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JORDAN ANALOGS OF THE BURNSIDE AND JACOBSON DENSITY THEOREMS

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If $\mathscr{A}$ is an (associative) algebra of linear operators on a vector space, it is well known that 2-transitivity for $\mathscr{A}$ implies density and, in certain situations, transitivity guarantees 2-transitivity. In this paper we consider analogs of these results for Jordan algebras of linear operators with the standard Jordan product.

0. Introduction. Let $\mathscr{L}(\mathscr{V})$ be the algebra of all linear operators on a vector space $\mathscr{V}$ over the field $\mathbb{F}$. A subset $\mathscr{S}$ of $\mathscr{L}(\mathscr{V})$ is called transitive if $\mathscr{S}x = \mathscr{V}$ for every nonzero x in $\mathscr{V}$. More generally, $\mathscr{S}$ is called k-transitive if given linearly independent vectors $x_1, x_2, \dots, x_k$ and arbitrary vectors $y_1, y_2, \dots, y_k$ in $\mathscr{V}$ there exists a member S of $\mathscr{S}$ such that $Sx_i = y_i$, $i = 1, 2, \dots, k$. If $\mathscr{S}$ is k-transitive for every k, then it is called (strictly) dense. It is a remarkable fact due to Jacobson [2] that if $\mathscr{S}$ is an (associative) subalgebra of $\mathscr{L}(\mathscr{V})$, then 2-transitivity implies density for arbitrary $\mathbb{F}$. In particular, if $\mathscr{V}$ is finite-dimensional, then $\mathscr{L}(\mathscr{V})$ is the only 2-transitive algebra on $\mathscr{V}$. There are transitive algebras that are not 2-transitive even if $\mathbb{F}$ is algebraically closed. In the presence of certain conditions (e.g., topological) transitivity implies density. The most well-known result of this kind is Burnside's theorem [3]: if $\mathscr{V}$ is finite-dimensional and $\mathbb{F}$ is algebraically closed, then the only transitive algebra over $\mathscr{V}$ is $\mathscr{L}(\mathscr{V})$.

In this paper we consider analogs of these results for Jordan algebras of operators: linear spaces $\mathscr{A}$ of operators such that A^2 and ABA belong to $\mathscr{A}$ for all A and B in $\mathscr{A}$. If the characteristic of the field $\mathbb{F}$ is different from 2, this is equivalent to the requirement that $\mathscr{A}$ be closed under the Jordan bracket $\{A, B\} = AB + BA$. Over this kind of field a Jordan algebra $\mathscr{A}$ may be equivalently defined as a linear space closed under taking positive integral powers. For the sake of completeness we include proofs of a few elementary facts obtainable from the general theory of Jordan algebras [4].

In what follows we often find it convenient to view members of $\mathscr{L}(\mathscr{V})$ as matrices over $\mathbb{F}$; this should cause no confusion. The set of all $n \times n$ matrices over $\mathbb{F}$ will be denoted by $\mathscr{M}_n(\mathbb{F})$. A member A of $\mathscr{L}(\mathscr{V})$ (or $\mathscr{M}_n(\mathbb{F})$) is called a projection or an idempotent element if $A^2 = A$.

1. Transitive Jordan algebras over arbitrary fields.

1.0. All the Jordan algebras $\mathscr{A}$ considered in this section are subalgebras of the algebra of all linear operators $\mathscr{L}(\mathscr{V})$ on a vector space $\mathscr{V}$ over a field $\mathbb{F}$. In finite dimensions Jacobson's theorem says that any 2-transitive associative algebra of linear operators of $\mathscr{V}$ is all of $\mathscr{L}(\mathscr{V})$ [2]. The proof of this result for Jordan algebras of operators needs some preparation.

1.1. Proposition. *Let $\mathscr{A}$ be a Jordan algebra of linear operators on a vector space $\mathscr{V}$. Then:*

(a) *$E\mathscr{A}E$ and $(I-E)\mathscr{A}(I-E)$ are Jordan subalgebras of $\mathscr{A}$ for every E in $\mathscr{A}$.*

(b) *If $\mathscr{V}$ is finite dimensional and $\mathscr{A}$ is 2-transitive, then for every subspace $\mathscr{W}$ of $\mathscr{V}$ there exists a projection $E \in \mathscr{A}$ such that $E\mathscr{V} = \mathscr{W}$.*

(c) *If $\mathscr{V}$ is finite dimensional and $\mathscr{A}$ is 2-transitive, then $I \in \mathscr{A}$.*

Proof. (a) follows directly from the definition and from the observation that $(I-E)\mathscr{A}(I-E) = A - EA - AE + EAE$.

(b) Assume first that $\mathscr{W}$ is a 1-dimensional subspace. By 2-transitivity there exists a singular $A \in \mathscr{A}$ such that $A\mathscr{W} = \mathscr{W}$. Choose A to be of minimal rank having this property and write it in the form $A = J \oplus N$, where J is invertible and N is nilpotent. As all the powers of A are in $\mathscr{A}$ and its rank is minimal, we have necessarily that $N = 0$. The minimal polynomial $p(t) = \sum_{0\le i\le m} a_i t^i$ of J has nonzero constant term a_0 because J is invertible. Thus, $I = -(\sum_{1\le i\le m} a_i J^i)/a_0$ is the identity operator on the range of A and the idempotent $E = -(\sum_{1\le i\le m} a_i A^i)/a_0 = I \oplus 0$ is in $\mathscr{A}$. Moreover, $\operatorname{rank} E = \operatorname{rank} A$ and $E\mathscr{W} = \mathscr{W}$. If the rank of E is strictly greater than 1, then let $E\mathscr{X}$ be a 1-dimensional subspace in the range of E distinct from $E\mathscr{W} = \mathscr{W}$. By 2-transitivity there is a $B \in \mathscr{A}$ such that $BE\mathscr{W} = \mathscr{W}$ and $BE\mathscr{X} = 0$. But then $EBE\mathscr{W} = \mathscr{W}$ and $EBE\mathscr{X} = 0$ so that the rank of EBE, which is in $\mathscr{A}$ by part (a), is strictly smaller than the rank of A contradicting the minimality assumption.

The rest follows by induction on the dimension of $\mathscr{W}$. Let $\mathscr{X}$ be a subspace of codimension 1 in $\mathscr{W}$ and $E \in \mathscr{A}$ a projection such that $E\mathscr{V} = \mathscr{X}$. Note that $\mathscr{Y} = \mathscr{W} \cap \ker E$ has dimension 1 and that $\mathscr{W} = \mathscr{X} \oplus \mathscr{Y}$. Let $F \in \mathscr{A}$ be a projection such that $F\mathscr{V} = \mathscr{Y}$; then $\mathscr{V} = \mathscr{W} \oplus \mathscr{U}$, where $\mathscr{U} = \ker E \cap \ker F$. Let P be a projection in $\mathscr{V}$ on $\mathscr{W}$ along $\mathscr{U}$; then $N = E + F - P$ has square equal to zero and therefore, $P = 2(E+F) - (E+F)^2$ is in $\mathscr{A}$. To get (c) take $\mathscr{W} = \mathscr{V}$ in (b). □

Some of the proofs of the following results could be shortened slightly, at the expense of keeping the paper self-contained, by using the Pierce decomposition associated with an idempotent.

1.2. THEOREM. *Let $\mathscr{A}$ be a Jordan algebra of linear operators on a finite dimensional vector space $\mathscr{V}$. Then $\mathscr{A}$ is 2-transitive if and only if $\mathscr{A} = \mathscr{L}(\mathscr{V})$.*

Proof. $\mathscr{L}(\mathscr{V})$ is clearly 2-transitive. The converse is proved by induction on the dimension of $\mathscr{V}$. The assertion obviously holds if $\dim \mathscr{V} = 2$. So, let $\dim \mathscr{V} > 2$. Let $\mathscr{X}$ be a 1-dimensional subspace of $\mathscr{V}$ and find by 1.1(b) a projection $E \in \mathscr{A}$ such that $E\mathscr{V} = \mathscr{X}$. Next, find a 1-dimensional subspace $\mathscr{Y} \subset \ker E$ and corresponding projection $F \in \mathscr{A}$ such that $F\mathscr{V} = \mathscr{Y}$. It is clear that $EF = 0$ and with no loss of generality we may assume that $FE = 0$ as well, since otherwise, we could replace F by $F - FE = (I-E)F(I-E) \in \mathscr{A}$. The Jordan subalgebra $\mathscr{B} = (I-E)\mathscr{A}(I-E)$, respectively $\mathscr{C} = (I-F)\mathscr{A}(I-F)$, of $\mathscr{A}$ can be viewed as a 2-transitive algebra of operators on $\ker E$, respectively $\ker F$, and by induction hypothesis $\mathscr{B} = \mathscr{L}(\ker \mathscr{E})$, respectively $\mathscr{C} = \mathscr{L}(\ker F)$. The subalgebra $\mathscr{C}$ is also called the Pierce zero-space relative to F. Choose now any $T \in \mathscr{L}(\mathscr{V})$ and let us show that $T \in \mathscr{A}$. By 2-transitivity we may assume with no loss of generality that $ETF = FTE = 0$. But, then, $T = R + S$, where $R = (I-E)T(I-E) \in \mathscr{B}$ and $S = ET + TE - ETE \in \mathscr{C}$. □

Theorem 1.2 can be generalized as follows for Jordan algebras of finite rank operators. For a further strengthening of this result see Theorem 3.4.

1.3. THEOREM. *Let $\mathscr{A}$ be a Jordan algebra of finite rank operators on a vector space $\mathscr{V}$. If $\mathscr{A}$ is 2-transitive, then it is dense, i.e. n-transitive for all $n \geq 1$.*

Proof. This can be done by reduction to the finite dimensional case. Observe that the proof of 1.1.(b) remains valid if we replace the condition " $\mathscr{V}$ is finite dimensional" by weaker conditions "$\mathscr{V}$ is finite dimensional and the elements of $\mathscr{A}$ have finite rank". Therefore, we can find for every finite dimensional subspace $\mathscr{W}$ of $\mathscr{V}$ a projection E in $\mathscr{A}$ such that $E\mathscr{V} = \mathscr{W}$. By 1.1.(a) $E\mathscr{A}E$ is a Jordan subalgebra of $\mathscr{A}$ and it is 2-transitive as a Jordan algebra of operators on $E\mathscr{V} = \mathscr{W}$. Thus, $E\mathscr{A}E = \mathscr{L}(\mathscr{W})$ by 1.2. If vectors $\{x_1, x_2, \dots, x_k\} \subset \mathscr{V}$ are linearly independent and vectors $\{y_1, y_2, \dots, y_k\} \subset \mathscr{V}$ are arbitrary, then apply this consideration to the span $\mathscr{W}$ of these two sets of vectors. □

2. Some characterizations of proper transitive Jordan algebras.

2.0. In this section we shall assume that the characteristic of the field $\mathbb{F}$ is different from 2. Let $\mathscr{S}_n(\mathbb{F})$ be the (transitive) Jordan algebra of all symmetric $n \times n$ matrices over $\mathbb{F}$. We give a proof that if $\mathbb{F}$ is algebraically closed, then $\mathscr{S}_n(\mathbb{F})$ is, up to similarlity, the only proper transitive Jordan subalgebra of $\mathscr{M}_n(\mathbb{F})$. This, of course, does not hold if $\mathbb{F}$ is not algebraically closed. However, for a formally real closed field the algebra $\mathscr{S}_n(\mathbb{F})$ has no proper transitive Jordan subalgebras. These results do not seem to be easily derivable from Jacobson's general structure theorems for Jordan matrix algebras [**4**]; our presentation here is self-contained and elementary.

2.1. THEOREM. *Let $\mathbb{F}$ be any formally real closed field. Then, the only transitive Jordan algebra of symmetric $n \times n$ matrices over $\mathbb{F}$ is $\mathscr{S}_n(\mathbb{F})$.*

Proof. We shall use induction on n. The assertion is trivial for $n = 1$. So, assume $\mathscr{A}$ is a transitive Jordan subalgebra of $\mathscr{S}_n(\mathbb{F})$ with $n \geq 2$. Let E be an idempotent of minimal positive rank in $\mathscr{A}$. Idempotents abound in $\mathscr{A}$ because in the spectral decomposition

$$A = \sum \lambda_i E_i, \quad E_i^2 = E_i, \quad E_i E_j = 0, \quad i \neq j,$$

for a member A of $\mathscr{A}$, every E_i corresponding to a nonzero λ_i is a polynomial in A (with constant term zero) and thus belongs to $\mathscr{A}$. The existence of spectral decompositions in $\mathscr{A}$ follows from the fact that $\mathbb{F}$ is real-closed [**5**].

The transitivity of $\mathscr{A}$ implies that it has nonscalar members, so that $E \neq I$. Since E is symmetric there exists an invertible matrix T with $T^{-1} = T^t$ such that

$$T^{-1}ET = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix},$$

where k is the rank of E, $0 < k < n$. Replacing $\mathscr{A}$ by $T^{-1}\mathscr{A}T$ we can assume that

$$E = \begin{pmatrix} I_k & 0 \\ 0 & 0 \end{pmatrix}.$$

Writing the corresponding matrix for a typical member of $\mathscr{A}$

$$A = \begin{pmatrix} X & Y' \\ Y & Z \end{pmatrix},$$

we observe that

$$B = EAE - (I-E)A(I-E) = \begin{pmatrix} X & 0 \\ 0 & -Z \end{pmatrix} \in \mathscr{A},$$

and thus

$$\frac{1}{2}(BE+EB) = \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix} - B = \begin{pmatrix} 0 & 0 \\ 0 & Z \end{pmatrix}$$

are in $\mathscr{A}$. By Proposition 1.1(a) we conclude that $E\mathscr{A}E$ and $(I-E)\mathscr{A}(I-E)$ are Jordan subalgebras of $\mathscr{A}$; they are also easily seen to be transitive on respective spaces $\operatorname{im} E$ and $\ker E$. Hence, by the inductive hypothesis, $E\mathscr{A}E = \mathscr{S}_k(\mathbb{F})$ and $(I-E)\mathscr{A}(I-E) = \mathscr{S}_{n-k}(\mathbb{F})$. This means that $\mathscr{A}$ contains all symmetric matrices of the form $\left(\begin{smallmatrix} L & 0 \\ 0 & M \end{smallmatrix}\right)$. In particular, $k = 1$ by minimality. To complete the proof observe that the transitivity of $\mathscr{A}$ forces it to contain a matrix with an arbitrarily assigned first column. Thus, for a given $(n-1)\times 1$ matrix N there is a member $\left(\begin{smallmatrix} L & N^t \\ N & M \end{smallmatrix}\right)$ in $\mathscr{A}$ with some L and M. Since by the argument above, $L = L^t$ and $M = M^t$ are arbitrary in this expression, we have that $\mathscr{A} = \mathscr{S}_n(\mathbb{F})$. □

The following example shows that the hypothesis of real closure in the theorem is needed. Let $\mathbb{F}$ be the field $\mathbb{Q}$ of rational numbers and

$$\mathscr{A} = \left\{ \begin{pmatrix} a+b & b \\ b & a-b \end{pmatrix} : a, b \in \mathbb{Q} \right\}.$$

Then, $\mathscr{A}$ is a proper Jordan subalgebra of $\mathscr{S}_2(\mathbb{Q})$. It is easily seen that $\mathscr{A}$ is transitive: it is generated by I and

$$A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

The minimal polynomial of A is irreducible over $\mathbb{Q}$, and thus if x is any nonzero vector, then the span of x and Ax is the whole underlying space.

Our next theorem is a more general result in the case of algebraically closed fields; it includes, as a corollary, the analog of the above theorem. We need the following lemmas.

2.2. LEMMA. *Let $\mathscr{A}$ be a transitive Jordan algebra of $n\times n$ matrices over an algebraically closed field $\mathbb{F}$. Then $\mathscr{A}$ contains an idempotent of rank* 1.

Proof. For $n=1$ this is trivially true. We first prove it for $n=2$. Assume there is no idempotent of rank 1 in this case. This implies, by considering spectral projections of matrices in $\mathscr{A}$, that every member of $\mathscr{A}$ has singleton spectrum, i.e., it is of the form $N+\alpha I$ with N nilpotent. Since the characteristic of $\mathbb{F}$ is different from 2, a member of $\mathscr{A}$ is nilpotent if and only if it has trace zero.

If $\mathscr{A}$ consists of nilpotents alone, then $0=(A+B)^2=A^2+B^2+AB+BA=AB+BA$ for all A, $B\in\mathscr{A}$ implying that if $B\neq 0$, then its range is invariant under every $A\in\mathscr{A}$. Thus, $\mathscr{A}$ is triangularizable; this contradicts the transitivity of $\mathscr{A}$. Thus, we can assume that $\mathscr{A}$ has an invertible member, which implies, by taking an appropriate polynomial, that $I\in\mathscr{A}$. Hence, N is in $\mathscr{A}$ for every $N+\alpha I$ in $\mathscr{A}$. Let $\mathscr{A}_0$ be the set of all nilpotent elements in the algebra. It follows that for A and B in $\mathscr{A}_0$ and $\alpha\in\mathbb{F}$, the matrix $A+\alpha B$ has trace zero and is thus nilpotent. In particular $A+B$ is nilpotent and hence $AB+BA=(A+B)^2-A^2-B^2=0$. This shows that $\mathscr{A}_0$ is a Jordan algebra. We see, as before, that $\mathscr{A}_0$ is triangularizable and so is $\mathscr{A}=\mathscr{A}_0+\mathbb{F}I$, contradicting the transitivity of $\mathscr{A}$.

We can now assume $n>2$. If $\mathscr{A}$ contains a nontrivial idempotent, i.e., an idempotent E with $0<\operatorname{rank}E<n$, then by (1.1)(a) $E\mathscr{A}E$ is a Jordan subalgebra of $\mathscr{A}$ which is forced, by the transitivity of $\mathscr{A}$, to be transitive as an algebra of operators acting on the range of E. We conclude, by induction on n, that $E\mathscr{A}E$ and thus $\mathscr{A}$ contain idempotents of rank 1. To complete the proof we must only show the existence of a nontrivial E.

Assume $\mathscr{A}$ contains no nontrivial idempotent. Then, as in the first paragraph of the proof, every member of $\mathscr{A}$ is seen to be of the form $N+\alpha I$ with N nilpotent. There must be nonzero nilpotent matrices in $\mathscr{A}$. (Just observe that if $N+\alpha I\in A$ with $\alpha\neq 0$, then, considering the characteristic polynomial of this matrix, we show that $I\in\mathscr{A}$ and hence $N\in\mathscr{A}$. Surely $\mathscr{A}$ cannot consist of scalar matrices.) Let N be a nilpotent member of $\mathscr{A}$ with minimal positive rank. Since N^2 has smaller rank than N, we must have $N^2=0$. We next show that N has rank 1. Considering members of $\mathscr{A}$ as operators on $\mathscr{V}$ and noting that the kernel of N contains its range, let $\mathscr{V}_1$ be the range of N, $\mathscr{V}_2$ a complement of $\mathscr{V}_1$ in the kernel of N, and $\mathscr{V}_3$ a complement of $\mathscr{V}_1 \oplus \mathscr{V}_2$ in $\mathscr{V}$. With respect to the decomposition $\mathscr{V} = \mathscr{V}_1 \oplus \mathscr{V}_2 \oplus \mathscr{V}_3$ (where $\mathscr{V}_2$ may of course be zero) and with an appropriate choice of basis, N will have the form

$$\begin{pmatrix} 0 & 0 & I_k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where k is the rank of N. If the corresponding block matrix of a typical A in $\mathscr{A}$ is $(A_{ij})_{i,j=1}^3$, then $NAN \in \mathscr{A}$ and its matrix equals

$$\begin{pmatrix} 0 & 0 & A_{31} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The matrix $NAN - \alpha N$ is a nilpotent member of $\mathscr{A}$ and all its blocks except $A_{31} - \alpha I_k$ are zero. The minimality of the rank k forces the block A_{31} of every A to be scalar. But this would contradict the transitivity of $\mathscr{A}$ if $k > 1$. Thus, $k = 1$. (These facts about algebras consisting of scalar translations of nilpotent operators can also be deduced from more sophisticated results on Jordan algebras [6].)

Finally, we shall exhibit a single member of $\mathscr{A}$ with an eigenvalue 1, showing that not every member of $\mathscr{A}$ is the sum of a nilpotent and a scalar. To this end, pick $x \in \mathscr{V}$ with $Nx \neq 0$. By transitivity, there is an $A \in \mathscr{A}$ such that $A(Nx) = x$. Then, $N^2 = 0$ implies $(NA + AN)(Nx) = NANx = Nx$. Since N has rank 1, the matrix $NA + AN$ has rank at most 2. Since $n > 2$, this matrix is a singular member of $\mathscr{A}$. □

2.3. Lemma. *If $\mathscr{A}$ satisfies the hypotheses of Lemma 2.2, then $\mathscr{A}$ contains idempotents E_i, $i = 1, 2, \dots, n$, of rank one with $E_iE_j = 0$ for $i \neq j$.*

Proof. We shall use induction on n. Let $n \geq 2$ and assume the assertion true for $n - 1$. Let $E_1 = E$ be an idempotent of rank one in $\mathscr{A}$ as in Lemma 1.1. We can assume with no loss of generality that $E = \operatorname{diag}(1, 0, \dots, 0)$. The Jordan algebra $(I - E)\mathscr{A}(I - E)$ is contained in $\mathscr{A}$ by Proposition 1.1(a). Since it also acts transitively on the range of $I - E$, which has dimension $n - 1$, we conclude from the inductive hypothesis that $(I - E)\mathscr{A}(I - E)$ contains idempotents $E_2, \dots, E_n$ with the desired property. The proof is completed by observing that $E_1E_j = E_jE_1 = 0$ for $j \geq 2$. □

2.4. THEOREM. *Let $\mathbb{F}$ be an algebraically closed field, and let $\mathscr{A}$ be a transitive Jordan algebra of $n \times n$ matrices over $\mathbb{F}$. Then either $\mathscr{A} = \mathscr{M}_n(\mathbb{F})$ or there exists an invertible matrix T such that $T^{-1}\mathscr{A}T = \mathscr{S}_n(\mathbb{F})$.*

Proof. We shall show first that $\mathscr{A}$ contains $\mathscr{S}_n(\mathbb{F})$ up to a similarity. By Lemma 2.3 we can assume that $\mathscr{A}$ contains diagonal idempotents E_j of rank one: $E_1 = \operatorname{diag}(1, 0, \dots, 0), \dots, E_n = \operatorname{diag}(0, \dots, 0, 1)$. Consider the special case of $n = 2$. In this case the transitivity of $\mathscr{A}$ implies that its dimension is either 3 or 4. If the dimension is 4, then $\mathscr{A} = \mathscr{M}_2(\mathbb{F})$; if it is 3, then $\mathscr{A}$ contains a nonzero matrix of the form $\left(\begin{smallmatrix} 0 & t \\ s & 0 \end{smallmatrix}\right)$ (after adding a suitable linear combination of E_1 and E_2). Now, both s and t have to be nonzero by transitivity. Thus, we have shown that, when $n = 2$, the algebra $\mathscr{A}$ must contain a matrix of the above form with $s = 1$ and $t \neq 0$.

Returning now to the general case, let $\{M_{ij}\}$ be the set of matrix units, i.e., the only nonzero entry of M_{ij} occurs at the (i, j) position and equals 1. Observe that for $j > 1$ the Jordan subalgebra $(E_1 + E_j)\mathscr{A}(E_1 + E_j)$ acts transitively on the 2-dimensional range of $E_1 + E_j$. As in the paragraph above, it must contain, together with E_1 and E_j, at least one matrix of the form $A_j = M_{1j} + t_j M_{j1}$. Letting $T = \operatorname{diag}(1, \sqrt{t_2}, \dots, \sqrt{t_n})$, we see that the Jordan algebra $T\mathscr{A}T^{-1}$ contains the symmetric matrices

$$B_{j1} = \frac{1}{\sqrt{t_j}} T A_j T^{-1} = M_{j1} + M_{1j}$$

(and, of course, $E_1, \dots, E_n$).

If 1, i, and j are distinct, then $B_{ij} = B_{i1}B_{j1} + B_{j1}B_{i1} \in T\mathscr{A}T^{-1}$. Observe that $B_{ij} = M_{ij} + M_{ji}$. We have shown that $T\mathscr{A}T^{-1}$ contains a basis for symmetric matrices. Hence, $T\mathscr{A}T^{-1} \supset \mathscr{S}_n(\mathbb{F})$.

To complete the proof of the theorem it suffices to show that if $\mathscr{A}$ contains $\mathscr{S}_n(\mathbb{F})$ properly, then $\mathscr{A} = \mathscr{M}_n(\mathbb{F})$. Thus, assume $\mathscr{A}$ contains a nonsymmetric matrix $C = (c_{ij})$. Some principal 2×2 submatrix must be nonsymmetric and by passing from $\mathscr{A}$ to $P^{-1}\mathscr{A}P$, where P is a permutation matrix, we can assume $c_{12} \neq c_{21}$. Observe that the matrix

$$M = (E_1 + E_2)C(E_1 + E_2) - c_{11}E_1 - c_{22}E_2 - c_{21}B_{21}$$

belongs to $\mathscr{A}$ and is a nonzero scalar multiple of M_{12}. We shall show that $M_{ij} \in \mathscr{A}$, $i, j = 1, \dots, n$. Every M_{ii} is in $\mathscr{A}$ and we have just seen that M_{12} and hence $M_{21} = B_{21} - M_{12}$ are in $\mathscr{A}$.

For $j > 2$, $M_{1j} = M_{12}B_{2j} + B_{2j}M_{12}$, and $M_{j1} = B_{j1} - M_{1j}$ which implies that M_{1j} and M_{j1} are in $\mathscr{A}$. Similarly, for $j > i > 1$, $M_{ji} = M_{1i}B_{j1} + B_{j1}M_{1i}$, and $M_{ij} = B_{ij} - M_{ji}$ so that M_{ji} and M_{ij} are in $\mathscr{A}$. □

2.5. Corollary. *Let* $\mathbb{F}$ *be an algebraically closed field. If* $\mathscr{A}$ *is a transitive Jordan algebra of symmetric* $n \times n$ *matrices over* $\mathbb{F}$, *then* $\mathscr{A} = \mathscr{S}_n(\mathbb{F})$.

The example given after Theorem 2.1 can be modified to show that the algebraic closure hypothesis is essential in the preceding result. Consider $\mathbb{Q}(i)$ instead of $\mathbb{Q}$ and let

$$\mathscr{A} = \left\{ \begin{pmatrix} a+b & b \\ b & a-b \end{pmatrix} : a, b \in \mathbb{Q}(i) \right\}.$$

Then, $\mathscr{A}$ is a proper Jordan subalgebra of $\mathscr{S}_2(\mathbb{Q}(i))$; it is also transitive.

The following example shows that the assumption $\operatorname{char}\mathbb{F} \neq 2$ is essential in the results above: the 3-dimensional Jordan algebra spanned over $\mathbb{F}_2$ by $\{I, M_{12}, M_{21}\}$ is transitive and contains no idempotent of rank 1.

3. Results on ideals.

3.0. We continue to assume that the characteristic of the field $\mathbb{F}$ is different from 2. In the associative algebra case some transitivity properties are inherited by ideals. This is of course trivial if $\dim \mathscr{V}$ is finite, since then $\mathscr{L}(\mathscr{V})$ is simple. In the Jordan case, restriction to ideals seems to be accompanied with some loss of transitivity. The following result is well known for general associative algebras [**2**].

3.1. Proposition. *Let* $\mathscr{J} \neq 0$ *be an ideal in an associative algebra* $\mathscr{A}$ *of operators on a vector space* $\mathscr{V}$. *If* $\mathscr{A}$ *is n-transitive, then so is* $\mathscr{J}$.

Here are our results on this question for Jordan algebras of linear operators and their Jordan ideals.

3.2. Theorem. *Every Jordan ideal* $\mathscr{J} \neq 0$ *of an* $(n+1)$*-transitive Jordan algebra* $\mathscr{A}$ *of operators on a vector space* $\mathscr{V}$ *is n-transitive,* $n \geq 1$.

Proof. If $\mathscr{V}$ is $(n+1)$-dimensional then $\mathscr{A} = \mathscr{L}(\mathscr{V})$ and, by [**1**, Theorem 1], we have $\mathscr{J} = \mathscr{A}$. So, assume with no loss of generality that $\mathscr{V}$ contains $n+2$ linearly independent vectors. Fix a linearly independent set of vectors $\{x_1, \dots, x_n\} \subset \mathscr{V}$ and let $\mathscr{X}$ and $\mathscr{Y}$ be the span of $\{x_1, \dots, x_n\}$ and of $\{x_1, \dots, x_{n-1}\}$, respectively. We will show first that

(a) $\exists J \in \mathscr{J}$ such that $J\mathscr{Y} = 0$, $Jx_n \notin \mathscr{X}$.

Assume the contrary; then

(b) $J \in \mathscr{J}$ and $J\mathscr{Y} = 0$ implies $Jx_n \in \mathscr{X}$.

Then, for any $A \in \mathscr{A}$ such that $A\mathscr{Y} = 0$ it holds that $K = JA + AJ$ is in $\mathscr{J}$ and that $K\mathscr{Y} = 0$. Therefore by (b), Kx_n belongs to $\mathscr{X}$. Let α be such that $Kx_n = (J+\alpha)Ax_n$. Since $\mathscr{A}$ is n-transitive, A may be chosen so that it satisfies the required conditions and that Ax_n is an arbitrary vector in $\mathscr{V}$. This shows that

(c) $J\mathscr{Y} = 0$ implies $(J+\alpha)\mathscr{V} \subseteq \mathscr{X}$ for some α.

Thus,

$$J = -\alpha + \sum_{1 \le i \le n} x_i \otimes f_i$$

for some linear functionals f_i, $i = 1, \dots, n$. But then, for an arbitrary $A \in \mathscr{A}$ with $A\mathscr{Y} = 0$ define K as above and use the expression for J to get

$$K = -2\alpha A + Ax_n \otimes f_n + \sum_{1 \le i \le n} x_i \otimes f_i A.$$

Thus K belongs to $\mathscr{J}$ and $K\mathscr{Y} = 0$. Hence, by (c), it must be of the same form as J, i.e.,

$$K = -\beta + \sum_{1 \le i \le n} x_i \otimes g_i.$$

Now, we choose vectors u, $v \in \mathscr{V}$ such that $\{x_1, \dots, x_n, u, v\}$ are linearly independent, and find an $A \in \mathscr{A}$ such that $A\mathscr{Y} = 0$, and that $Ax_n = u$, $Au = v$. Then

$$Ku = -2\alpha v + f_n(u)u + \sum_{1 \le i \le n} f_i(v)x_i = -\beta u + \sum_{1 \le i \le n} g_i(u)x_i,$$

which forces $\alpha = 0$. A similar argument with K playing the role of J shows that $\beta = 0$. Also, using the freedom in the choice of u, we conclude that f_n is trivial. Thus, from the fact that (b) holds for every $J \in \mathscr{J}$ we obtain

(d) $J \in \mathscr{J}$ and $J\mathscr{Y} = 0$ implies $J\mathscr{V} \subset \mathscr{Y}$.

The conclusion (d) contradicts the assumption that $\mathscr{J} \neq 0$ in case $n = 1$. In other words, we have shown that given $x \neq 0$, there is a $J \in \mathscr{J}$ such that x and Jx are linearly independent. Observe that this proves the theorem for $n = 1$: if $x \neq 0$ and y are given and $J \in \mathscr{J}$ is such that x and Jx are linearly independent, then by 2-transitivity choose an $A \in \mathscr{A}$ with $Ax = x$ and $AJx = y - Jx$. Then $AJ + JA \in \mathscr{J}$ and $(AJ + JA)x = y$.

Assume now for $n > 1$ inductively that $\mathscr{J}$ is $(n-1)$-transitive and find an $E \in \mathscr{J}$ such that $Ex_i = x_i$, $i = 1, \dots, n-1$. As $E^2 \in \mathscr{J}$ and equals E on $\mathscr{Y}$, we have by (d) that $(E - E^2)\mathscr{V} \subset \mathscr{Y}$. Assume now that there exists a vector $u \in \mathscr{V}$ such that u and Eu do not belong to $\mathscr{Y}$. It follows from $(E - E^2)\mathscr{V} \subset \mathscr{Y}$ that E^2u equals the sum of Eu and a vector from $\mathscr{Y}$ (so $E^2u \neq \mathscr{Y}$). By n-transitivity of $\mathscr{A}$ find an $A \in \mathscr{A}$ such that $Ax_i = x_i$, $i = 1, \dots, n-1$, and $AEu = 0$. This implies for $K = EA + AE \in \mathscr{J}$ that $Kx_i = 2x_i$, $i = 1, \dots, n-1$ and $KEu \in \mathscr{Y}$. Hence, $K - 2E$ annihilates $\mathscr{Y}$ and its image is not contained in $\mathscr{Y}$, because $(K - 2E)Eu$ equals the sum of $-2E^2u$ and a vector from $\mathscr{Y}$, contradicting (d). The freedom in the choice of u shows that for every $E \in \mathscr{J}$ such that $Ex_i = x_i$, $i = 1, \dots, n-1$, we have necessarily that $E\mathscr{V} \subset \mathscr{Y}$ and E is a projection on $\mathscr{Y}$. Choose now a nonzero vector $u \in \ker E$ and find by $(n+1)$-transitivity of $\mathscr{A}$ an $A \in \mathscr{A}$ such that $Ax_1 = u$, $Ax_i = 0$, $i = 2, \dots, n-1$, and $Au = x_1$. Then, $K = EA + AE - 2EAE \in \mathscr{J}$, and $Kx_1 = u$, $Kx_i = 0$, $i = 2, \dots, n-1$, and $Ku = x_1$. It follows for $L = K^2 - EK^2E \in \mathscr{J}$ that $L\mathscr{Y} = 0$ and $Lu = u$ contradicting (d). Consequently, we have shown that (b) leads to a contradiction and we have (a). So, fix a $J \in \mathscr{J}$ such that $J\mathscr{Y} = 0$ and $Jx_n \notin \mathscr{X}$. Now, pick by $(n+1)$-transitivity of $\mathscr{A}$ an $A \in \mathscr{A}$ such that $A\mathscr{X} = 0$, and $AJx_n = u$ an arbitrary vector from $\mathscr{V}$. Thus, for $K = AJ + JA \in \mathscr{J}$ we have that $K\mathscr{Y} = 0$, and $Kx_n = JAx_n + AJx_n = u$. The n-transitivity of $\mathscr{J}$ in the theorem now follows easily by cyclicly permuting the vectors x_i, $i = 1, \dots, n$ and taking sums of corresponding operators K. □

The following result can also be obtained from work of Osborn and Racine [7].

3.3. COROLLARY. *Every Jordan ideal of a dense Jordan algebra is dense.*

3.4. THEOREM. *Let $\mathscr{A}$ be a 2-transitive Jordan algebra of operators on a vector space $\mathscr{V}$. If $\mathscr{A}$ contains at least one operator of finite rank, then the Jordan ideal $\mathscr{J}$ of all finite rank operators of $\mathscr{A}$ is strictly dense, and so is $\mathscr{A}$.*

Proof. Assume with no loss of generality that $\mathscr{V}$ is not finite dimensional. By 3.2 $\mathscr{J}$ is transitive. Thus, we may find an $E \in \mathscr{J}$

such that $Ex = x$ and such that it is of minimal rank with this property. Similarly as in the proof of 1.1.(b), we may find that E is a projection of rank one and that for every finite dimensional subspace $\mathscr{W}$ of $\mathscr{V}$ we may find a projection $E \in \mathscr{J}$ such that $E\mathscr{V} = \mathscr{W}$. Now, for any $\{x, y\}$ linearly independent and $\{u, v\}$ arbitrary vectors of $\mathscr{V}$, let $\mathscr{W}$ denote the linear span of these four vectors and let $E \in \mathscr{J}$ be the corresponding projection. Use 2-transitivity of $\mathscr{A}$ to find $A \in \mathscr{A}$ such that $Ax = u$ and $Ay = v$, use 1.1.(a) to see that $B = EAE \in \mathscr{J}$, and observe that again $Bx = u$ and $By = v$. Thus, $\mathscr{J}$ is 2-transitive and it is strictly dense by 1.3. □

The reader will no doubt have noticed that we left the following questions unanswered.

Question 1. Is there an n-transitive Jordan algebra $\mathscr{A}$ with a Jordan ideal $\mathscr{J} \neq 0$ which is not n-transitive?

Question 2. Is there an n-transitive Jordan algebra which is not $(n+1)$-transitive for any $n \geq 2$?

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