2$ and $K \geq 1$ let $\alpha = K^{1/(1-n)} = 1/\beta$, $L = (\lambda_n/2)^{1-\alpha}$, $l = (\lambda_n/2)^{1-\beta}$, $l_1 = 2l/(1+l^2)$. Then*

$$(1) \qquad \varphi_{K,n}(r) \leq \tanh(2\operatorname{arctanh}(LA(r)^\alpha))$$

for $r \in (0, l_1)$, and

$$(2) \qquad \varphi_{1/K,n}(r) \geq \tanh(2\operatorname{arctanh}(lA(r)^\beta))$$

for all $r \in (0, 1)$, where $A(r) = r/(1+r')$, $r' = \sqrt{1-r^2}$. For $K = 1$, both (1) and (2) reduce to equality.

The following local Hölder continuity theorem simplifies and improves earlier results in [G], [R, pp. 82–83], [MRV], [AVV2], [Ca].

1.12. THEOREM. *For $K \geq 1$, $n \geq 2$ let $f \in QR_K(B^n)$, $0 < r < 1$, $\alpha = K^{1/(1-n)}$, and let λ_n be as in (1.8).*
(1) If $x, y \in \overline{B}^n(r)$ then

$$|f(x) - f(y)| \leq \lambda_n^{1-\alpha}(1-r^2)^{-\alpha}|x-y|^\alpha.$$

(2) If $x \in \overline{B}^n(r)$ and $y \in B^n$, then

$$|f(x) - f(y)| \leq 2^\alpha \lambda_n^{1-\alpha}(1-r)^{-\alpha}|x-y|^\alpha.$$

Next, the hyperbolic metric $\rho(x, y)$ on B^n is given by

$$(1.13) \qquad \tanh^2 \frac{\rho(x, y)}{2} = \frac{|x-y|^2}{|x-y|^2 + (1-|x|^2)(1-|y|^2)}$$

for $x, y \in B^n$ (cf. [B, p. 40], [Vu2, 2.47]). The following distortion theorem for the hyperbolic metric is a consequence of Theorem 1.9.

1.14. THEOREM. *For $K \geq 1$, $n \geq 2$, let $f \in QR_K(B^n)$. Then for all $x, y \in B^n$,*

$$\tanh \frac{\rho'}{4} \leq \left(\frac{\lambda_n}{2}\right)^{1-\alpha} \left(\tanh \frac{\rho}{4}\right)^\alpha \leq K \left(\tanh \frac{\rho}{4}\right)^\alpha \leq K \left(\tanh \frac{\rho}{4}\right)^{1/K},$$

where $\rho = \rho(x, y)$, $\rho' = \rho(f(x), f(y))$, *and* $\alpha = K^{1/(1-n)}$. *These estimates are sharp as K tends to 1.*

1.15. COROLLARY. *For $K \geq 1$, $n \geq 2$, let $f \in QC_K(B^n)$. Then for all $x, y \in B^n$,*

$$\tanh \frac{\rho'}{4} \leq \min\{2, K\} \left(\tanh \frac{\rho}{4}\right)^{1/K},$$

where $\rho = \rho(x, y)$ and $\rho' = \rho(f(x), f(y))$.

The results in §3 will show that for $n \geq 3$ the functions $\varphi_{K,n}(r)$, $\varphi_{K,n}^*(r)$, and $\psi_{K,n}(r)$ in (1.7) behave differently and that the behavior of $\varphi_{K,n}^*(r)$ differs drastically from that of $\varphi_{K,2}(r)$ (cf. 3.7(1) and 3.14).

In §4 we study the *linear dilatation*

$$(1.16) \quad H_n(K) = \sup\left\{ \frac{|f(x)|}{|f(y)|} : |x| = |y| > 0, \right.$$

$$\left. f \in QC_K(\mathbb{R}^n), f(0) = 0 \right\},$$

obtaining the following asymptotic estimate.

1.17. THEOREM. *For each $K > 1$, $\limsup_{n\to\infty} H_n(K) \leq K^4$.*

An explicit expression for $H_n(K)$ is known only for $n = 2$, and

$$H_2(K) = \left(\frac{\mu^{-1}(\pi/(2K))}{\mu^{-1}(\pi K/2)}\right)^2$$

[**LV**, (6.4), p. 81] is also denoted by $\lambda(K)$. It follows from [**AVV5**, Theorems 1.1, 1.2] that

$$(1.18) \quad K^{\pi\sqrt{K}} < e^{\pi(K-1)} < \lambda(K) = H_2(K) < e^{\pi(K-(1/K))}$$

for $K > 1$ and that $(\lambda(K))^{1/K}$ is strictly increasing to e^π as K tends to ∞. Thus 1.18 and Theorem 1.17 imply that

$$(1.19) \quad H_n(K) < \left(1 + \frac{\log H_2(K)}{\pi}\right)^4 < (H_2(K))^{4/\pi}, \qquad K > 1,$$

for n sufficiently large. On the other hand by [**AVV1**, Theorem 1.14], $H_n(K) \geq \lambda(K^{1/(n-1)}) > 1$ for each $K > 1$ and $n \geq 2$. Whether the upper bound in Theorem 1.17 can be replaced by 1 remains an open problem (see 4.12(1) below). In any case, Theorem 1.17 shows that quasiconformal mappings become more rigid in high dimensions. If

one wants to extend the theory of quasiconformal mappings to infinite-dimensional Banach spaces then the definition of a K-quasiconformal mapping must be chosen carefully, as e.g. in J. Väisälä [V3].

1.20. NOTATION. For $n \geq 2$, $\mathbb{R}^n$ denotes the n-dimensional euclidean space, and $\overline{\mathbb{R}}^n$ its one-point compactification $\mathbb{R}^n \cup \{\infty\}$. Unit vectors along the coordinate axes in $\mathbb{R}^n$ are denoted by $e_1, e_2, \ldots, e_n$. For x in $\mathbb{R}^n$ and $r > 0$ we let $B^n(x, r) = \{z \in \mathbb{R}^n : |x - z| < r\}$, $S^{n-1}(x, r)$ its boundary sphere, $B^n(r) = B^n(0, r)$, $S^{n-1}(r) = S^{n-1}(0, r)$, $B^n = B^n(1)$, and $S^{n-1} = S^{n-1}(1)$. For any $E \subset \overline{\mathbb{R}}^n$ we let $\overline{E}$ denote its closure. For $a, b \in \mathbb{R}^n$ we let $[a, b] = \{(1 - t)a + tb : 0 \leq t \leq 1\}$ and for $a \in \mathbb{R}^n \setminus \{0\}$, $[a, \infty) = \{ta : t \geq 1\}$, $[a, \infty] = [a, \infty) \cup \{\infty\}$. Whenever $0 < r < 1$, by r' we shall mean $\sqrt{1 - r^2}$.

Given E, F, and G, subsets of $\overline{\mathbb{R}}^n$, we let $\Delta(E, F; G)$ denote the family of all curves joining E and F in G [V1]. The conformal modulus of a family of curves Γ in $\overline{\mathbb{R}}^n$ is defined by

$$M(\Gamma) = \inf_\rho \int_{\mathbb{R}^n} \rho^n \, dm,$$

where dm represents n-dimensional Lebesgue measure and the infimum is taken over all nonnegative Borel-measurable functions ρ satisfying $\int_\gamma \rho \, ds \geq 1$ for each locally rectifiable curve $\gamma \in \Gamma$ [V1].

Let G be a domain (open connected set) in $\mathbb{R}^n$. A continuous function $f : G \to \mathbb{R}^n$ is said to be K-*quasiregular*, $1 \leq K < \infty$, if f is ACL^n, i.e. f is absolutely continuous on almost all lines parallel to the coordinate axes and the first partial derivatives are locally L^n-integrable, and if

$$(1.21) \quad \begin{cases} |f'(x)|^n \leq K J_f(x), & |f'(x)| \equiv \max_{|h|=1} |f'(x)h|, \\ J_f(x) \leq K l(f'(x))^n, & l(f'(x))^n \equiv \min_{|h|=1} |f'(x)h| \end{cases}$$

hold a.e. in G. Here $f'(x)$ denotes the formal derivative of f at x, i.e. $f'(x)e_i = \nabla f_i$, $i = 1, \ldots, n$, $f = (f_1, f_2, \ldots, f_n)$, and $J_f(x) = \det(f'(x))$. A K-quasiregular homeomorphism is called K-*quasiconformal*. A mapping f is called *quasiregular* or *quasiconformal* if it is K-quasiregular or K-quasiconformal, respectively, for some $K \in [1, \infty)$. For properties of these mappings the reader is referred to [V1], [V2], [Vu2].

2. Modulus of the Grötzsch ring. A *ring* $R = R(C_0, C_1)$ in $\overline{\mathbb{R}}^n$ is a domain whose complement consists of two components C_0 and C_1, where $\infty \notin C_0$. Two extremal rings, the Grötzsch ring

$$R_{G,n}(s) = \overline{\mathbb{R}}^n \backslash (\overline{B}^n \cup [se_1, \infty]), \qquad s > 1,$$

and the Teichmüller ring

$$R_{T,n}(t) = \overline{\mathbb{R}}^n \backslash ([-e_1, 0] \cup [te_1, \infty]), \qquad t > 0,$$

are important in the study of distortion [G], [Vu1], [Vu2], [AVV1], [AVV2], [AVV5].

The *conformal capacity* cap R and *modulus* mod R of a ring $R = R(C_0, C_1)$ are given [V1] by

$$\operatorname{cap} R = M(\Gamma) = \omega_{n-1} (\operatorname{mod} R)^{1-n},$$

where $\Gamma = \Delta(C_0, C_1; R)$ and ω_{n-1} is the $(n-1)$-dimensional measure of S^{n-1}.

The function $M_2(r)$ defined in formula (1.2), usually denoted by $\mu(r)$, is given explicitly [LV, (2.2), p. 60] by

$$(2.1) \qquad\qquad \mu(r) = \frac{\pi}{2} \frac{\mathcal{K}'(r)}{\mathcal{K}(r)},$$

where

$$\mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{-1/2} \, dt, \quad \mathcal{K}'(r) = \mathcal{K}(r'), \quad r' = \sqrt{1 - r^2},$$

are complete elliptic integrals. We also use the dual integrals

$$\mathcal{E}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{1/2} \, dt, \quad \mathcal{E}'(r) = \mathcal{E}(r').$$

The next result is an analog of l'Hôpital's rule and will be useful in establishing monotoneity of a ratio of two functions.

2.2. LEMMA. *For* $-\infty < a < b < \infty$ *let* $f, g : [a, b) \to \mathbb{R}$ *be differentiable functions such that* $g'(x) \neq 0$ *for* $x \in (a, b)$.

If $f'(x)/g'(x)$ is increasing (decreasing) on (a, b) then so is $(f(x) - f(a))/(g(x) - g(a))$.

Proof. We may assume that $g'(x) > 0$ for all $x \in (a, b)$ and that $f'(x)/g'(x)$ is increasing on (a, b). By the Cauchy mean value theorem, for $x \in (a, b)$ there exists $y \in (a, x)$ such that

$$\frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(y)}{g'(y)} \leq \frac{f'(x)}{g'(x)},$$

which implies that $(f(x) - f(a))/(g(x) - g(a))$ has a positive derivative on (a, b). $\qquad\qquad\square$

2.3. *Proof of Theorem 1.9.* The monotoneity appears in [**A1**, Theorem 5, p. 15], but for completeness we include the proof here. For $0 < a < b < 1$ let R be the ring whose boundary components are S^{n-1} and the segment $[-ae_1, ae_1]$. Then $S^{n-1}(a/b)$ separates R into two rings R_1 and R_2, where R_1 is conformally equivalent to $R_{G,n}((1 + b^2)/2b)$ and R_2 is the spherical annulus $\{x: a/b < |x| < 1\}$. By superadditivity of the modulus [**F**] we get $\mod R \geq \mod R_1 + \mod R_2$, that is,

$$M_n\left(\frac{2a}{1 + a^2}\right) \geq M_n\left(\frac{2b}{1 + b^2}\right) + \log\frac{b}{a}.$$

Next, putting $r_1 = 2a/(1 + a^2)$, $r_2 = 2b/(1 + b^2)$, we conclude that f_n is decreasing. The limit as r tends to 0 follows from (1.8), while the limit at 1 is trivial. In [**A2**, §8] a different proof of this monotoneity is based on Hölder's inequality, which, by [**A2**, Lemma 4], cannot reduce to equality; hence the monotoneity is strict.

Next, let $n = 2$ and $f(r) = f_2(r)$. Then by [**AVV6**, Lemma 2.1(2)],

$$f'(r) = \frac{1}{rr'}\left(1 - \frac{\pi^2}{4r'\mathcal{K}(r)^2}\right),$$

which is negative for $0 < r < 1$ since $r'\mathcal{K}(r)^2$ is strictly decreasing on $(0, 1)$ [**AVV6**, Theorem 2.2(3)] and $\mathcal{K}(0) = \pi/2$. Thus f is strictly decreasing on $(0, 1)$. Next, $-f'(r) = g(r)h(r)/r$, where

$$g(r) = \frac{\dfrac{\pi}{2\sqrt{r'}\mathcal{K}(r)} + 1}{r'} \quad \text{and} \quad h(r) = \frac{\pi}{2\sqrt{r'}\mathcal{K}(r)} - 1.$$

Since $g(r)$ is increasing [**AVV6**, Theorem 2.1(3)], we need only prove that $h(r)/r$ is increasing, and by Lemma 2.2 it is sufficient to show

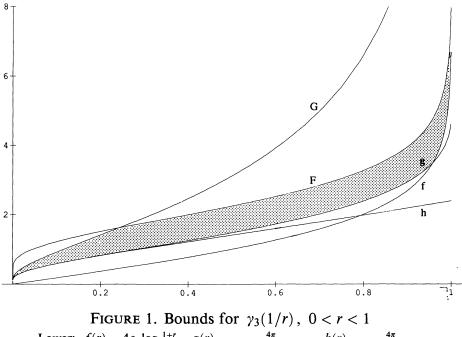that $h'(r)$ is increasing. First by [**BF**, 710.00]

$$\frac{h'(r)}{1 + h(r)} = \frac{1}{2r'^2 \mathscr{K}} \cdot \frac{H(r)}{r}, \quad \text{where } H(r) = (\mathscr{K} - \mathscr{E}) - (\mathscr{E} - r'^2 \mathscr{K}(r)).$$

By [**AVV6**, Theorem 2.2(3)] the first factor is increasing; thus by Lemma 2.2 it is sufficient to show that $H'(r)$ is increasing. Now, by [**BF**, 710.04-.05], $H'(r) = (\mathscr{E} - r'^2 \mathscr{K})rr'^{-2}$, which is clearly increasing.   □

2.4. REMARK. For $0 < r < 1$, by [**A1**, Theorem 2], [**G**, Lemma 8] (cf. [**AVV4**, (1.10), (1.11)]), and 1.10(1) we have

$$(2.5) \quad \begin{cases} 2^{n-1} c_n \log \frac{1+r}{1-r} \leq \gamma_n(1/r) \leq 2^{n-1} c_n \mu \left(\frac{1-r}{1+r} \right), \\ \omega_{n-1} \left(\log \frac{\lambda_n(1+r')}{2r} \right)^{1-n} \leq \gamma_n(1/r) \leq \omega_{n-1} \mu(r)^{1-n}, \end{cases}$$

where c_n is a constant [**V1**, p. 31]. Here equality holds on the right sides for $n = 2$. Let $f(r)$, $g(r)$ denote the minorants and $F(r)$, $G(r)$ the majorants in (2.5) and let $h(r) = \omega_{n-1}(\log(\lambda_n/r))^{1-n}$. Then, for $n = 3$, the graphs of these functions are in Figure 1, where the graph of $\gamma_3(1/r)$ lies in the shaded region.

FIGURE 1. Bounds for $\gamma_3(1/r)$, $0 < r < 1$

Lower: $f(r) = 4c_3 \log \frac{1+r}{1-r}$, $g(r) = \frac{4\pi}{\log^2(4.9501(1+r')/r)}$, $h(r) = \frac{4\pi}{\log^2(9.9002/r)}$,

Upper: $F(r) = 4c_3 \mu \left(\frac{1-r}{1+r} \right)$, $G(r) = \frac{4\pi}{\mu(r)^2}$.

From Figure 1 we see that the new lower bound $g(r)$ for $\gamma_3(1/r)$ given by Corollary 1.10(1) is the best of the present lower bounds when $0 < r < r_0$, $r_0 \approx 0.95$.

2.6. THEOREM. *For $n \geq 2$ the function*

$$f_n(r) \equiv M_n(r) \log((1 + \sqrt{r})/(1 - \sqrt{r}))$$

is strictly increasing from $(0, 1)$ onto I_n, where $I_2 = (0, \pi^2/2)$ and $I_n = (0, \infty)$ for $n \geq 3$.

Proof. First, by [**LV**, (2.3), p. 60], $f_2(r) = \mu(t) \log((1 + t)/(1 - t))$, where $t = 2\sqrt{r}/(1 + r)$, and the result follows from [**AVV4**, Lemma 2.6(4)]. Hence for $n \geq 3$ the result follows from [**AVV4**, Lemma 2.6(5)].   □

As shown in [**AVV4**, 1.21], the function M_n satisfies several non-linear functional inequalities. Next we shall prove additional results of this type.

2.7. THEOREM. *For each $n \geq 2$ and $s, t \in (0, 1)$,*

$$M_n \left(\frac{2s}{1 + s^2} \right) + M_n \left(\frac{2t}{1 + t^2} \right) \leq 2M_n \left(\frac{2\sqrt{st}}{1 + st} \right).$$

There is equality when $s = t$.

Proof. The statement about equality is trivial. For the inequality, let R denote the ring in $\mathbb{R}^n$ whose complementary components are the sets $[-e_1/(st), \infty] \cup [e_1/(st), \infty]$ and $[-e_1, e_1]|$. Then R is conformally equivalent to the Teichmüller ring $R_{T,n}((1-st)^2/(4st))$, and the sphere $S^{n-1}(1/s)$ separates R into two rings R_1 and R_2 that are conformally equivalent to $R_{G,n}((1+s^2)/(2s))$ and $R_{G,n}((1+t^2)/(2t))$, respectively. The result now follows from (2.1) and the superadditivity of the modulus [**F**].   □

2.8. COROLLARY. *For $n \geq 2$, $a, b \in (0, 1)$,*

$$M_n(a) + M_n(b) \leq 2M_n(\sqrt{ab}),$$

where equality holds if and only if $a = b$.

Proof. For $a = b$, equality is clear. Next, for $a \neq b$, let $s = a/(1 + a')$, $t = b/(1 + b')$, where $a' = \sqrt{1 - a^2}$, $b' = \sqrt{1 - b^2}$. Then

$a = 2s/(1 + s^2)$, $b = 2t/(1 + t^2)$, and Theorem 2.7 gives

$$M_n(a) + M_n(b) \leq 2M_n\left(\frac{2\sqrt{st}}{1 + st}\right) < 2M_n\left(\frac{2\sqrt{st}}{\sqrt{(1 + s^2)(1 + t^2)}}\right)$$
$$= 2M_n(\sqrt{ab}),$$

since M_n is strictly decreasing.   □

2.9. REMARK. Corollary 2.8 strengthens [**AVV4**, (1.22)].

2.10. THEOREM. *For each* $n \geq 2$, *let*

$$f_n(r) \equiv \frac{M_n(r)}{\log((1 + r')/r)} \quad and \quad g_n(r) \equiv \frac{M_n(r)}{\log(\lambda_n(1 + r')/(2r))}.$$

Then $f_n(r)$ *is strictly increasing from* $(0, 1)$ *onto* $(1, \infty)$ *and* $g_n(r)$ *is decreasing from* $(0, 1)$ *onto* $(0, 1)$.

Proof. By [**LV**, (2.5), p. 60] we have

$$f_2(r) = \frac{\mu(r)}{\log\left(\frac{1+r'}{r}\right)} = \frac{2\mu(r)}{\log\left(\frac{1+r'}{1-r'}\right)} = \frac{\mu(t)}{\log\left(\frac{1}{t}\right)},$$

where $t = (1 - r')/(1 + r')$, and it follows from [**AVV6**, Theorem 4.3(4)] and $f_2(r)$ is increasing from $(0, 1)$ onto $(1, \infty)$. For $n \geq 3$ it follows from [**AVV4**, Lemma 2.6(5)] that the function

$$f_n(r) = f_2(r) \cdot \frac{M_n(r)}{\mu(r)}$$

has the desired properties.

Next,

$$1 - g_n(r) = \frac{\log\frac{\lambda_n}{2} - \left(M_n(r) + \log\frac{r}{1+r'}\right)}{\log\frac{\lambda_n(1+r')}{2r}},$$

which, by Theorem 1.9, is increasing from $(0, 1)$ onto itself.   □

3. Distortion inequalities in $\mathbb{R}^n$. In this section we obtain estimates for the distortion functions $\varphi_{K,n}$, $\varphi^*_{K,n}$, and $\psi_{K,n}$ introduced in (1.4) and (1.5). In [**AVV3**, Theorem 2.2] it was shown that $\varphi^*_{K,n} \neq \varphi_{K,n}$ for $n \geq 3$, $K \neq 1$. We now show that $\varphi^*_{K,n} \neq \psi_{K,n}$ and also that $\varphi_{K,n}$ and $\psi_{K,n}$ are not comparable. In our first result we apply a method of O. Hübner [**H**] (cf. [**LV**, pp. 64, 65], [**AVV1**, p. 698]) along with our Theorem 1.9 to derive inequalities that improve earlier distortion estimates [**AVV1**, Theorem 4.10].

3.1. THEOREM. *For* $n \geq 2$, $K \geq 1$, $0 < r < 1$, *let* $\alpha = K^{1/(1-n)}$, $\beta = 1/\alpha$, $r' = \sqrt{1-r^2}$, $A(r) = r/(1+r')$. *Then*

$$(1) \qquad A(\varphi_{K,n}(r)) \leq \left(\frac{\lambda_n}{2}\right)^{1-\alpha} A(r)^\alpha \leq K A(r)^\alpha,$$

$$(2) \qquad A(\varphi_{1/K,n}(r)) \geq \left(\frac{\lambda_n}{2}\right)^{1-\beta} A(r)^\beta \geq K^{-\beta} A(r)^\beta,$$

$$(3) \qquad \varphi_{K,n}(r) \leq \left(\frac{\lambda_n}{2}\right)^{1-\alpha} (1+r')^{1-\alpha} r^\alpha \leq K(1+r')^{1-\alpha} r^\alpha,$$

$$(4) \qquad \varphi_{1/K,n}(r) \geq \left(\frac{\lambda_n}{2}\right)^{1-\beta} (1+r')^{1-\beta} r^\beta \geq K^{-\beta}(1+r')^{1-\beta} r^\beta.$$

Proof. Let $s = \varphi_{K,n}(r)$. Then $M_n(s) = \alpha M_n(r)$, $s \geq r$. Hence by Theorem 1.9

$$\log A(s) \leq (1-\alpha)(M_n(r) + \log A(r)) + \alpha \log A(r)$$

$$\leq (1-\alpha) \log\left(\frac{\lambda_n}{2}\right) + \alpha \log A(r) \leq \log K + \alpha \log A(r),$$

and (1) follows. Since $\varphi_{K,n}^{-1} = \varphi_{1/K,n}$, (2) follows from (1). Finally, (1) implies (3), and (2) implies (4). □

3.2. COROLLARY. For $n \geq 2$, $K_2 \geq K_1 > 0$, $\alpha = (K_2/K_1)^{1/(1-n)}$, $0 < r < 1$,

$$\varphi_{K_2,n}(r) \leq \left(\frac{\lambda_n}{2}\right)^{1-\alpha} \varphi_{K_1,n}^\alpha(r) \left(1 + \sqrt{1 - \varphi_{K_1,n}^2(r)}\right)^{1-\alpha}.$$

Proof. Since $\varphi_{K_2,n}(r) = \varphi_{K_2/K_1,n}(\varphi_{K_1,n}(r))$, the result follows from 3.1(3). □

3.3. REMARK. We can obtain analogous inequalities for $\varphi_{K,n}^*$ by combining (1.7) with Theorem 3.1.

3.4. *Proof of Theorem* 1.11. Part (1) follows from Theorem 3.1(1), since $2 \operatorname{arctanh} A(r) = \operatorname{arctanh} r$ and $LA(r)^\alpha < 1$ if $0 < r < r_1 = \tanh(2 \operatorname{arctanh} l)$. Part (2) follows similarly from Theorem 3.1(2). □

3.5. **Theorem.** *For* $n \geq 2$, $K \geq 1$, $\alpha = K^{1/(1-n)}$, $\beta = 1/\alpha$, $0 < r < 1$, $r' = \sqrt{1-r^2}$,

(1) $\tanh(\beta \operatorname{arctanh} r) \leq \varphi_{K,n}^*(r)$
$$\leq \tanh(\operatorname{arctanh} r + (\beta - 1)M_n(r'))$$
$$\leq \tanh(\beta \operatorname{arctanh} r + (\beta - 1)\log(\lambda_n/2))$$

(2) $\tanh(\alpha \operatorname{arctanh} r) \geq \varphi_{1/K,n}^*(r)$
$$\geq \tanh(\operatorname{arctanh} r + (\alpha - 1)M_n(r'))$$
$$\geq \tanh(\alpha \operatorname{arctanh} r + (\alpha - 1)\log(\lambda_n/2)).$$

The inequalities reduce to equality when $K = 1$*. Moreover,*

(3) $$\liminf_{r \to 0} r^{-\alpha} \operatorname{arctanh} \varphi_{K,n}^*(r) \geq 4^{1-\alpha},$$
$$\limsup_{r \to 0} r^{-\beta} \operatorname{arctanh} \varphi_{1/K,n}^*(r) \leq 4^{1-\beta},$$

(4) $$\lim_{r \to 1} \frac{\operatorname{arctanh} \varphi_{K,n}^*(r)}{\operatorname{arctanh} r} = \beta, \quad \lim_{r \to 1} \frac{\operatorname{arctanh} \varphi_{1/K,n}^*(r)}{\operatorname{arctanh} r} = \alpha,$$

whereas

(5) $$\lim_{r \to 0} r^{-\alpha} \operatorname{arctanh} \varphi_{K,n}(r) = \lambda_n^{1-\alpha},$$
$$\lim_{r \to 0} r^{-\beta} \operatorname{arctanh} \varphi_{1/K,n}(r) = \lambda_n^{1-\beta},$$

(6) $$\lim_{r \to 1} \frac{\operatorname{arctanh} \varphi_{K,n}(r)}{\operatorname{arctanh} r} = K, \quad \lim_{r \to 1} \frac{\operatorname{arctanh} \varphi_{1/K,n}(r)}{\operatorname{arctanh} r} = 1/K.$$

Proof. From [**AVV2**, 2.18] and [**AVV1**, 4.4]
$$\varphi_{K,n}^*(r) \geq \varphi_{\beta,2}(r) \geq \tanh(\beta \operatorname{arctanh} r)$$
and
$$\varphi_{1/K,n}^*(r) \leq \varphi_{\alpha,2}(r) \leq \tanh(\alpha \operatorname{arctanh} r).$$

Next, with $s = \varphi_{K,n}^*(r)$, $s' = \sqrt{1-s^2} \leq r' = \sqrt{1-r^2}$, Theorem 1.9 and (1.7) give
$$M_n(r') - \operatorname{arctanh} r \leq M_n(s') - \operatorname{arctanh} s \leq \beta M_n(r') - \operatorname{arctanh} s,$$

and the second inequality in (1) follows. Then Corollary 1.10(1) yields the third inequality in (1). The proof of the first and second inequalities in (2) is similar.

The first limit in (3) follows from [**AVV1**, Theorem 4.9] and [**LV**, p. 65]; and the second follows from the first by inversion. The limits in

(4) follow directly from (1) and (2); (5) follows from [**AVV4**, (3.7)]; and (6) is implied by [**AVV1**, Theorem 4.4]. □

Theorem 3.5 yields the following sharp quasiconformal distortion result for the hyperbolic metric in B^n.

3.6. COROLLARY. *If* $f \in QC_K(B^n)$ *and* $\beta = K^{1/(n-1)} = 1/\alpha$, *then*

$$(1) \qquad \rho(f(x), f(y)) \le \rho(x, y) + 2(\beta - 1)M_n \left(\operatorname{sech} \frac{\rho(x, y)}{2} \right)$$

$$\le \beta \rho(x, y) + 2(\beta - 1) \log \frac{\lambda_n}{2}$$

for all $x, y \in B^n$. *Moreover, if* $f(B^n) = B^n$, *then we also get the reverse inequality,*

$$(2) \qquad \alpha \rho(x, y) - 2(1 - \alpha) \log \frac{\lambda_n}{2}$$

$$\le \rho(x, y) - 2(1 - \alpha)M_n \left(\operatorname{sech} \left(\frac{\rho(x, y)}{2} \right) \right)$$

$$\le \rho(f(x), f(y)).$$

Proof. Denoting $\tanh(\rho/2) = r$, $\tanh(\rho'/2) = s$, $\rho = \rho(x, y)$, $\rho' = \rho(f(x), f(y))$, then applying 3.5(1) and [**Vu2**, 11.2], [**MRV**], we get

$$\rho' = 2 \operatorname{arctanh} s \le 2 \operatorname{arctanh} \varphi_{K, n}^*(r)$$

$$\le \rho(x, y) + 2(\beta - 1)M_n \left(\operatorname{sech} \left(\frac{\rho(x, y)}{2} \right) \right)$$

$$\le \beta \rho(x, y) + 2(\beta - 1) \log \frac{\lambda_n}{2},$$

proving (1). Inequality (2) follows from (1) applied to f^{-1}. □

3.7. COROLLARY. *For* $K \ge 1$, $n \ge 2$, $\beta = K^{1/(n-1)} = 1/\alpha$, $0 < r < 1$,

$$(1) \qquad \left(\frac{1+r}{1-r} \right)^\beta \le \frac{1 + \varphi_{K,n}^*(r)}{1 - \varphi_{K,n}^*(r)} \le \left(\frac{\lambda_n}{2} \right)^{2(\beta-1)} \left(\frac{1+r}{1-r} \right)^\beta,$$

$$(2) \qquad \left(\frac{\lambda_n}{2} \right)^{2(\alpha-1)} \left(\frac{1+r}{1-r} \right)^\alpha \le \frac{1 + \varphi_{1/K,n}^*(r)}{1 - \varphi_{1/K,n}^*(r)} \le \left(\frac{1+r}{1-r} \right)^\alpha.$$

Proof. Inequalities (1), (2) follow from Theorem 3.5(1), (2) respectively. □

In a recent paper D. Cooper [C] has shown that, for a class of home-omorphisms of B^n onto itself keeping 0 fixed, the image of $S^{n-1}(r)$ is close to $S^{n-1}(r)$ in a certain sense. (For an alternative proof see J. Väisälä [V3].) More precisely, Cooper showed, for a given mapping f in this class and for r close to 1 and $\varepsilon > 0$, that $S^{n-1}(r) = A \cup E$, where $A \cap E = \varnothing$ and

$$f(A) \subset \bigcup_{|z|=r} D(z, \varepsilon); \qquad D(z, \varepsilon) = \{w \in B^n : \rho(z, w) < \varepsilon\}$$

and the exceptional set E is small in the $(n-1)$-dimensional measure. Cooper's class of mappings includes quasiconformal mappings in particular. We shall now apply Corollary 3.7 to show that for every $\varepsilon > 0$ Cooper's exceptional set E is nonempty for r close to 1, even for K-quasiconformal mappings when $K > 1$. Indeed, with $s = \varphi^*_{K,n}(r)$ from Corollary 3.7 we obtain

$$\rho(S^{n-1}(r), S^{n-1}(s)) = \log\left(\frac{1+s}{1-s}\frac{1-r}{1+r}\right) \geq (\beta - 1) \cdot \log\frac{1+r}{1-r},$$

which tends to ∞ as $r \to 1$. Thus given $K > 1$ there exists a K-quasiconformal mapping f of B^n onto B^n keeping 0 fixed and for which there are points on $S^{n-1}(r)$ whose images under f are at hyperbolic distance at least $((\beta - 1)/2)\log((1 + r)/(1 - r))$ from $S^{n-1}(r)$. In other words, Cooper's exceptional set is nonempty for all $K > 1$ and for all r sufficiently close to 1.

3.8. *Proof of Theorem* 1.14. Let $r = \tanh(\rho/2)$, $s = \tanh(\rho'/2)$, $r' = \sqrt{1 - r^2}$, $s' = \sqrt{1 - s^2}$, and $\alpha = K^{1/(1-n)}$. Then by Theorem 3.1, [AVV4, (3.22)], and the Schwarz lemma [MRV, 3.1],

$$\tanh\frac{\rho'}{4} = \frac{s}{1+s'} \leq \frac{\varphi_{K,n}(r)}{1 + \sqrt{1 - \varphi_{K,n}(r)^2}} \leq \left(\frac{\lambda_n}{2}\right)^{1-\alpha}\left(\frac{r}{1+r'}\right)^\alpha$$

$$= \left(\frac{\lambda_n}{2}\right)^{1-\alpha}\left(\tanh\left(\frac{\rho}{4}\right)\right)^\alpha \leq K\left(\tanh\frac{\rho}{4}\right)^\alpha \leq K\left(\tanh\frac{\rho}{4}\right)^{1/K}.$$

When $K = 1$ the inequality in the theorem reduces to $\rho' \leq \rho$, which by conformal invariance is an equality if f is one-to-one and $f(B^n) = B^n$. $\qquad\square$

3.9. *Proof of Corollary* 1.15. This follows immediately from Theorem 1.14 and [AVV2, (2.5)]. $\qquad\square$

We shall require the following simple inequality (cf. e.g. [**Vu2**, (2.27)]):

$$(3.10) \qquad |x - y| \leq 2 \tanh \frac{\rho(x, y)}{4} \leq \frac{|x - y|}{\sqrt{(1 - |x|^2)(1 - |y|^2)}}$$

for $x, y \in B^n$.

3.11. *Proof of Theorem* 1.12. For (1), by Theorem 1.14 and (3.10) for $x, y \in \overline{B}^n$, with $\rho = \rho(x, y)$, $\rho' = \rho(f(x), f(y))$, we have

$$|f(x) - f(y)| \leq 2 \tanh \frac{\rho'}{4} \leq 2 \left(\frac{\lambda_n}{2}\right)^{1-\alpha} \left(\tanh \frac{\rho}{4}\right)^\alpha$$

$$\leq \lambda_n^{1-\alpha}(1 - r^2)^{-\alpha}|x - y|^\alpha.$$

The proof of (2) is similar.   □

The next result is elementary.

3.12. LEMMA. *For $K \geq 1$ and $r \in (0, 1)$, $r' = \sqrt{1 - r^2}$, the following inequalities hold*:

$$(1) \qquad r < \sqrt{1 - r'^{2K}} \leq \tanh(K \operatorname{arctanh} r)$$

$$\leq \min\{Kr, \sqrt{1 - 4^{1-K} r'^{2K}}\},$$

$$(2) \qquad \sqrt{\max\left\{\left(\frac{r}{K}\right)^2, 1 - 4^{1-1/K} r'^{2/K}\right\}} \leq \tanh\left(\frac{1}{K} \operatorname{arctanh} r\right)$$

$$\leq \sqrt{1 - r'^{2/K}} < r.$$

3.13. THEOREM. *For $n \geq 2$, $0 \leq r \leq 1$, $1 \leq K < \infty$, $\beta = K^{1/(n-1)}$, $r' = \sqrt{1 - r^2}$*,

$$\sqrt{1 - r'^{2\beta}} \leq \varphi^*_{K,n}(r) \leq \psi_{K,n}(r) \leq \sqrt{1 - \lambda_n^{2(1-\beta)} r'^{2\beta}}.$$

Equality holds for $K = 1$. The upper bound for $\psi_{K,n}(r)$ is asymptotically sharp as r tends to 1.

Proof. By Theorem 3.5(1) and Lemma 3.12 we obtain

$$\varphi^*_{K,n}(r) \geq \tanh(\beta \operatorname{arctanh} r) \geq \sqrt{1 - r'^{2\beta}}.$$

The second inequality follows from (1.7). Next, by (1.4) and Theorem 3.1(4) we have

$$\psi^2_{K,n}(r) = 1 - \varphi^2_{1/K,n}(r') \leq 1 - \lambda_n^{2(1-\beta)} r'^{2\beta},$$

proving the third inequality. The statement about equality is clear. Finally, the third inequality is asymptotically sharp as $r \to 1$ since

$$\lim_{r \to 1-} r'^{-2\beta}(1 - \psi^2_{K,n}(r)) = \lim_{r \to 1-} r'^{-2\beta}\varphi^2_{1/K,n}(r') = \lambda_n^{2(1-\beta)}$$

by (1.4) and [**AVV1**, (4.12)]. □

3.14. Corollary. *Let* $n \geq 3$, $m \geq 2$, $1 < K < \infty$, *and* $0 < r < 1$.
(1) *If* r *is close to* 1 *then* $\varphi^*_{K,n}(r) < \varphi_{K,m}(r)$.
(2) *If* $m > n$ *and* r *is close to* 0, *then the above inequality is reversed.*

Proof. For (1) choose r such that $(1 - \lambda_n^c)^{1/2} < r < 1$, where $c = 2(1 - \beta)/(K - \beta)$, $\beta = K^{1/(n-1)}$. Then

$$1 - \lambda_n^{2(1-\beta)}r'^{2\beta} < 1 - r'^{2K}$$

and the result follows from Lemma 3.12 and Theorem 3.13.
By [**AVV1**, Theorem 4.10] and [**AVV2**, (2.18)] we get

$$\lim_{r \to 0} \varphi^*_{K,n}(r)/\varphi_{K,m}(r) = \infty,$$

and hence (2) follows. □

4. High dimensions. In this section we prove Theorem 1.17, by using the following result from [**Vu3**].

4.1. Theorem. *The linear dilatation in* (1.16) *has the following majorant*:

$$H_n(K) \leq \inf_{0 < t < 1} \frac{A_{K,n}(t)}{B_{K,n}(t)},$$

where

$$A_{K,n}(t) = \frac{\varphi^2_{K,n}(\sqrt{t})}{1 - \varphi^2_{K,n}(\sqrt{t})}, \quad B_{K,n}(t) = \varphi^2_{1/K,n}(\sqrt{t/(1+t)}).$$

4.2. *Proof of Theorem* 1.17. Fix $t \in (0, 4^{1-K}K^{-2})$. Then by Theorem 3.1(3), (4) and [**AVV4**, (3.23)],

$$\limsup_{n \to \infty} A_{K,n}(t) \leq \lim_{n \to \infty} \frac{\lambda_n^{2(1-\alpha)}t^\alpha}{1 - \lambda_n^{2(1-\alpha)}t^\alpha} = \frac{K^2 t}{1 - K^2 t},$$

$$\liminf_{n \to \infty} B_{K,n}(t) \geq \lim_{n \to \infty} \lambda_n^{2(1-\beta)}\left(\frac{t}{1+t}\right)^\beta = \frac{t}{K^2(1+t)}.$$

Hence

$$\limsup_{n\to\infty} \frac{A_{K,n}(t))}{B_{K,n}(t)} \le \frac{K^4(1+t)}{1-K^2t},$$

so that

$$\limsup_{n\to\infty} H_n(K) \le \limsup_{n\to\infty} \left(\inf_{0<t<1} \frac{A_{K,n}(t)}{B_{K,n}(t)} \right) \le K^4. \qquad \Box$$

4.3. Conjectures. (1) $\lim_{n\to\infty} H_n(K) = 1$ for all $K > 1$.

(2) $\lim_{n\to\infty} \varphi^*_{K,n}(r) = r$ for all $K > 1$ and $0 < r < 1$.

(3) $M_n(r)$ can be replaced by $M_n^{n-1}(r)$ in Theorem 2.6.

4.4. REMARK. We observe that [AVV1, (4.5), (4.11)] implies that

$$r < \liminf_{n\to\infty} \varphi_{K,n}(r) \le \limsup_{n\to\infty} \varphi_{K,n}(r) \le Kr$$

for all $K > 1$ and $0 < r < 1$, so that the analog of 4.3(2) is false for $\varphi_{K,n}(r)$ if $K > 1$.

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