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ON THE METHOD OF CONSTRUCTING IRREDUCIBLE FINITE INDEX SUBFACTORS OF POPA

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Let $U^s(Q)$ be the universal Jones algebra associated to a finite von Neumann algebra Q and $R^s \subset R$ be the Jones subfactors, $s \in \{4\cos^2\frac{\pi}{n} | n \geq 3\} \cup [4, \infty)$. We consider for any von Neumann subalgebra $Q_0 \subset Q$ the algebra $U^s(Q, Q_0)$ defined as the quotient of $U^s(Q)$ through its ideal generated by $[Q_0, R]$ and we construct a Markov trace on $U^s(Q, Q_0)$. If $\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0) = \mathbb{C}$ and Q contains $n \geq s+1$ unitaries $u_1 = 1$, $u_2, \dots, u_n$, with $E_{Q_0}(u_i^* u_j) = \delta_{ij} 1$, $1 \leq i, j \leq n$, then we get a family of irreducible inclusions of type II_1 factors $N^s \subset M^s$, with $[M^s : N^s] = s$ and minimal higher relative commutant. Although these subfactors are nonhyperfinite, they have the Haagerup approximation property whether $Q_0 \subset Q$ is a Haagerup inclusion and if either Q_0 is finite dimensional or $Q_0 \subset \mathscr{Z}(Q)$.

Introduction. Let M be a finite factor with the normal finite faithful trace τ and denote by $L^2(M, \tau)$ the completion of M in the Hilbert norm $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in M$. For $N \subset M$ subfactor of M $(1_N = 1_M)$, the Jones index $[M; N]$ is defined as the Murray-von Neumann coupling constant $\dim_N L^2(M)$ of N in its representation on the Hilbert space $L^2(M, \tau)$. Jones **[J]** proved that $[M : N]$ can only take the values $\{4\cos^2\frac{\pi}{n} | n \geq 3\} \cup [4, \infty]$ and constructed a one parameter family R^s of subfactors of the hyperfinite type II_1 factor R with $[R : R^s] = s$, $s \in \{4\cos^2\frac{\pi}{n} | n \geq 4\} \cup [4, \infty)$.

When $s = [M : N] = 4\cos^2\frac{\pi}{n}$, $n \geq 3$, the properties of the local index **[J]** imply that the pair $N \subset M$ is irreducible (i.e. $N' \cap M = \mathbb{C}$). For $s \geq 4$ Jones' inclusions $R^s \subset R$ are reducible and the problem of characterizing the values $s \geq 4$ with the property that there exist inclusions $R_0 \subset R$ with $[R : R_0] = s$ and $R_0' \cap R = \mathbb{C}$ remained open.

The problem of finding all possible values of indices of irreducible finite index subfactors in arbitrary II_1 factors was completely answered by Popa, who constructed in **[P2]** irreducible inclusions of nonhyperfinite type II_1 factors $N^s \subset M^s$, with $[M^s : N^s] = s$, for all $s \in \{4\cos^2\frac{\pi}{n} | n \geq 4\} \cup [4, \infty)$. His method consists in constructing certain traces, that he called Markov traces, on some universal algebras $U^s(Q)$ canonically associated with a given finite von Neumann algebra Q and to the Jones sequence of projections $\{e_i\}_{i\geq 1}$ of trace $\tau(e_i) = s^{-1}$, subject to the commutation relations $[Q, e_i] = 0$, $i \geq 2$. The algebras $U^s(Q)$ were called in **[P2]** universal Jones algebras. An interesting feature that this method of constructing subfactors is shown to have in **[P2]** is that any pair of subfactors (in particular hyperfinite) $N \subset M$ and $[M : N] = s$ arises this way, for an appropriate Markov trace tr on some universal algebra $U^s(Q)$. The enveloping algebra M_∞ is in this case $\pi_{\mathrm{tr}}(U^s(Q))''$, M the smallest algebra containing $\tau_{\mathrm{tr}}(Q)$ and on which e_1 implements by reduction a conditional expectation and N the commutant of e_1 in M. Then he considered on $U^s(Q)$ the free trace τ defined by $\tau(w) = 0$ for all words w with alternating letters $x_i \in Q$, $y_i \in R$ with $\tau_Q(x_i) = E_{R^s}(y_i) = 0$ and proved that this is indeed a Markov trace with $\pi_\tau(U^s(Q))'' = M^s_\infty = (R^s \otimes Q) *_{R^s} R$, where $R = vN\{e_i\}_{i\geq 1}$ and $R^s = vN\{e_i\}_{i\geq 2}$ are the Jones factors, and that for any *nonatomic* finite von Neumann algebra Q and any $s \in \{4\cos^2\frac{\pi}{n} | n \geq 4\} \cup [4, \infty)$, the appropriate inclusion $N^s \subset M^s$ is an irreducible inclusion of II_1 nonhyperfinite factors with standard matrix A_n for $s = 4\cos^2\frac{\pi}{n}$ and A_∞ for $s \geq 4$. Moreover, the factors M^s are always non Γ in the sense of Murray and von Neumann and do not have the property T of Connes **[C]**.

In this paper we look for other Markov traces by factoring through certain ideals of $U^s(Q)$ which require parts of Q to commute with $R = vN\{e_i\}_{i\geq 1}$. More precisely, given a von Neumann subalgebra Q_0 of Q, denote by $U^s(Q, Q_0)$ the quotient of the universal Jones algebra $U^s(Q)$ through the ideal generated by Q and R subject to the commutation relations $[Q, R^s] = [Q_0, R] = 0$. Then, we prove in §1 that the trace τ on $U^s(Q, Q_0)$ defined by $\tau(w) = 0$ for all words w with alternating letters $x_i \in Q$, $y_i \in R$ with $E_{Q_0}(x_i) = E_{R^s}(y_i) = 0$ and $\tau(q_0 r) = \tau_Q(q_0)\tau_R(r)$ for all $q_0 \in Q_0$, $r \in R^s$ is a Markov trace. Following **[P2]**, the algebras M^s and N^s are then defined as the smallest subalgebra of $\pi_\tau(U^s(Q, Q_0))''$ containing $\pi_\tau(Q)$ and on which e_1 implements by reduction a conditional expectation and respectively as the commutant of e_1 in M^s.

We prove in §2 that if $\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0) = \mathbb{C}$ and there exist $n \geq s+1$ unitaries $u_1 = 1, u_2, \ldots, u_n$ in Q such that $E_{Q_0}(u_i^* u_j) = \delta_{ij} 1$, $1 \leq i, j \leq n$, then we obtain irreducible inclusions of type II_1 factors $N^s \subset M^s$ with $[M^s : N^s] = s$ and standard matrix A_n for $s = 4\cos^2\frac{\pi}{n}$ and A_∞ for $s \geq 4$.

Our initial motivation was to look for "finer" Markov traces on $U^s(Q)$ that would get us closer to the construction of irreducible hyperfinite subfactors. We fail in doing this and our subfactors are still non Γ (hence nonhyperfinite) and contain copies of the II_1 factor associated with the free group on two generators $\mathscr{L}(\mathbb{F}_2)$, since our Markov traces are still free in some sense, namely the enveloping algebra of $N^s \subset M^s$ is in this case the amalgamated product von Neumann algebra $M^s_\infty = \pi_\tau(U^s(Q, Q_0))'' = (R^s \otimes Q) *_{R^s \otimes Q_0} (R \otimes Q_0)$, defined as in **[P2**, §3] (see also **[V1]** for the C^*-definition), with the free trace $\tau = \tau_{R^s \otimes Q} * \tau_{R \otimes Q_0}$ (in fact M^s_∞ is also a factor under the previous required conditions on $Q_0 \subset Q$). However, we prove in §3 that in many cases (e.g. when $Q_0 \subset Q$ has the relative Haagerup property and either Q_0 is finite dimensional or $Q_0 \subset \mathscr{Z}(Q)$), all the von Neumann algebras from the Jones tower $N^s \subset M^s \subset^{e_1} M^s_1 \subset^{e_2} M^s_2 \subset^{e_3} \cdots$ have the Haagerup approximation property, namely for any s there exists a net $(\Phi_\iota)_{\iota \in I}$ of trace preserving unital completely positive maps $\Phi_\iota \colon M^s \to M^s$ converging to id_{M^s} in the point-$\| \; \|_2$ topology and inducing compact operators on $L^2(M^s, \tau_{M^s})$; hence M^s are not very far from being hyperfinite and for any $s \geq 4$ there exists an irreducible inclusion of II_1 factors with index s, $N^s \subset M^s$, with the Haagerup approximation property.

An important problem which is still open at this moment is to decide whether the factors M^s, or at least the enveloping algebras M^s_∞, are isomorphic or not for nonrigid Q. This problem seems to be related, at least for the isomorphism of the associated M^s_∞ in the case $Q = \mathscr{L}(\mathbb{F}_\infty)$ to Voiculescu's type isomorphisms $\mathscr{L}(\mathbb{F}_\infty * \mathbb{Z}_n) \simeq \mathscr{L}(\mathbb{F}_\infty)$ (**[V4]**, **[D]**, **[R]**).

We would like to thank Professor Sorin Popa for suggesting this problem and the idea of extending the results of **[P2]** in this way and Professor Edward Effros for useful discussions concerning §3.

1. The construction of $N^s \subset M^s$ and the Markov property of the free trace. Let $s \in \{4\cos^2 \frac{\pi}{n} | n \geq 3\} \cup [4, \infty)$, denote $\lambda = s^{-1}$ and let $R = vN\{e_i\}_{i \geq 1}$, $R^s = vN\{e_i\}_{i \geq 2}$, $R^s_{-1} = vN\{e_i\}_{i \geq 3}$ be the appropriate Jones factors. Then $R^s_{-1} \subset R^s \subset^{e_1} R$ is the basic construction for $R^s_{-1} \subset R^s$. Let Q and Q_0 ($1_Q = 1_{Q_0}$) be finite von Neumann algebras with a normal faithful trace τ_Q and denote by E_{Q_0} the trace preserving conditional expectation from Q onto Q_0. Then $E_1 = E_{Q_0} \otimes \mathrm{id}_{R^s} \colon Q \otimes R^s \to Q_0 \otimes R^s$ and $E_2 = \mathrm{id}_{Q_0} \otimes E_{R^s} \colon Q_0 \otimes R \to Q_0 \otimes R^s$ are trace preserving conditional expectations. Denote by M^s_∞ the reduced amalgamated product $(Q \otimes R^s) *_{Q_0 \otimes R^s} (Q_0 \otimes R)$ of $(Q \otimes R^s, E_1)$ and $(Q_0 \otimes R, E_2)$, by τ the free trace on M^s_∞ and by E the τ-preserving conditional expectation from M^s_∞ onto $Q_0 \otimes R^s$. The algebras Q and R are identified with $Q \otimes \mathbb{C}1$ and respectively $\mathbb{C}1 \otimes R$ in M^s_∞.

Note that, denoting by $U^s(Q, Q_0)$ the algebra generated by R and Q with the relations $[R, Q_0] = [R^s, Q] = 0$ and by τ the trace on $U^s(Q, Q_0)$ defined by $\tau(w) = 0$ for all words w with alternating letters $x_i \in Q$, $y_i \in R$ with $E_{Q_0}(x_i) = E_{R^s}(y_i) = 0$ and $\tau(q_0 x) = \tau(q_0)\tau(x)$ for all $q_0 \in Q_0$, $x \in R^s$, the von Neumann algebra M^s_∞ can be also defined as $M^s_\infty = \pi_\tau(U^s(Q, Q_0))''$.

Let $\{m_k\}_k$ be a Pimsner-Popa orthonormal basis of R^s over R^s_{-1} with $m_1 = 1$ and consider the unital completely positive map Φ: $M^s_\infty \to M^s_\infty$, $\Phi(x) = \sum_k m_k e_1 x e_1 m_k^*$, $x \in M^s_\infty$. Then M^s is defined as the smallest Φ-invariant von Neumann subalgebra of M^s_∞ that contains Q, i.e. if $B_0 = Q$ and $B_{i+1} = \mathrm{Alg}(B_i, \Phi(B_i))$, $i \geq 0$, then $M^s = \overline{\bigcup_i B_i}$. Let $N^s = \{e_1\}' \cap M^s$. One can easily check the following properties of the averaging map Φ as in [**P2**, 6.1–6.3]:

LEMMA 1.1. (i) $\Phi((R^s)' \cap M^s_\infty) \subset (R^s)' \cap M^s_\infty$. *In particular* $[M^s, R^s] = 0$ *and* $\Phi|_{N^s} = \mathrm{id}_{N^s}$.

(ii) $e_1\Phi(x) = \Phi(x)e_1 = e_1 x e_1$, $x \in M^s_\infty$. *Consequently* $\Phi(M^s) \subset N^s$.

The free amalgamated trace τ on M^s_∞ has the remarkable property that it is a Markov trace, i.e. $\tau(xe_1) = \lambda\tau(x)$ for all $x \in M^s$. This can be proved following step by step the arguments in [**P2**, §5].

Sums of type $\sum_{k_1,\ldots,k_r} f(m_{k_1}, \ldots, m_{k_r}, m^*_{k_1}, \ldots, m^*_{k_r})$ are denoted by $\sum' f(m_{k_1}, \ldots, m_{k_r}, m^*_{k_1}, \ldots, m^*_{k_r})$.

DEFINITION 1.2 ([**P2**, §5]). A *homogeneous reduced closed element* is an element of the form $x = \sum' w$ where $w = x_0 y_1 x_1 \ldots y_n x_n \in M^s_\infty$ is an *alternating word* (i.e. $x_i \in Q$, $y_j \in R$) such that there exists a partition $\{1, \ldots, n\} = I \cup I^* \cup I_0$ with a bijection $I \ni i \leftrightarrow i^* \in I^*$ that satisfy:

(i) $i < i^*$, $\forall i \in I$;

(ii) If $i_1, i_2 \in I$, $i_1 < i_2$ then either $i_i^* < i_2$ or $i_2^* < i_1^*$;

(iii) For each $i_0 \in I_0$ there exists $i \in I$ with $i < i_0 < i^*$;

(iv) If $i \in I$ then $y_i = m_k(e - \lambda 1)$, $y_{i^*} = (e - \lambda 1)m_k^*$ for some k;

(v) If $i_0 \in I_0$ then $y_{i_0} = e - \lambda 1$;

(vi) $E_{Q_0}(x_i) = 0$ for $0 < i < n$ and for $0 \leq i \leq n$ either $x_i \in Q_0$ or $E_{Q_0}(x_i) = 0$.

The set of homogeneous reduced closed elements is denoted by $H_{r,c}$.

Following the arguments in **[P2**, §5] one can check that $\bigcup_i B_i = Q + \operatorname{span} H_{r,c}$ and since $E(x(e_1 - \lambda 1)) = 0$ for all $x \in H_{r,c}$ one obtains

PROPOSITION 1.3. *$E(xe_1) = \lambda E(x)$ for all $x \in M^s$. In particular τ is a Markov trace on M^s.*

Another proof of the markovianity of τ may be found in **[B3]**.

2. Factoriality, index and irreducibility for $N^s \subset M^s$. Let M be a finite von Neumann algebra with a normal faithful trace (nff) τ. For each $a = a^* \in M$ denote, following **[V1]**, by μ_a the linear functional $\mu_a : \mathbb{C}[X] \to \mathbb{C}$, $\mu_a(X^n) = \tau(a^n)$, $n \geq 0$. Then μ_a can be viewed as a probability measure with compact support on $\mathbb{R}$.

LEMMA 2.1. *Let $e_1, \dots, e_n$ be τ-free projections in M with $\tau(e_i) = \lambda \in (0, 1)$, $1 \leq i \leq n$. Then*

$$\sigma(e_1 + \cdots + e_n) \subset \{0\} \cup [a(n, \lambda), b(n, \lambda)] \cup \{n\}$$

and

$$\mu_{e_1+\cdots+e_n} = c_0\delta_0 + c_n\delta_n + \phi(t)\, dt,$$

where

$$a(n, \lambda) = (\sqrt{1-\lambda} - \sqrt{(n-1)\lambda})^2;$$
$$b(n, \lambda) = (\sqrt{1-\lambda} + \sqrt{(n-1)\lambda})^2;$$
$$c_0 = \max(1 - \lambda n, 0); \quad c_n = \max(1 - (1-\lambda)n, 0);$$
$$\phi(t) = \frac{n\sqrt{(1-\lambda n)^2 + 2(1 + (n-2)\lambda)t - t^2}}{2\pi t(n-t)},$$

for any $t \in [a(n, \lambda), b(n, \lambda)]$.

Proof. Denote $\mu = \mu_{e_1+\cdots+e_n}$. An elementary computation relying on the formulae and notations from **[V2]** yields:

$$R_{\mu_{e_i}}(z) = \frac{-1 + z \pm \sqrt{(1-z)^2 + 4\lambda z}}{2z}, \qquad 1 \leq i \leq n;$$
$$R_\mu(z) = nR_{\mu_{e_i}}(z) = K_\mu(z^{-1}) - z^{-1};$$
$$K_\mu(z^{-1}) = \frac{nz - (n-2) \pm \sqrt{(1-z)^2 + 4\lambda z}}{2z}.$$

Since $G_\mu(z) = \int_{\mathbb{R}} \frac{d\mu(t)}{z-t}$ is the inverse of the function $z \to K_\mu(z^{-1})$ one obtains

$$G_\mu(z) = \frac{-(n-2)z - n(1-\lambda n) \pm n\sqrt{z^2 - 2(1 + (n-2)\lambda)z + (1-\lambda n)^2}}{2z(z-n)},$$

the choice of the branch of the square root obeying the rule $\operatorname{Im} z > 0 \Rightarrow \operatorname{Im} G_\mu(z) \leq 0$. The measure μ is easily recovered from its Cauchy transform G_μ as in the statement. $\square$

COROLLARY 2.2. *If $n \geq \max(\frac{1}{\lambda}, \frac{1}{1-\lambda})$ in Lemma 2.1, then $a = a^* = e_1 + \cdots + e_n$ has support 1 and absolutely continuous spectrum; hence $\{a\}''$ is completely nonatomic.*

Let P_1 and P_2 be finite von Neumann algebras with nff traces τ_1 and τ_2, $B \subset P_1$, $B \subset P_2$ be a common von Neumann subalgebra, E_i be the τ_i-preserving conditional expectation from P_i onto B, $P = P_1 *_B P_2$ be the reduced amalgamated product of (P_1, E_1) and (P_2, E_2) and E the conditional expectation from P onto B which invariates the trace $\tau = \tau_1 * \tau_2$. Denote $\|x\|_2 = \tau(x^*x)^{1/2}$, $x \in P$.

If $P_i^0 = \operatorname{Ker} E_i$, $i = 1, 2$, then $P_i = B \oplus P_i^0$ as B-bimodules and let P^0 be the *-subalgebra of P spanned by the formal reduced words, i.e.

$$P^0 = B \oplus \bigoplus_{k \geq 1, i_1 \neq \cdots \neq i_k} P_{i_1}^0 \otimes_B \cdots \otimes_B P_{i_k}^0 .$$

Assume that there exist $u_1 = 1, u_2, \ldots, u_n \in \mathscr{U}(P_1)$ with $E(u_i^* u_j) = \delta_{ij} 1$, $1 \leq i, j \leq n$. Then any $a \in P_1$ can be written as $a = \sum_{i=1}^n u_i E(u_i^* a) + a'$, where $E(u_i^* a') = 0$, $1 \leq i \leq n$, and similarly $a = \sum_{i=1}^n E(a u_i) u_i^* + a''$, with $E(a'' u_i) = 0$, $1 \leq i \leq n$. Since $u_1 = 1$ we can talk about reduced elements from P^0 beginning with u_i, $2 \leq i \leq n$, or with $a \in P_1$ orthogonal to $\{u_i\}_{1 \leq i \leq n}$, i.e. $E(u_i^* a) = 0$, $1 \leq i \leq n$, or ending with u_i^*, $2 \leq i \leq n$, or with $a \in P_1$ orthogonal to $\{u_i\}_{1 \leq i \leq n}$, i.e. $E(a u_i) = 0$, $1 \leq i \leq n$.

For each reduced word $w = a_1 \cdots a_m \in P^0$, $a_j \in P_{i_j}^0$, $i_1 \neq \cdots \neq i_m$ denote $a_1 = o(w)$, $a_m = t(w)$ and define

$$\begin{aligned}
P^{ij} &= \{w \in P^0 | i_1 = i, \ i_m = j\}, \qquad i, j = 1, 2; \\
P_{u_i}^{12} &= \operatorname{span}\{w \in P^{12} | o(w) = u_i b, \ b \in B\}, \qquad 2 \leq i \leq n; \\
P_{u_i}^{21} &= \operatorname{span}\{w \in P^{21} | t(w) = b u_i^*, \ b \in B\}, \qquad 2 \leq i \leq n; \\
P_*^{12} &= \operatorname{span}\{w \in P^{12} | E(u_i^* o(w)) = 0, \ 1 \leq i \leq n\}; \\
P_*^{21} &= \operatorname{span}\{w \in P^{21} | E(t(w) u_i) = 0, \ 1 \leq i \leq n\}.
\end{aligned}$$

These subspaces give rise to the following direct sum of orthogonal vector spaces

$$L^2(P, \tau) = L^2(N, \tau) \oplus \overline{P^{11}} \oplus \overline{P^{22}} \oplus \overline{P_*^{12}} \oplus \overline{P_*^{21}} \oplus \bigoplus_{i=2}^n (\overline{P_{u_i}^{12}} \oplus \overline{P_{u_i}^{21}})$$

with the suitable decomposition for each $x \in P^0$:

$$x = E(x) + x^{11} + x^{22} + x_*^{12} + x_*^{21} + \sum_{i=2}^{n}(x_{u_i}^{12} + x_{u_i}^{21}).$$

LEMMA 2.3. *If there exist* $u_1, \dots, u_n \in \mathscr{U}(P_1)$ *with* $E_1(u_i^* u_j) = \delta_{ij} 1$, $1 \le i, j \le n$, *and* $e \in \mathscr{P}(P_2)$ *with* $E_2(e) = \lambda \in (0, 1)$, *then* $\{u_i e u_i^*\}_{1 \le i \le n}$ *is a* τ*-free family in* P.

Proof. Denote $e^0 = e - \lambda 1$ and remark that

$$\begin{aligned}&\mathrm{Alg}\{u_i e u_i^*\} = \{\alpha 1 + \beta u_i e^0 u_i^* | \alpha, \beta \in \mathbb{C}\}, \quad 1 \le i \le n \quad \text{and}\\ &E(u_{i_1} e^0 u_{i_1}^* u_{i_2} e^0 u_{i_2}^* \dots u_{i_m} e^0 u_{i_m}^*) = 0,\\ &\qquad\qquad \text{for } i_1, \dots, i_m \in \{1, \dots, n\},\ i_1 \ne \dots \ne i_m. \qquad \square\end{aligned}$$

LEMMA 2.4. *With the assumptions of Lemma 2.3 and* $n \ge 1 + \max(\frac{1}{\lambda}, \frac{1}{1-\lambda})$, *pick a* $k_0 \in \{1, \dots, n\}$ *and denote* $a = \sum_{k \ne k_0} u_k e u_k^*$.
Then, there exists a $u \in \mathscr{U}(\{a\}'')$ *with* $E(u^m) = 0$ *for all* $m \ne 0$ *and* $\{u, u_{k_0} e u_{k_0}^*\}$ *a* τ*-free family. Moreover, if* $i, j \in \mathbb{Z}$, $j \ne 0$ *and* $x \in P^{22} + P_{u_{k_0}}^{12} + P_*^{12}$, $x' \in P^{22} + P^{21}$, *then* $E(u^i x u^j x') = 0$.

Proof. Remark that for each nonnegative integer m there exists a finite set

$$F_m \subset \{I = (i_1, \dots, i_l) | i_1 \ne \dots \ne i_l \in \{1, \dots, n\} \setminus \{k_0\}\}$$

such that

$$\begin{aligned}a^m &= \left(\sum_{k \ne k_0} u_k e u_k^*\right)^m\\ &= \tau(a^m)1 + \sum_{I=(i_1, \dots, i_l) \in F_m} \alpha_I u_{i_1} e^0 u_{i_1}^* \cdots u_{i_l} e^0 u_{i_l}^*, \qquad \alpha_I \in \mathbb{C}.\end{aligned}$$

Then for each $f \in \mathbb{C}[X]$, $f(a)$ can be written as

$$f(a) = \tau(f(a))1 + \sum_{I=(i_1, \dots, i_k) \in F_f} \beta_I u_{i_1} e^0 u_{i_1}^* \cdots u_{i_k} e^0 u_{i_k}^*,$$

with F_f finite set and $i_1, \dots, i_k \in \{1, \dots, n\} \setminus \{k_0\}$, $i_1 \ne \dots \ne i_k$. Consequently, for any $x \in P^{22} + P_{u_{k_0}}^{12} + P_*^{12}$, $x' \in P^{22} + P^{21}$, $f, g \in$

$\mathbb{C}[X]$ we get

$$\begin{aligned}
&E(f(a)xg(a)x') \\
&= E\left(\left(\tau(f(a))1 + \sum_{I\in F_f} \beta_I u_{i_1}\dots u_{i_k}e^0u^*_{i_k}\right)\right. \\
&\qquad \left.\cdot x\left(\tau(g(a))1 + \sum_{J\in F_g} \gamma_J u_{j_1}e^0u^*_{j_1}\dots u_{j_l}e^0u^*_{j_l}\right)x'\right) \\
&= \tau(f(a))\tau(g(a))E(xx') \\
&\quad + \tau(g(a))E\left(\sum_{I\in F_f} \beta_I u_{i_1}e^0u^*_{i_1}\dots u_{i_k}e^0u^*_{i_k}xx'\right) \\
&= \tau(g(a))E(f(a)xx').
\end{aligned}$$

The normality of E and τ yields $E(x_1xx_2x') = \tau(x_2)E(x_1xx')$ for $x_1, x_2 \in \{a\}''$ and $E(x) = \tau(x)1$ for $x \in \{a\}''$. By Lemmas 2.1 and 2.3 $\{a\}''$ is completely nonatomic; hence $\{a\}'' \simeq L^\infty(\mathbb{T}, d\lambda)$ and there exists a $u \in \mathscr{U}(\{a\}'')$ with the required properties. □

PROPOSITION 2.5. *Let $P = P_1 *_B P_2$ be the reduced amalgamated product of (P_1, E_1) and (P_2, E_2) and assume that there exist $e \in \mathscr{P}(P_2)$ with $\lambda = E_2(e) \in (0, 1)$ and $u_1 = 1$, $u_2, \dots, u_n \in \mathscr{U}(P_1)$ with $n \geq 1 + \max(\frac{1}{\lambda}, \frac{1}{1-\lambda})$, $E_1(u_i^*u_j) = \delta_{ij}1$, $1 \leq i, j \leq n$. Then there exist $x_1, \dots, x_n \in P$ such that for any $\varepsilon > 0$, there is a $\delta > 0$ with $x \in P$, $\|x\| \leq 1$, $\|[x, x_i]\|_2 \leq \delta$, $1 \leq i \leq n \Rightarrow \|x - E_B(x)\|_2 \leq \varepsilon$.*

Proof. For each $i \in \{2, \dots, n\}$ denote by v_i the suitable unitary for i given by 2.3. Take $x_1 = e^0 = e - \lambda 1$ and $x_i = v_i$, $2 \leq i \leq n$. Since P^0 is dense in P we can assume by Kaplansky's density theorem that $x \in P^0$. Let $\varepsilon, \varepsilon' > 0$ such that $\varepsilon' < \varepsilon^2/\phi(n, \lambda)$, where

$$\phi(n, \lambda) = 8(n-1) + \frac{1 + 4(1-\lambda)\sqrt{2(n-1)}}{2(\lambda - \lambda^2)}$$

and r be an integer with $1 \leq (2r+1)\varepsilon'$. Assume that $x \in P^0$ satisfies $\|[x, v_i]\|_2 \leq \frac{2\varepsilon'}{r+1}$, $2 \leq i \leq n$ and $\|[x, e]\|_2 \leq \sqrt{\varepsilon'}$.

Denote $x_i'' = x^{22} + x^{12}_* + x^{12}_{u_i}$, $x_i' = x - x_i''$, for $2 \leq i \leq n$. Since

$$\begin{aligned}
\|[x, v_i^k]\|_2 = \|x - v_i^k x v_i^{-k}\|_2 &\leq \|x - v_i^{k-1}xv_i^{-(k-1)}\|_2 + \|x - v_ixv_i^*\|_2 \\
&= \|[x, v_i^{k-1}]\|_2 + \|[x, v_i]\|_2,
\end{aligned}$$

we obtain that $\|[x, v_i^k]\|_2 \le |k| \cdot \|[x, v_i]\|_2$ and

$$\begin{aligned}\left\|x - \frac{1}{2r+1}\sum_{k=-r}^{r} v_i^k x v_i^{-k}\right\|_2 &\le \frac{1}{2r+1}\sum_{k=-r}^{r}\|x - v_i^k x v_i^{-k}\|_2 \\ &\le \frac{\sum_{k=-r}^{r}|k|}{2r+1}\|[x, v_i]\|_2 \\ &= \frac{r(r+1)}{2r+1}\|[x, v_i]\|_2 \le \varepsilon' .\end{aligned}$$

Consequently

$$\begin{aligned}\|x\|_2 &\le \left\|\frac{1}{2r+1}\sum_{k=-r}^{r} v_i^k x v_i^{-k}\right\|_2 + \left\|x - \frac{1}{2r+1}\sum_{k=-r}^{r} v_i^k x v_i^{-k}\right\|_2 \\ &\le \varepsilon' + \left\|\frac{1}{2r+1}\sum_{k=-r}^{r} v_i^k x_i' v_i^{-k}\right\|_2 + \left\|\frac{1}{2r+1}\sum_{k=-r}^{r} v_i^k x_i'' v_i^{-k}\right\|_2 \\ &\le \varepsilon' + \|x_i'\|_2 + \left\|\frac{1}{2r+1}\sum_{k=-r}^{r} v_i^k x_i'' v_i^{-k}\right\|_2 .\end{aligned}$$

By Lemma 2.4 $\{v_i^k x_i'' v_i^{-k}\}_{-r\le k\le r}$ are mutually orthogonal in $\|\ \|_2$; hence

$$\begin{aligned}\left\|\frac{1}{2r+1}\sum_{k=-r}^{r} v_i^k x_i'' v_i^{-k}\right\|_2^2 &= \frac{1}{(2r+1)^2}\sum_{k=-r}^{r}\|v_i^k x_i'' v_i^{-k}\|_2^2 \\ &= \frac{\|x_i''\|_2^2}{2r+1} \le \frac{1}{2r+1} \le \varepsilon'\end{aligned}$$

and

$$\|x\|_2 \le 2\varepsilon' + \|x_i'\|_2 .$$

The last inequality shows that

$$\begin{gathered}\|x_i''\|_2^2 = (\|x\|_2 - \|x_i'\|_2)(\|x\|_2 + \|x_i'\|_2) \le 4\varepsilon' \quad \text{and} \\ \max(\|x^{22}\|_2, \|x_*^{12}\|_2, \|x_{u_i}^{12}\|_2) \le \|x_i''\|_2 \le 2\sqrt{\varepsilon'}, \quad \text{for } 2 \le i \le n .\end{gathered}$$

Since $\{v_i^k(x^{22} + x_*^{21} + x_{u_i}^{21})v_i^{-k}\}_{-r\le k\le r}$ are still mutually orthogonal by Lemma 2.4, a similar computation yields

$$\max(\|x^{22}\|_2, \|x_*^{21}\|_2, \|x_{u_i}^{21}\|_2) \le 2\sqrt{\varepsilon'} .$$

We obtain

$$\begin{aligned}\|x - E_B(x) - x^{11}\|_2^2 &= \|x^{22}\|_2^2 + \|x_*^{12}\|_2^2 + \|x_*^{21}\|_2^2 \\ &\quad + \sum_{i=2}^{n}(\|x_{u_i}^{12}\|_2^2 + \|x_{u_i}^{21}\|_2^2) \\ &\le 4\varepsilon' + 4\varepsilon' + 4(2n-4)\varepsilon' = \varepsilon_0^2 .\end{aligned}$$

The end of the proof is the same as in [**P2**, Theorem 7.1]. Since $x^{11}x_1 \in P^{12}$ and $x_1 x^{11} \in P^{21}$, they are orthogonal in $\| \ \|_2$; hence

$$\begin{aligned}\|[x^{11}, x_1]\|_2^2 &= \|x^{11}x_1\|_2^2 + \|x_1 x^{11}\|_2^2 \\ &= \tau(x^{11}x_1x_1^*x^{11*}) + \tau(x^{11*}x_1^*x_1x^{11}) \\ &= \tau(x^{11}E_B(x_1x_1^*)x^{11*}) + \tau(x^{11*}E_B(x_1^*x_1)x^{11}) \\ &= 2\|x_1\|_2^2\|x^{11}\|_2^2 = 2(\lambda - \lambda^2)\|x^{11}\|_2^2 .\end{aligned}$$

Since $[E_B(x), x_1] \in P_2^0$ is orthogonal on x_1x^{11} and on $x^{11}x_1$, one obtains

$$\begin{aligned}\sqrt{\varepsilon'} &\ge \|[x, e]\|_2 = \|[x, x_1]\|_2 \\ &\ge \|[E_B(x) + x^{11}, x_1]\|_2 - \|[x - E_B(x) - x^{11}, x_1]\|_2 \\ &\ge \|[E_B(x) + x^{11}, x_1]\|_2 - 2\varepsilon_0\|x_1\| \\ &= (\|[E_B(x), x_1]\|_2^2 + \|[x^{11}, x_1]\|_2^2)^{1/2} - 2\varepsilon_0\|x_1\| \\ &= (\|[E_B(x), x_1]\|_2^2 + 2(\lambda - \lambda^2)\|x^{11}\|_2^2)^{1/2} - 2\varepsilon_0\|x_1\| .\end{aligned}$$

In particular

$$\|x^{11}\|_2^2 \le \frac{(\sqrt{\varepsilon'} + 2\varepsilon_0\|x_1\|)^2}{2(\lambda - \lambda^2)} = \frac{(1 + 4\sqrt{2(n-1)}(1-\lambda))\varepsilon'}{2(\lambda - \lambda^2)}$$

and

$$\|[E_B(x), x_1]\|_2 = \|[E_B(x), e]\|_2 \le (1 + 4\sqrt{2(n-1)}(1-\lambda))\sqrt{\varepsilon'} .$$

Consequently

$$\|x - E_B(x)\|_2^2 = \|x - E_B(x) - x^{11}\|_2^2 + \|x^{11}\|_2^2 \le \phi(n, \lambda)\varepsilon' \le \varepsilon^2 . \quad \square$$

COROLLARY 2.6. *If there exist $e \in \mathscr{P}(P_2)$ with $E_2(e) = \lambda 1 \in (0, \frac{1}{2}]$, n unitaries $u_1 = 1, u_2, \dots, u_n \in \mathscr{U}(P_1)$ with $E_1(u_i^* u_j) = \delta_{ij}1$, $1 \le i, j \le n$, $n \ge 1 + \frac{1}{\lambda}$ and $\mathscr{Z}(B) \cap \mathscr{Z}(P_1) = \mathbb{C}$ or $\mathscr{Z}(B) \cap \mathscr{Z}(P_2) = \mathbb{C}$, then P is a factor and contains a copy of $\mathscr{L}(\mathbb{F}_2)$.*

Proof. By the previous proposition there exist $x_1, \dots, x_n \in P$ such that $\{x_1, \dots, x_n\}' \cap P \subset B$; hence $\mathscr{Z}(P) = P' \cap P \subset B \cap P' \subset B \cap P_i' = \mathscr{Z}(B) \cap \mathscr{Z}(P_i) = \mathbb{C}$.

By Lemma 2.3 we obtain a τ-free family of n projections $\{u_i e u_i^*\}_{1\le i\le n}$. According to Corollary 2.2 $\{u_i e u_i^*\}''_{2\le i\le n}$ contains a copy of $L^\infty(\mathbb{T}, d\lambda) \simeq \mathscr{L}(\mathbb{Z})$; hence P contains a copy of the von Neumann algebra $\mathscr{L}(\mathbb{Z}) *_{\mathbb{C}} (\mathbb{C}e \oplus \mathbb{C}(1-e))$.

Pick $2N$ unitaries $u_i \in \mathscr{L}(\mathbb{Z})$ with $N \ge \frac{1}{\lambda}$ and $\tau(u_i^* u_j) = \delta_{ij} 1$, $1 \le i, j \le n$. Using again Lemma 2.3 we obtain $2N$ τ-free projections $\{e_i\}_{1\le i\le 2N}$ of trace $\frac{1}{\lambda}$ and by Corollary 2.2 ($N \ge \frac{1}{\lambda}$) two unitaries $u \in \mathscr{U}(\{e_1 + \cdots + e_N\}'')$ and $v \in \mathscr{U}(\{e_{N+1} + \cdots + e_{2N}\}'')$ such that $\tau(u^k) = \tau(v^k) = 0$, $\forall k \ne 0$.

Since $\{u, v\}$ is τ-free, it follows that $\{u, v\}'' \simeq \mathscr{L}(\mathbb{Z} * \mathbb{Z}) = \mathscr{L}(\mathbb{F}_2)$. □

Define

$$M_{-1}^s = N^s, \quad M_0^s = M^s, \quad M_1^s = vN(M^s, e_1),$$
$$M_k^s = vN(M_{k-1}^s, e_k) = vN(M^s, e_1, \dots, e_k), \qquad k \ge 1.$$

Sometimes we simply denote M_k instead of M_k^s, $k \ge -1$.

COROLLARY 2.7. *If Q contains $n \ge s+1$ unitaries $u_1 = 1, u_2, \dots, u_n$ with $E_{Q_0}(u_i^* u_j) = \delta_{ij} 1$, $1 \le i, j \le n$, then*

$$\mathscr{Z}(M_k^s) = \mathscr{Z}(Q) \cap \mathscr{Z}(Q_0), \quad \textit{for all } k \ge 1.$$

Proof. Applying Proposition 2.5 for $P_1 = Q \otimes R^s$, $P_2 = Q_0 \otimes R$, $B = Q_0 \otimes R^s$ we get $x_1, \dots, x_n \in M_1$ with $\{x_1, \dots, x_n\}' \cap M_\infty \subset Q_0 \otimes R^s$; hence

$$\begin{aligned} \mathscr{Z}(M_1) = M_1' \cap M_1 &\subset (Q_0 \otimes \mathscr{N}) \cap Q' \cap \{e\}' \cap M_1 \\ &= ((\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes \mathscr{N}_{-1}) \cap M_1 . \end{aligned} \tag{2.1}$$

Let B_i, $i \ge 0$ as in §1. Clearly

$$[\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0), B_0] = [\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0), Q] = 0.$$

Assume that $[\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0), B_i] = 0$ for $i \ge 0$. Since $[Q_0, R] = 0$ it follows that for any $x \in \mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)$, $y \in B_i$ we have

$$\begin{aligned} x\Phi(y) &= \sum_k x m_k e y e m_k^* = \Phi(xy) = \Phi(yx) \\ &= \sum_k m_k e y e m_k^* x = \Phi(y) x . \end{aligned}$$

Thus $[\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0), \Phi(B_i)] = 0$. But $B_{i+1} = \mathrm{Alg}(B_i, \Phi(B_i))$ and $M_0 = \overline{\bigcup_i B_i}$; hence $[\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0), M_0] = 0$ and therefore $[\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0), M_1] = 0$.

By Lemma 1.1 $[R^s_{-1}, M_1] = 0$ and thus
$$[(\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes R^s_{-1}, M_1] = 0.$$
We get
$$\begin{aligned}((\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes R^s_{-1}) \cap M_1 &\subset \mathscr{Z}((\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes R^s_{-1}) \\ &= \mathscr{Z}(Q) \cap \mathscr{Z}(Q_0);\end{aligned}$$
hence according to (2.1) $\mathscr{Z}(M_1) \subset \mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)$.

The other inclusion was already proved.

Denote $A^i_k = vN\{e_j\}_{i \le j \le k}$ for $1 \le k \le \infty$. Arguing as before we obtain
$$\begin{aligned}\mathscr{Z}(M_k) &\subset ((\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes R^s_{-1}) \cap (A^1_k)' \cap M_k \\ &= ((\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes R^s_{-k}) \cap M_k.\end{aligned} \tag{2.2}$$

Since $[\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0), M] = [Q_0, A^1_k] = 0$ and $[R^s_{-k}, A^1_k] = [R^s_{-k}, M] = 0$, we get $[(\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes R^s_{-k}, M_k] = 0$; hence
$$\begin{aligned}((\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes R^s_{-k}) \cap M_k &\subset \mathscr{Z}((\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes R^s_{-k}) \\ &= \mathscr{Z}(Q) \cap \mathscr{Z}(Q_0).\end{aligned}$$
According to (2.2) this yields $\mathscr{Z}(M_k) \subset \mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)$.

The other inclusion is straightforward since $Q \subset M_k$, $[Q_0, A^1_k] = 0$ and $[\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0), M] = 0$. □

The following is a rewriting of Lemma 6.4 in **[P2]**.

LEMMA 2.8. *If Q contains a partition of unity $\{p_i\}_{i \in I}$ with $E_{Q_0}(p_i) = \tau(p_i) \le \lambda$ or if $Q_0 \subset Q$ is as in 2.7, then*

(i) $M_i = \overline{\operatorname{span}}\, M_{i-1} e_i M_{i-1}$, $i \ge 1$ $(M_0 = M, M_{-1} = N)$;
(ii) $\tau(e_i x) = \tau(e_i)\tau(x) = \lambda\tau(x)$, $x \in M_{i-1}$, $i \ge 1$;
(iii) $e_i x e_i = E_{M_{i-2}}(x) e_i$, $x \in M_{i-1}$ *and* $M_{i-2} = \{e_i\}' \cap M_{i-1}$, $i \ge 1$.

Proof. In the first case the computation from the end of **[V3]** shows that $\tau(s(p_i e p_i)) = \tau(p_i)$; hence one obtains $a \in \overline{\operatorname{span}}\, Q e_1 Q$ with support 1. When $Q_0 \subset Q$ is as in 2.7, such an element is produced by 2.1 and 2.3. Then the proof in **[P2]** applies literally. □

REMARK 2.9. Under the assumptions of Lemma 2.8 the tower of von Neumann algebras
$$M = M_0 \subset^{e_1} M_1 \subset^{e_2} \cdots \subset^{e_{i-1}} M_{i-1} \subset^{e_i} M_i \subset \cdots \quad \text{satisfies}$$
$$e_i x e_i = E_{M_{i-2}}(x) e_i, \quad x \in M_{i-1}; \qquad M_i = \overline{\operatorname{span}}\, M_{i-1} e_i M_{i-1};$$
$$[e_i, M_{i-2}] = 0; \qquad E_{M_{i-1}}(e_i) = \lambda 1, \quad i \ge 1;$$

hence the arguments of **[PiPo**, Proposition 2.1] apply and we get

$$\lambda = \max\{\mu \in \mathbb{R}_+ | E_{M_{i-1}}(x) \geq \mu x, \ x \in M_i^+\}.$$

This shows that the probabilistic index of the trace preserving conditional expectation $E_{M_{i-1}}$ from M_i onto M_{i-1} is always s when $Q_0 \subset Q$ is as in 2.8.

COROLLARY 2.10. *If the hypotheses of 2.7 are fulfilled then*

$$M_i' \cap M_j = (\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes A_j^{i+2}, \qquad 1 \leq i \leq j \leq \infty.$$

Proof. It is obvious that

$$M_1' \cap M_\infty \subset (\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes R_{-1}^s = (\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes A_\infty^3.$$

Since $A_k^3 = \operatorname{span} A_{k-1}^3 e_k A_{k-1}^3$, $A_{k-1}^1 \subset M_{k-1}$ and $E_{M_{k-1}}(e_k) = \lambda 1$, it follows that $E_{M_{k-1}}(A_k^3) = A_{k-1}^3$, $k \geq 1$; thus for any $k \geq i$ we obtain $E_{M_1' \cap M_i}(A_k^3) = E_{M_1' \cap M_1} E_{M_i} E_{M_{i+1}} \cdots E_{M_{k-1}}(A_k^3) = \cdots = E_{M_1' \cap M_i}(A_i^3) = A_i^3$ and consequently $E_{M_1' \cap M_i}(A_\infty^3) = A_i^3$.

Moreover, since $M_1' \cap M_\infty \subset (\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes A_\infty^3$ and $\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0) \subset M_1' \cap M_i$, it follows that

$$\begin{aligned} M_1' \cap M_i &= E_{M_1' \cap M_i}(M_\infty) = E_{M_1' \cap M_i}(M_1' \cap M_\infty) \\ &\subset E_{M_1' \cap M_i}((\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes A_\infty^3) = (\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes A_i^3. \end{aligned}$$

On the other side, the inclusion $(\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes A_i^3 \subset M_1' \cap M_i$ is obvious. A similar argument yields

$$M_i' \cap M_j = (\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes A_j^{i+2}, \qquad 1 \leq i \leq j \leq \infty. \qquad \square$$

COROLLARY 2.11. *Let $Q_0 \subset Q$ as in 2.7 with $\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0) = \mathbb{C}$. Then $N^s \subset M^s$ are II_1 factors with $[M^s : N^s] = s$. Moreover, $(N^s)' \cap M_i^s = A_i^1$, $i \geq 0$ $(M_0^s = M^s)$ and the enveloping algebra of $N^s \subset M^s$ is $M_\infty^s = (R^s \otimes Q) *_{R^s \otimes Q_0} (R \otimes Q_0)$.*

Proof. The arguments from the proof of Theorem 6.7 in **[P2]** apply in our case, due to 2.7, 2.8 and 2.10. $\square$

COROLLARY 2.12. *Let $Q_0 \subset Q$ as in 2.7 and $\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0) = \mathbb{C}$. Then M_i^s, $i \geq -1$, $(M_{-1}^s = N^s)$ are non Γ factors.*

Proof. By 2.5, if $x \in M_1^s$ almost commutes with the elements $x_1 = e_1, x_2, \ldots, x_n \in M_i^s$, then x is "concentrated" on $(\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes$

R^s and the arguments in [**P2**, 7.1] show that x is "concentrated" on $(\mathscr{Z}(Q) \cap \mathscr{Z}(Q_0)) \otimes R^s_{-1} = R^s_{-1}$.

Since $[M_1^s, R^s_{-1}] = 0$ it follows that $E_{R^s_{-1}}(a) = \tau(a)1$ for all $a \in M_1^s$; hence x is "concentrated" on $\mathbb{C}$. By [**PiPo**, 1.11] each M_i^s, $i \geq -1$ is non Γ. □

REMARK 2.13. The analogue of Theorem 6.10 in [**P2**] is also true, namely if $Q_0 \subset Q \subset Q_1$ are finite von Neumann algebras and $N^{s,i} \subset M^{s,i} \subset M_1^{s,i} \subset \cdots \subset M_\infty^{s,i}$, $i = 1, 2$, is the tower of factors associated to $Q_0 \subset Q_1$ respectively $Q_0 \subset Q_2$ and the inclusion $M_\infty^{s,1} \subset M_\infty^{s,2}$ is implemented by $Q_1 \subset Q_2$, then $M_j^{s,1} \subset M_j^{s,2}$, for all $j \geq -1$ $(M_{-1}^{s,i} = N^{s,i})$ and

(i) $E_{M_j^{s,1}} E_{Q_2} = E_{Q_1}$, $0 \leq j \leq \infty$;

(ii) $E_{M_j^{s,1}} E_{M_i^{s,2}} = E_{M_i^{s,1}}$, $-1 \leq i \leq j \leq \infty$.

3. Haagerup type approximation property for M^s. Since the hyperfinite II_1-factor $\mathscr{R}$ is an increasing limit of finite dimensional matrix algebras, its identity can be approximated in the point-$\| \ \|_2$ topology by a net of conditional expectations of $\mathscr{R}$ onto finite dimensional subalgebras. One can replace this property for a finite von Neumann algebra M with trace τ, assuming only the existence of a net of τ-preserving unital completely positive maps $\Phi_\iota : M \to M$, $\iota \in I$, such that $\lim_{\iota \in I} \|\Phi_\iota(x) - x\|_2 = 0$, $x \in M$, and each Φ_ι induces a compact operator on $L^2(M, \tau)$. An important example, the II_1-factor $\mathscr{L}(\mathbb{F}_n)$ associated with the free group on n generators ($n \in \mathbb{N} \cup \{\infty\}$, $n \geq 2$), was pointed out by Haagerup ([**H**]). It is known by [**CJ**] (see also [**P1**]) that the von Neumann algebras with this property don't contain subfactors with the property T of Connes.

In this section we prove that the algebras M_i^s, $i \geq -1$, constructed in §1 from pairs $Q_0 \subset Q$ satisfying a property that we call the relative Haagerup property with Q_0 finite dimensional or with $Q_0 \subset \mathscr{Z}(Q)$ have the Haagerup approximation property. In particular the subfactors M^s constructed in [**P2**] starting with a nonatomic finite von Neumann algebra Q (or with an algebra Q that contains $n \geq s+1$ unitaries orthogonal in the trace as in Chapter 2) have the Haagerup approximation property if and only if Q has this property. In order to do this, we shall use the method of construction of completely positive maps on amalgamated C^*-products from [**B1**] and [**B2**]. As a consequence, it follows (from [**CJ**] or [**P1**]) that in these cases the von Neumann algebras M_i^s don't contain subfactors with the property T.

We define first a Haagerup type property for inclusions of finite von Neumann algebras. Let $N \subset M$ be finite von Neumann algebras, τ be a fixed normal faithful trace on M, which acts by left multiplication on $L^2(M, \tau)$ in the GNS representation of τ. Let $x_\tau \in L^2(M, \tau)$ be the appropriate vector for each $x \in M$ and let E_N be the τ-preserving conditional expectation from M onto N.

Let $\Phi: M \to M$ be a E_N-preserving N-bimodule unital completely positive map. Then the Cauchy-Schwarz type inequality $\Phi(x)^*\Phi(x) \leq \Phi(x^*x)$, $x \in M$, yields the contraction $T_\Phi \in \mathscr{B}(L^2(M, \tau))$, $T_\Phi(x_\tau) = (\Phi x)_\tau$, $x \in M$.

The N-linearity of Φ yields $T_\Phi(x_\tau) = \Phi(x)_\tau = x_\tau$, $x \in N$; hence $T|_{L^2(N,\tau)} = I_{L^2(N,\tau)}$. We check that $T_\Phi^*(x_\tau) = x_\tau$, $x \in N$. Indeed, for $a \in M$, $x \in N$ one obtains

$$\begin{aligned}\langle T_\Phi(a_\tau), x_\tau\rangle_{2,\tau} &= \tau(x^*\Phi(a)) = \tau(E_N(x^*\Phi(a))) \\ &= \tau(x^*E_N(a)) = \tau(x^*a) = \langle a_\tau, x_\tau\rangle_{2,\tau}.\end{aligned}$$

Consequently $T = \left(\begin{smallmatrix} I & 0 \\ 0 & T^0 \end{smallmatrix}\right)$ subject to the orthogonal decomposition $L^2(M, \tau) = L^2(N, \tau) \oplus L^2(N, \tau)^\perp$. Note also that $T_{E_N} = e_N = P^{L^2(M,\tau)}_{L^2(N,\tau)}$. An operator ae_Nb, $a, b \in M$, acts on $L^2(M, \tau)$ by $ae_Nbx_\tau = (aE_N(bx))_\tau$, $x \in M$.

Set $\mathscr{F}_N(M) = \{T \in N' \cap \mathscr{B}(L^2(M, \tau)) | T = \sum_{i\in F} a_i e_N b_i$, F finite set, $a_i, b_i \in M\}$ and let $\mathscr{K}_N(M)$ be the norm closure of $\mathscr{F}_N(M)$ in $\mathscr{B}(L^2(M, \tau))$.

Definition 3.1. The finite von Neumann algebra inclusion $N \subset M$ has the *Haagerup property* (or is of Haagerup type) if there exists a net $\{\Phi_\iota\}_{\iota\in I}$ of E_N-preserving N-bimodules unital completely positive maps $\Phi_\iota: M \to M$ such that:

(i) $\lim_\iota \|\Phi_\iota(x) - x\|_2 = 0$, $x \in M$;
(ii) $T_{\Phi_\iota} \in \mathscr{K}_N(M)$.

Remark 3.2. If $N = \mathbb{C}$, the usual definition of the Haagerup approximation property of M is recovered. Note that, in the literature, the condition $\tau\Phi = \tau$ is sometimes replaced by $\tau(\Phi(x^*x)) \leq \tau(x^*x)$, $x \in M$, that ensures the contractivity of T_Φ.

Remark 3.3. Assume that the maps Φ_ι are as in Definition 3.1 and let $\Phi_{\iota,\varepsilon} = \frac{1}{1+\varepsilon}(\Phi_\iota + \varepsilon E_N)$, $\varepsilon \geq 0$.

Clearly $\Phi_{\iota,\varepsilon}$ are E_N-preserving N-linear unital completely positive maps with $\lim_{(\iota,\varepsilon)\in I_*} \|\Phi_{\iota,\varepsilon}(x) - x\|_2 = 0$, $x \in M$, where $I_* = I \times \mathbb{R}_+$ endowed with the order $(\iota_1, \varepsilon_1) \leq (\iota_2, \varepsilon_2) \Leftrightarrow \iota_1 \leq \iota_2$ and $\varepsilon_2 \leq \varepsilon_1$.

We obtain $T^0_{\Phi_{\iota,\varepsilon}} = \frac{1}{1+\varepsilon} T^0_{\Phi_\iota}$ and consequently $\|T^0_{\Phi_{\iota,\varepsilon}}\| < 1$. This remark shows that we always are allowed to assume that $\|T^0_{\Phi_\iota}\| < 1$ in the definition of the Haagerup property.

REMARK 3.4. Let $N \subset M$ be a Haagerup inclusion and P be a von Neumann algebra with $N \subset P \subset M$. Then $N \subset P$ is still a Haagerup inclusion (with respect to the trace induced on P from M).

Proof. Let $\Phi_\iota : M \to M$ be as in Definition 3.1 and let $\Psi_\iota : P \to P$, $\Psi_\iota = E_P \Phi_\iota|_P$. Then

$$\begin{aligned} \|\Psi_\iota(x) - x\|_2 &= \|E_P(\Phi_\iota x) - x\|_2 \\ &= \|E_P(\Phi_\iota x - x)\|_2 \le \|\Phi_\iota x - x\|_2, \qquad x \in P; \\ T_{\Psi_\iota} &= e_P T_{\Phi_\iota}|_{L^2(P,\tau)} = e_P T_{\Phi_\iota} e_P. \end{aligned}$$

Since $e_P x e_N y e_P = e_P x e_P e_N e_P y e_P = E_P(x) e_N E_P(y)$, $x, y, \in M$, we get

$$\begin{aligned} \|T_{\Psi_\iota} - \sum_i E_P(a_i) e_N E_P(b_i)\| &= \left\| e_P T_{\Phi_\iota} e_P - \sum_i e_P (a_i e_N b_i) e_P \right\| \\ &\le \left\| T_{\Phi_\iota} - \sum_i a_i e_N b_i \right\| . \end{aligned}$$

Moreover, since $e_P \in N'$ we get $\sum_i E_P(a_i) e_N E_P(b_i)|_{L^2(P,\tau))} \in N' \cap \mathscr{B}(L^2(P, \tau))$ and $T_{\Psi_\iota} \in \mathscr{K}_N(P)$.

At this moment we recall some facts about completely positive maps on amalgamated products. Let P_1 and P_2 be finite von Neumann algebras with fixed traces τ_1 and respectively τ_2 and let N be a common von Neumann subalgebra of P_1 and P_2. Denote by $E_i : P_i \to N$, $i = 1, 2$, the τ_i-preserving conditional expectations of P_i onto N.

Denote $P_j^0 = \operatorname{Ker} E_j$, $j = 1, 2$, and consider the $*$-algebra

$$P_0^0 = N \oplus \bigoplus_{n \ge 1;\, i_1 \ne \cdots \ne i_n} P_{i_1}^0 \otimes_N \cdots \otimes_N P_{i_n}^0 .$$

Following **[P2**, §3], consider the canonical "projection" E_0 from P_0^0 onto N, that agrees with E_i when restricted to P_i, defined by

$$E_0(x) = \begin{cases} x, & \text{for } x \in N, \\ 0, & \text{for } x = a_1 \dots a_n, \quad a_j \in P^0_{i_j}, \ i_1 \neq \dots \neq i_n, \end{cases}$$

the trace $\tau = \tau_1 E_0 = \tau_2 E_0$ on P^0_0 and the finite von Neumann algebra $P = P_1 *_N P_2 = \pi_\tau(P^0_0)''$ acting on

$$\begin{aligned} L^2(P, \tau) = L^2(N, \tau) \\ \oplus \bigoplus_{n \geq 1; i_1 \neq \dots \neq i_n} (L^2(P_{i_1}, \tau) \ominus L^2(N, \tau)) \\ \otimes_N \dots \otimes_N (L^2(P_{i_n}, \tau) \ominus L^2(N, \tau)). \end{aligned}$$

Then P^0_0 is a weakly dense $*$-subalgebra of P and E_0 extends to a τ-preserving conditional expectation $E: P \to N$.

The following lemma shows a proof of the Cauchy-Schwarz type inequality for unital completely positive maps on (unital) $*$-algebras without using Stinespring dilations.

LEMMA 3.5. *If A is a unital $*$-algebra and $\Phi: A \to \mathscr{B}(\mathscr{H})$ is a unital completely positive map, then $\Phi(x)^*\Phi(x) \leq \Phi(x^*x)$, $x \in A$.*

Proof. Consider $K: A \times A \to \mathscr{B}(\mathscr{H})$ defined by $K(x, y) = \Phi(y^*x)$, $x, y \in A$. Then the kernel K is positively defined, since

$$\sum_{i,j=1}^n \langle K(a_i, a_j)\xi_i, \xi_j\rangle = \sum_{i,j=1}^n \langle \Phi(a_j^* a_i)\xi_i, \xi_j\rangle \geq 0, \qquad a_1, \dots, a_n \in A, \ \xi_1, \dots, \xi_n \in \mathscr{H}.$$

Consequently Kolmogorov's theorem yields a vector space $\mathscr{K}$ and $V_x \in \mathscr{B}(\mathscr{H}, \mathscr{K})$, $x \in A$, such that $K(x, y) = V_y^* V_x$, $x, y \in A$.

Since $K(1, 1) = I_{\mathscr{H}}$, V_1 is an isometry and we obtain

$$\begin{aligned} \Phi(x)^*\Phi(x) &= K(x, 1)^* K(x, 1) = (V_1^* V_x)^* V_1^* V_x \\ &= V_x^* V_1 V_1^* V_x \leq V_x^* V_x = \Phi(x^*x), \qquad x \in A. \qquad \square \end{aligned}$$

Let $\Phi_i: P_i \to P_i$, $i = 1, 2$, be E_N-preserving N-bimodule unital completely positive maps. Consider the N-linear map $\Phi_0: P^0_0 \to P^0_0$

defined by

$$\Phi_0(x) = \begin{cases} x, & \text{for } x \in N, \\ \Phi_{i_1}(a_1)\dots\Phi_{i_n}(a_n), & \text{for } x = a_1 \dots a_n,\ a_j \in P^0_{i_j}, \\ & i_1 \neq \dots \neq i_n. \end{cases}$$

Then Φ_0 is known to be completely positive on P^0_0 ([**B1**], [**B2**]) and since $\Phi_j(P^0_j) \subset P^0_j$, $j = 1, 2$, we get $E_0\Phi_0 = E_0$. This last equality together with Lemma 3.5 yield $E(\Phi_0(x)^*\Phi_0(x)) \leq E(x^*x)$, $x \in P^0_0$, and since E is τ-preserving we get $\tau(\Phi_0(x)^*\Phi_0(x)) \leq \tau(x^*x)$, $x \in A$. The following lemma shows that Φ_0 extends to a strongly continuous N-linear unital completely positive map $\Phi = \Phi_1 *_E \Phi_2 \colon P_1 *_N P_2 \to P_1 *_N P_2$ and has been proved in [**B2**].

LEMMA 3.6. *Let P be a finite von Neumann algebra with a* nff *trace τ, acting on $L^2(P, \tau)$ by left multiplication, let P_0 be a unital weakly dense *-subalgebra of P and $\Phi_0 \colon P_0 \to P_0$ be a unital linear map such that $\omega_{1_\tau}\Phi_0 = \omega_{1_\tau}$ and $\Phi_0(x)^*\Phi_0(x) \leq \Phi_0(x^*x)$, $x \in P_0$. Then Φ_0 extends to a strongly continuous contractive map $\Phi \colon P \to P$. If Φ_0 is completely positive on the $*$-algebra P_0, then Φ is completely positive on P.*

For any contraction $T_i \in \mathscr{B}(L^2(P_i, \tau))$ with $T_i = I_{L^2(N,\tau)} \oplus T^0_i$ in the decomposition $L^2(P_i, \tau) = L^2(N, \tau) \oplus L^2(P^0_i, \tau)$, we define as in [**V1**, §5] the contraction $T = T_1 * T_2 \in \mathscr{B}(L^2(P, \tau))$ by

$$T|_{L^2(N,\tau)=I_{L^2(N,\tau)}};$$
$$T|_{L^2(P^0_{i_1},\tau)\otimes\dots\otimes L^2(P^0_{i_n},\tau)} = T^0_{i_1} \otimes \dots \otimes T^0_{i_n} \quad \text{for } i_1 \neq \dots \neq i_n.$$

Denote $E_1(b_1x_1) = E_N(b_1x_1)$ and $E_n(b_1, \dots, b_n, x_1, \dots, x_n) = E_N(b_nE_{n-1}(b_1, \dots, b_{n-1}, x_1, \dots, x_{n-1})x_n)$ for $n \geq 2$, $x_j, b_j \in P$, $1 \leq j \leq n$.

LEMMA 3.7. *If $x_j \in P^0_{i_j}$, $b_j \in P_{i_j}$, $1 \leq j \leq n$, $i_1 \neq \dots \neq i_n$, then $E_N(b_n \cdots b_1x_1 \cdots x_n) = E_n(b_1, \dots, b_n, x_1, \dots, x_n)$.*

Proof. For any $x \in P$, denote $x^0 = x - E_N(x)$. Since the length of $(b_1x_1)^0x_2 \cdots x_n$ is n and $b_n \cdots b_2$ is a sum of words of length $\leq n-1$, then $E_N(b_n \cdots b_2(b_1x_1)^0x_2 \cdots x_n) = 0$ and

$$E_N(b_n \cdots b_1x_1 \cdots x_n) = E_N(b_n \cdots b_2E_N(b_1x_1)x_2 \cdots x_n).$$

Denote

$$\begin{aligned}\widetilde{E}_i(b_1, \dots, b_n, x_1, \dots, x_n) \\ = E(b_n \cdots b_{i+1} E_i(b_1, \dots, b_i, x_1, \dots, x_i) x_{i+1} \cdots x_n), \\ 1 \le i \le n-1,\end{aligned}$$

and assume that

$$E_N(b_n \cdots b_1 x_1 \cdots x_n) = \widetilde{E}_i(b_1, \dots, b_n, x_1, \dots, x_n)$$

for some $i \le n-2$.

The length of $(b_{i+1}E_i(b_1, \dots, b_i, x_1, \dots, x_i)x_{i+1})^0 x_{i+2} \cdots x_n$ is $n-i$ and $b_n \cdots b_{i+2}$ is a sum of words of length $\le n-i-1$, thus $E_N(b_n \cdots b_{i+2}(b_{i+1}E_i(b_1, \dots, b_i, x_1, \dots, x_i)x_{i+1})^0 x_{i+2} \cdots x_n) = 0$ and

$$\begin{aligned}E_N(b_n \cdots b_1 x_1 \cdots x_n) \\ = E_N(b_n \cdots b_{i+2} E_N(b_{i+1} E_i(b_1, \dots, b_i, x_1, \dots, x_i) x_{i+1}) x_{i+2} \cdots x_n) \\ = \widetilde{E}_{i+1}(b_1, \dots, b_n, x_1, \dots, x_n).\end{aligned}$$

Finally

$$\begin{aligned}E_N(b_n \cdots b_1 x_1 \cdots x_n) &= \widetilde{E}_{n-1}(b_1, \dots, b_n, x_1, \dots, x_n) \\ &= E_n(b_1, \dots, b_n, x_1, \dots, x_n). \qquad \square\end{aligned}$$

LEMMA 3.8. *Let* $X_j^0 = \sum_{k_j \in F_j} a_{jk_j} e_N b_{jk_j} \in N' \cap B(L^2(P_j^0, \tau))$ *with* F_j *finite sets and* $a_{jk_j}, b_{jk_j} \in P_j$. *Then*

$$(3.1) \quad \begin{aligned}X_{i_1}^0 \otimes \cdots \otimes X_{i_n}^0 \\ = \sum_{j=1}^{n} \sum_{k_j \in F_j} a_{i_1 k_{i_1}} \cdots a_{i_n k_{i_n}} e_N b_{i_n k_{i_n}} \cdots b_{i_1 k_{i_1}} |_{L^2(P_{i_1}^0, \tau) \otimes \cdots \otimes L^2(P_{i_n}^0, \tau)}\end{aligned}$$

for all $i_1 \ne \cdots \ne i_n$, $n \ge 1$.

Proof. The equality is done by induction on n. The case $n = 1$ is obvious. Assume that (3.1) is true for $i_1 \ne \cdots \ne i_n$ and take $i_{n+1} \ne i_n$. Using the N-linearity of $X_{i_{n+1}}^0$ and Lemma 3.7 we obtain for any $x_j \in P^0_{i_j}$, $1 \le j \le n+1$:

$$
\begin{aligned}
(X^0_{i_1} &\otimes \cdots \otimes X^0_{i_{n+1}})((x_1)_\tau \otimes \cdots \otimes (x_{n+1})_\tau) \\
&= X^0_{i_1}((x_1)_\tau) \otimes \cdots \otimes X^0_{i_{n+1}}((x_{n+1})_\tau) \\
&= \Bigg(\sum_{k_1, \dots, k_n} a_{i_1 k_{i_1}} \cdots a_{i_n k_{i_n}} E_N(b_{i_n k_{i_n}} \cdots b_{i_1 k_{i_1}} x_1 \cdots x_n) \\
&\qquad\qquad \cdot \sum_{k_{n+1}} E_N(b_{i_{n+1} k_{i_{n+1}}} x_{n+1}) \Bigg)_\tau \\
&= \Bigg(\sum_{k_1, \dots, k_{n+1}} a_{i_1 k_{i_1}} \cdots a_{i_{n+1} k_{i_{n+1}}} E_N(b_{i_{n+1} k_{i_{n+1}}} \cdots b_{i_1 k_{i_1}} x_1 \cdots x_{n+1}) \Bigg)_\tau \\
&= \sum_{k_1, \dots, k_{n+1}} a_{i_1 k_{i_1}} \cdots a_{i_{n+1} k_{i_{n+1}}} e_N b_{i_{n+1} k_{i_{n+1}}} \\
&\qquad\qquad \cdots b_{i_1 k_{i_1}} ((x_1)_\tau \otimes \cdots \otimes (x_{n+1})_\tau). \qquad \square
\end{aligned}
$$

PROPOSITION 3.9. *If $N \subset P_1$, $N \subset P_2$ have the Haagerup property, then the inclusion $N \subset P = P_1 *_N P_2$ has the Haagerup property (with respect to the free trace $\tau_{P_1} * \tau_{P_2}$).*

Proof. Using the product net we can assume that the completely positive maps which approximate the unit in P_1 and respectively P_2 are indexed by the same set I. Let $(\Phi_{1,\iota})_{\iota \in I}$ and $(\Phi_{2,\iota})_{\iota \in I}$ be the appropriate nets of completely positive maps for $N \subset P_1$ and $N \subset P_2$ according to Definition 3.1. By a previous remark we can also assume that $\rho_\iota = \max(\|T^0_{\Phi_{1,\iota}}\|, \|T^0_{\Phi_{2,\iota}}\|) < 1$, $\iota \in I$, where

$$T_{\Phi_{j,\iota}} = \begin{pmatrix} I & 0 \\ 0 & T^0_{\Phi_{j,\iota}} \end{pmatrix}$$

according to $L^2(P_j, \tau) = L^2(N, \tau) \oplus L^2(P^0_j, \tau)$, $j = 1, 2$.

Denote $\Phi_\iota = \Phi_{1,\iota} * \Phi_{2,\iota}$. By the previous comments $E_N \Phi_\iota = E_N$, $\Phi_\iota \colon P \to P$ is a N-bimodule unital completely positive map and

$$T_{\Phi_\iota} = T_{\Phi_{1,\iota}} * T_{\Phi_{2,\iota}} = I_{L^2(N,\tau)} \oplus \bigoplus_{n \ge 1, i_1 \ne \cdots \ne i_n} T^0_{\Phi_{i_1,\iota}} \otimes \cdots \otimes T^0_{\Phi_{i_n,\iota}}.$$

Since $\|T_{\Phi_\iota}\| \le 1$, the equality $\lim_{\iota \in I} \|\Phi_\iota(x) - x\|_2 = 0$, $x \in P$, should be checked only on finite sums of reduced words. Since $L^2(P, \tau)$ decomposes in an orthogonal direct sum according to the type of the words, it is enough to check that equality only for reduced words x and in this case it follows by the definition of Φ_ι and by $\lim_{\iota\in I}\|\Phi_{j,\iota}(a)-a\|_2=0$, $a\in P_j$, $j=1,2$.

It remained to check only that $T_{\Phi_\iota}\in\mathscr{K}_N(P)$, $\iota\in I$. Fix an index $\iota\in I$ and denote $T_j=T_{\Phi_{j,\iota}}$, $j=1,2$. For any $0<\varepsilon<1-\rho_\iota$ let $X_j\in\mathscr{F}_N(M)$ with $\|T_j-X_j\|\le\varepsilon$, $\|X_j\|<1$, $j=1,2$, and denote $X_j^0=(1-e_N)X_j(1-e_N)$.

Then $X_j^0\in\mathscr{F}_N(M)$, $X_j^0(L^2(P_j^0,\tau))\subset L^2(P_j^0,\tau)$, $\|X_j^0\|<1$, $\|T_j^0-X_j^0\|\le\varepsilon$, $j=1,2$, and we get for all $i_1\ne\cdots\ne i_n$

$$\begin{aligned}
&\|T_{i_1}^0\otimes\cdots\otimes T_{i_n}^0-X_{i_1}^0\otimes\cdots\otimes X_{i_n}^0\|\\
&\quad\le\|T_{i_1}^0-X_{i_1}^0\|\|T_{i_2}^0\|\cdots\|T_{i_n}^0\|+\|X_{i_1}^0\|\|T_{i_2}^0-X_{i_2}^0\|\|T_{i_3}^0\|\cdots\|T_{i_n}^0\|\\
&\qquad+\cdots+\|X_{i_1}^0\|\cdots\|X_{i_{n-1}}^0\|\|T_{i_n}^0-X_{i_n}^0\|\\
&\quad\le\varepsilon(\rho_\iota^{n-1}+\rho_\iota^{n-2}+\cdots+\rho_\iota+1)\le\frac{\varepsilon}{1-\rho_\iota},
\end{aligned}$$

hence $\|T_{\Phi_\iota}-Y_m\|\le\max(\frac{\varepsilon}{1-\rho_\iota},\rho_\iota^{m+1})$.

By Lemma 3.8 $Y_m=I_{L^2(N,\tau)}\oplus\bigoplus_{n\le m;\,i_1\ne i_n}X_{i_1}^0\otimes\cdots\otimes X_{i_m}^0\in\mathscr{F}_N(P)$ and consequently $T_{\Phi_\iota}\in\mathscr{K}_N(P)$. □

The following lemma contains a couple of immediate examples of inclusions of von Neumann algebras with the Haagerup property.

LEMMA 3.10. (i) *If $Q_0\subset Q$ is an inclusion with the Haagerup property, then $N\otimes Q_0\subset N\otimes Q$ is a Haagerup pair for any finite von Neumann algebra N.*

(ii) *If $N\subset M$ is an inclusion of finite factors with $[M:N]$ finite, then $N\subset M$ is a Haagerup pair.*

(iii) *If $N\otimes Q_0\subset M$ is a Haagerup pair with N, M finite von Neumann algebras and Q_0 is finite dimensional then $N\subset M$ is a Haagerup pair.*

(iv) *If $N\subset M$ is a Haagerup pair with N finite factor and $N_0\subset N$ is a subfactor with finite index, then $N_0\subset M$ is a Haagerup pair.*

(v) *If $P_0\subset P_1\subset P_2\subset\cdots$ are type II_1 factors with $[P_i:P_0]<\infty$, $i\ge0$, and $P_\infty=\overline{\bigcup P_n}$, then $P_0\subset P_\infty$ has the Haagerup property.*

(vi) *If Q is finite dimensional, then $Q_0\subset Q$ is a Haagerup pair.*

Proof. (i) Let $\Phi_\iota\colon Q\to Q$ be E_{Q_0}-preserving Q_0-bimodule unital completely positive maps with $T_{\Phi_\iota}\in\mathscr{K}_{Q_0}(Q)$ and

$$\lim_{\iota\in I}\|\Phi_\iota(x)-x\|_2=0,\qquad x\in Q.$$

Then $\Psi_\iota = \mathrm{id}_N \otimes \Phi_\iota$, $\iota \in I$, satisfy the required properties for the pair $N \otimes Q_0 \subset N \otimes Q$.

(ii) Follows from the equality $\mathrm{id}_M = \sum_i m_i e_N m_i^*$, where $\{m_i\}_i$ is an orthonormal basis of M over N with $m_1 = 1$.

(iii) Since Q_0 is finite dimensional, it is isomorphic to $\bigoplus_{k=1}^m M_{k_\iota}$. Let $\{e_{irs}\}_{1\le i\le m, 1\le r,s\le k_\iota}$ be a matrix unit for Q_0. Since the conditional expectation onto $N \otimes Q_0$ is

$$E_{N\otimes Q_0}(x) = \sum_{i=1}^{m} \sum_{r,s=1}^{k_\iota} \frac{1}{\tau(e_{irr})} e_{irs} E_N(e_{isr}x), \qquad x \in M,$$

then

$$e_{N\otimes Q_0} = \sum_{i=1}^{m} \sum_{r,s=1}^{k_\iota} \frac{1}{\tau(e_{irr})} e_{irs} e_N e_{isr}$$

as operators on $L^2(M, \tau)$. Consequently $\mathscr{F}_{N\otimes Q_0}(M) \subset \mathscr{F}_N(M)$; thus any net that approximates the unit in $N \otimes Q_0$ as in Definition 3.1 satisfies automatically the same property for $N \subset M$.

(iv) Since $e_N = \sum_i m_i e_{N_0} m_i^*$ as operators on $L^2(M, \tau)$ for any orthonormal basis $\{m_i\}_i$ of N over N_0, then $\mathscr{F}_N(M) \subset \mathscr{F}_{N_0}(M)$.

(v) The trace preserving conditional expectations $\Phi_i = E_{P_\iota}^{P_\infty}$, $i \ge 0$, satisfy Definition 3.1.

(vi) By Proposition 3.1.5(iv) in **[J]**, the central support of e_{Q_0} in $\langle Q, e_{Q_0}\rangle$ is 1 i.e. $\bigvee_{u\in\mathscr{U}(Q)} u e_{Q_0} u^* = 1$ and since Q is finite dimensional it follows that $Q = \mathrm{span}\{\sum_{i\in F} a_i e_{Q_0} b_i | a_i, b_i \in Q, F \text{ finite}\}$. Consequently $\mathrm{id}_Q \in \mathscr{F}_{Q_0}(Q)$ and we set $\Phi_\iota = \mathrm{id}_Q$, $\iota \in I$. □

PROPOSITION 3.11. *Let $(Q \subset N \subset M; Q \subset P \subset M)$ be a commutative square such that $N \subset M$ has the Haagerup property and Q is finite dimensional. Then P has the Haagerup approximation property.*

Proof. Let $(\Phi_\iota)_{\iota\in I}$ be a net of unital N-linear completely positive maps $\Phi_\iota\colon M \to M$ with $E_N\Phi_\iota = E_N$, $\lim_{\iota\in I} \|\Phi_\iota(x) - x\|_2 = 0$, $x \in M$ and $T_{\Phi_\iota} \in \mathscr{K}_N(M)$. Consider $\widetilde{\Phi}_\iota = E_P^M \Phi_\iota|_P\colon P \to P$, $\iota \in I$, which are unital Q-linear completely positive maps, $\tau\widetilde{\Phi}_\iota = \tau$ and $\|\widetilde{\Phi}_\iota(x) - x\|_2 = \|E_P^M(\Phi_\iota(x)) - E_P^M(x)\|_2 \le \|\Phi_\iota(x) - x\|_2$, $x \in P$.

Finally, we have to check that $T_{\widetilde{\Phi}_\iota} \in \mathscr{K}(L^2(P, \tau))$. Since $T_{\Phi_\iota} \in \mathscr{F}_N(M)$, it follows that for any $\varepsilon > 0$ there exists $T = \sum_i a_i e_N^M b_i \in \mathscr{F}_N(M)$ such that $\|T_{\Phi_\iota} - T\| \le \varepsilon$ and consequently $\|T_{\widetilde{\Phi}_\iota} - e_P^M T e_P^M\| = \|e_P^M(T_{\Phi_\iota} - T)e_P^M\| \le \varepsilon$.

Therefore we have only to check that $e_P^M T e_P^M \in \mathscr{K}(L^2(P, \tau))$. Let $(\eta_j)_{j\in J} \subset L^2(N, \tau)$ be an orthonormal basis of N over Q (cf. **[P3, 1.1.3]**) with $f_j = E_Q(\eta_j^* \eta_j) \in \mathscr{P}(Q)$. Since $x = \sum_j \eta_j E_Q(\eta_j^* x)$ for all $x \in N$, it follows that

$$e_N^M \leq p = \sum_j \eta_j e_P^M \eta_j^*,$$

where p is the orthogonal projection from $L^2(M, \tau)$ onto $\bigoplus_j \eta_j L^2(P, \tau)$.

For any $a, b \in M$ we get

$$\begin{aligned} e_P^M a e_N^M b e_P^M &= \sum_{j,k} e_P^M a \eta_j e_P^M \eta_j^* e_N^M \eta_k e_P^M \eta_k^* b e_P^M \\ &= \sum_{j,k} E_P(a\eta_j) e_P^M e_N^M \eta_j^* \eta_k e_N^M e_P^M E_P(\eta_k^* b) \\ &= \sum_{j,k} E_P(a\eta_j) E_Q(\eta_j^* \eta_k) e_Q^P E_P(\eta_k^* b) \\ &= \sum_j E_P(a\eta_j) f_j e_Q^P f_j E_P(\eta_j^* b) \\ &= \sum_j E_P(a\eta_j) e_Q^P E_P(\eta_j^* b). \end{aligned}$$

Since Q is finite dimensional and

$$\sum_j \|E_P(a\eta_j)\|_2^2 = \left\| P_{\bigoplus_j L^2(P, \tau)\eta_j}^{L^2(M, \tau)} (a_\tau) \right\|_2^2 \leq \|a\|_{2,\tau}^2 < \infty,$$

$$\sum_j \|E_P(\eta_j^* b)\|_2^2 = \left\| P_{\bigoplus_j \eta_j L^2(P, \tau)}^{L^2(M, \tau)} (b_\tau) \right\|_2^2 \leq \|b\|_{2,\tau}^2 < \infty$$

it follows that $e_P^M T e_P^M$ is a compact operator on $L^2(P, \tau)$. □

REMARKS. (1) The previous computations show that if $Q \subset N$ has a finite orthonormal basis, then $Q \subset P$ is also a Haagerup inclusion.

(2) The proof didn't use the fact that $T_{\Phi_i} \in N'$. In fact that condition is important only to achieve Proposition 3.9.

COROLLARY 3.12. *If $N \subset M$ is a Haagerup inclusion and the center of N is finite dimensional, then the relative commutant $N' \cap M$ has the Haagerup approximation property.*

COROLLARY 3.13. *If Q_0 is finite dimensional and $Q_0 \subset Q$ is a Haagerup inclusion, then all the von Neumann algebras $M_{-1}^s = N^s$,*

$M_0^s = M^s$, $M_{i+1}^s = vN(M_i^s, e_{i+1})$, $i \geq -1$, *with* M^s *and* N^s *defined as in Chapter* 1 (*not necessarily factors*), *have the Haagerup approximation property* (*the trace on each* M_i^s *is the restriction of the free trace on* M_∞^s).

Proof. By Lemma 3.10 $R^s \otimes Q_0 \subset R^s \otimes Q$ and $R^s \otimes Q_0 \subset R \otimes Q_0$ are Haagerup inclusions and by Proposition 3.9 the inclusion $R^s \otimes Q_0 \subset M_\infty^s = (R^s \otimes Q) *_{R^s \otimes Q_0} (R \otimes Q_0)$ is also of Haagerup type. Since Q_0 is finite dimensional and $[R^s : R_{-i}^s] < \infty$, $i \geq 0$, the same property is still true for $R_{-i}^s = A_\infty^{i+2} \subset M_\infty^s$. According to Corollary 3.12, the von Neumann algebra $(R_{-i}^s)' \cap M_\infty^s$ has the Haagerup approximation property. But $[M^s, R^s] = 0$; hence $M_i^s \subset (R_{-i}^s)' \cap M_\infty^s$ and M_i^s has itself the Haagerup property for all $i \geq -1$. □

Corollary 3.14. *If* Q *is finite dimensional, then all the von Neumann algebras* M_i^s, $i \geq -1$, *have the Haagerup approximation property.*

Proof. It follows by Corollary 3.13 and by Lemma 3.10(vi). □

Corollary 3.15. *If* $Q_0 \subset \mathscr{Z}(Q)$ *and* $Q_0 \subset Q$ *is a Haagerup inclusion, then the von Neumann algebras* M_i^s, $i \geq -1$, *have the Haagerup property.*

Proof. Since Q commutes with Q_0, it follows that $[M^s, R^s \otimes Q_0] = 0$ and consequently $[M_i^s, R_{-i}^s \otimes Q_0] = 0$. Since $R_{-i}^s \otimes Q_0 \subset R^s \otimes Q_0$ is a Haagerup inclusion it follows that M_i^s has the Haagerup property. □

Corollary 3.16. *If* $Q_0 = \mathbb{C}$, *then the corresponding algebras* M_i^s, $i \geq -1$, *have the Haagerup property if and only if* Q *has this property.*

Corollary 3.17. *If* Q_0 *and* Q *are as in Corollary* 3.13 *or Corollary* 3.15, *then none of the von Neumann algebras* M_i^s, $i \geq -1$, *contains a rigid subfactor.*

Proof. It follows by Corollaries 3.13 and 3.15 and by the arguments from [**CJ**, Theorem 3] or [**P1**, Theorem 4.3.1]. □

At the end of this chapter we show that the Haagerup property for an inclusion of group von Neumann algebras is related to the existence of certain positive definite functions on the group, with some special properties.

Let G be a discrete countable group with unit e. The group von Neumann algebra $\mathscr{L}(G)$ associated with G is defined as follows: let G acting on $l^2(G)$ by $g \cdot f = \delta_g * f$, $f \in l^2(G)$, $g \in G$ (δ_g is the evaluation function in g) and denote by u_g the unitary operator on $l^2(G)$ given by $u_g f = \delta_g * f$, $f \in l^2(G)$, $g \in G$.

Then $\mathscr{L}(G)$ is the bicommutant of $\{u_g\}_{g\in G}$ in $\mathscr{B}(l^2(G))$. Note that this action of G on $l^2(G)$ extends to a *-representation $\lambda\colon \mathbb{C}[G] \to \mathscr{B}(l^2(G))$ of the group algebra $\mathbb{C}[G]$ on $l^2(G)$ defined by:

$$\lambda\left(\sum_{g\in G}{}' a_g\delta_g\right) = \sum_{g\in G}{}' a_g u_g\,, \qquad a_g \in \mathbb{C}\,.$$

The use of the notation $\sum'$ signifies that only a finite number of a_g's are nonzero.

The linear functional $\tau\colon \lambda(\mathbb{C}[G]) \to \mathbb{C}$ defined by $\tau(\sum'_{g\in G} a_g u_g) = a_e$, $a_g \in \mathbb{C}$, extends to a nff trace on $\mathscr{L}(G)$ and this will be the trace considered on $\mathscr{L}(G)$ from now on.

For G_0 subgroup of G, $\mathscr{L}(G_0)$ is isomorphic to the weak closure of $\lambda(\mathbb{C}[G_0])$ in $\mathscr{L}(G)$ and the map $E\colon \lambda(\mathbb{C}[G]) \to \mathscr{L}(G_0)$, defined by $E(\sum'_{g\in G} a_g u_g) = \sum'_{g\in G_0} a_g u_g$, $a_g \in \mathbb{C}$, extends to the τ-preserving conditional expectation $E = E_{\mathscr{L}(G_0)}\colon \mathscr{L}(G) \to \mathscr{L}(G_0)$.

Proposition 3.18. *If $\mathscr{L}(G_0) \subset \mathscr{L}(G)$ is a Haagerup inclusion, then there exists a net $(\phi_\iota)_{\iota\in I}$ of G_0-bivariant positive defined functions on G such that*

(i) $\phi_\iota(e) = 1$, $\iota \in I$;

(ii) $\lim_{\iota\in I} \tilde{\phi}_\iota(x) = \lim_{\iota\in I} \tilde{\tilde{\phi}}_\iota(y) = 1$, *for all* $x \in G/G_0$, $y \in G_0\backslash G$, *where* $\tilde{\phi}_\iota$ *(respectively* $\tilde{\tilde{\phi}}_\iota$*) denotes the map induced by the* G_0*-invariant map* ϕ_ι *on the left cosets* G/G_0 *(respectively on the right cosets* $G_0\backslash G$*).*

(iii) *Each* $\tilde{\phi}_\iota$ *(respectively* $\tilde{\tilde{\phi}}_\iota$*) vanishes at infinity on* G/G_0 *(respectively on* $G_0\backslash G$*) i.e. for any* $\iota \in I$, $\varepsilon > 0$, *there exists a finite set* $F^1_{\iota,\varepsilon} \subset G/G_0$ *(respectively* $F^2_{\iota,\varepsilon} \subset G_0\backslash G$*) such that* $|\tilde{\phi}_\iota(x)| < \varepsilon$, *for* $x \in (G/G_0)\backslash F^1_{\iota,\varepsilon}$ *(respectively* $|\tilde{\tilde{\phi}}_\iota(y)| < \varepsilon$, *for* $x \in (G_0\backslash G)\backslash F^2_{\iota,\varepsilon}$*).*

Proof. Let $\Phi_\iota\colon \mathscr{L}(G) \to \mathscr{L}(G)$, $\iota \in I$, be a net of $\mathscr{L}(G_0)$-linear unital completely positive maps and define the functions $\phi_\iota\colon G \to \mathbb{C}$, $\phi_\iota(g) = \tau(u_g^*\Phi_\iota(u_g))$, $g \in G$.

The $\mathscr{L}(G_0)$-linearity of Φ_ι yields for any $g \in G$, $g_0 \in G_0$:

$$\begin{aligned}\phi_\iota(gg_0) &= \tau(u_{g_0}^* u_g^* \Phi_\iota(u_g)u_{g_0}) = \tau(u_g^*\Phi_\iota(u_g)) \\ &= \phi_\iota(g) = \tau((u_{g_0}u_g)^*\Phi_\iota(u_{g_0}u_g)) = \phi_\iota(g_0 g).\end{aligned}$$

Since ϕ_ι are G_0-invariant, (ii) is equivalent to $\lim_{\iota\in I}\phi_\iota(g) = 1$, for all $g \in G$, and this follows by

$$|\phi_\iota(g) - 1| = |\langle \Phi_\iota(u_g) - u_g\,, u_g\rangle_\tau| \leq \|\Phi_\iota(u_g) - u_g\|_2$$

and by $\lim_{\iota\in I}\|\Phi_\iota(u_g) - u_g\|_2 = 0$, $g \in G$.

Fix $\iota \in I$ and denote $\Phi = \Phi_\iota$, $\phi = \phi_\iota$. Clearly $\phi(e) = 1$ and the complete positivity of Φ yields

$$\begin{aligned}\sum_{i,j=1}^n \lambda_i\bar{\lambda}_j\phi(g_j^{-1}g_i) &= \sum_{i,j=1}^n \tau((\lambda_j u_{g_j})\Phi(u_{g_j}^* u_{g_i})\bar{\lambda}_i u_{g_i^{-1}}) \\ &= \sum_{i,j=1}^n \langle \Phi(u_{g_j}^* u_{g_i})\bar{\lambda}_i u_{g_i^{-1}}\,, \bar{\lambda}_j u_{g_j^{-1}}\rangle_\tau \geq 0,\end{aligned}$$

for all $g_1, \dots, g_n \in G$, $\lambda_1, \dots, \lambda_n \in \mathbb{C}$; hence ϕ is positive defined on G.

Finally, let us check that $\tilde{\phi}$ vanishes at infinity on G/G_0. Let S be a complete system of representations in G for G/G_0. Since $\{u_g\}_{g\in G}$ is a left orthonormal basis of $\mathscr{L}(G)$ over $\mathscr{L}(G_0)$, we get

$$\begin{aligned} b &= \sum_{g\in S} u_g E_{\mathscr{L}(G_0)}(u_g^* b), \qquad b \in \mathscr{L}(G) \quad \text{and} \\ \|b\|_2 &= \sum_{g\in S}\|u_g E_{\mathscr{L}(G_0)}(u_g^* b)\|_2^2 = \sum_{g\in S}\|E_{\mathscr{L}(G_0)}(u_g^* b)\|_2^2.\end{aligned}$$

Let $\varepsilon > 0$. Then there exists a finite set $F_{b,\varepsilon} \subset S$ such that

$$\sum_{g\in S\setminus F_{b,\varepsilon}} \|E_{\mathscr{L}(G_0)}(u_g^* b)\|_2^2 \leq \varepsilon^2.$$

Since $T_\Phi \in \mathscr{K}_{\mathscr{L}(G_0)}(\mathscr{L}(G))$, there exist $a_i, b_i \in \mathscr{L}(G)$, $1 \leq i \leq n$, such that $\|T_\Phi - \sum_{i=1}^n a_i e_{\mathscr{L}(G_0)} b_i\| \leq \varepsilon$. In particular

$$\left\|\Phi(u_g) - \sum_{i=1}^{n} a_i E_{\mathscr{L}(G_0)}(b_i u_g)\right\|_2 \leq \varepsilon, \qquad g \in G$$

and

$$\begin{aligned}
|\phi(g)| &= |\tau(\Phi(u_g)u_g^*)| \\
&\leq \left|\tau\left(\left(\Phi(u_g) - \sum_{i=1}^{n} a_i E_{\mathscr{L}(G_0)}(b_i u_g)\right) u_g^*\right)\right| \\
&\quad + \sum_{i=1}^{n} |\tau(a_i E_{\mathscr{L}(G_0)}(b_i u_g) u_g^*)| \\
&\leq \varepsilon + \sum_{i=1}^{n} \|a_i\|_2 \|E_{\mathscr{L}(G_0)}(b_i u_g)\|_2 \\
&= \varepsilon + \sum_{i=1}^{n} \|a_i\|_2 \|E_{\mathscr{L}(G_0)}(u_g^* b_i^*)\|_2, \quad \text{for all } g \in G.
\end{aligned}$$

Denote $M = \max_{1\leq i\leq n} \|a_i\|_2$ and let $\varepsilon_0 = \frac{\varepsilon}{M\sqrt{n}}$, $F_\varepsilon = \bigcup_{i=1}^n F_{b_i^*, \varepsilon_0}$. The previous inequality yields $|\phi(g)| \leq 2\varepsilon$, $\forall g \in G \backslash F_\varepsilon G_0$. □

Remark that in order to prove the previous statement we used only that $T_{\Phi_\iota} \in \overline{\text{span}}^{\|\ \|}\{a e_{\mathscr{L}(G_0)} b | a, b \in \mathscr{L}(G)\}$.

Consider the action $h \cdot gG_0 = hgG_0$, $h \in G_0$, $g \in G$ of G_0 on the left cosets G/G_0. The orbit of an element $gG_0 \in G/G_0$ under this action is $\{wgG_0 | w \in G_0\}$. If G_0 is normal in G, then G_0 acts trivially on G/G_0, the orbit of each element gG_0 being $\{gG_0\}$.

PROPOSITION 3.19. *Let G_0 be a subgroup of the countable discrete group G. Assume that the orbit of each element $gG_0 \in G/G_0$ under the action of G_0 is finite and there exists a net of G_0-bivariant positive defined functions $\phi_\iota \colon G \to \mathbb{C}$ such that*

(i) $\phi_\iota(e) = 1$;
(ii) $\lim_{\iota\in I} \phi_\iota(g) = 1$, $g \in G$;
(iii) *each $\tilde{\phi}_\iota$ vanishes at infinity on G/G_0.*

Then the inclusion $\mathscr{L}(G_0) \subset \mathscr{L}(G)$ has the Haagerup property.

Proof. Define $\Phi_\iota : \lambda(\mathbb{C}[G]) \to \lambda(\mathbb{C}[G])$ by

$$\Phi_\iota\left(\sum_{g\in G}{}' a_g u_g\right) = \sum_{g\in G}{}' \phi_\iota(g) a_g u_g\,, \qquad a_g \in \mathbb{C}.$$

In order to check that each Φ_ι is completely positive on $\lambda(\mathbb{C}[G])$, let $\xi_i \in l^2(G)$ and

$$x_i = \sum_{g\in G}{}' a_{g,i} u_g \in \lambda(\mathbb{C}[G])\,, \qquad 1 \le i \le n.$$

Denote $\eta_g = \sum_{i=1}^n a_{g,i} u_g \xi_i \in l^2(G)$, $g \in G$. We obtain

$$\begin{aligned} \sum_{i,j=1}^n \langle \Phi_\iota(x_j^* x_i)\xi_i, \xi_j\rangle &= \sum_{i,j=1}^n \sum_{g,h\in G}{}' \langle \overline{a_{g,j}} a_{g,i} \Phi_\iota(u_h^* u_g)\xi_i, \xi_j\rangle \\ &= \sum_{i,j=1}^n \sum_{g,h\in G}{}' \overline{a_{h,j}} a_{g,i} \phi_\iota(h^{-1}g)\langle u_h^* u_g \xi_i, \xi_j\rangle \\ &= \sum_{g,h\in G}{}' \phi_\iota(g^{-1}h)\langle \eta_g, \eta_h\rangle \ge 0, \end{aligned}$$

since the (finite) matrix $[\phi_\iota(g^{-1}h)\cdot\langle\eta_g, \eta_h\rangle]_{g,h\in G}$ is positive, as the pointwise product of the positive matrices $[\phi_\iota(g^{-1}h)]_{g,h\in G}$ and $[\langle\eta_g, \eta_h\rangle]_{g,h\in G}$ and the sum of the entries of a positive matrix is positive.

Since $\phi_\iota(g) = 1$, $g \in G_0$, it follows that $E_{\mathscr{L}(G_0)}\Phi_\iota = E_{\mathscr{L}(G_0)}$ and each Φ_ι extends to a $E_{\mathscr{L}(G_0)}$-preserving unital completely positive map $\Phi_\iota : \mathscr{L}(G) \to \mathscr{L}(G)$ (cf. [**H**, Proposition 1]). The G_0-bivariance of ϕ_ι implies the $\mathscr{L}(G_0)$-linearity of Φ_ι.

In order to check $\lim_{\iota\in I}\|\Phi_\iota(x) - x\|_2 = 0$, $x \in \mathscr{L}(G)$, note that, since $\|T_{\Phi_\iota}\| \le 1$, it is enough to consider only the case $x = u_g$, $g \in G$, and the equality follows by $\lim_{\iota\in I}\phi(g) = 1$, $g \in G$, and by

$$\begin{aligned} \|\Phi_\iota(u_g) - u_g\|_2^2 &= \|\Phi_\iota(u_g)\|_2^2 + \|u_g\|_2^2 - 2\,\mathrm{Re}\langle\Phi_\iota(u_g), u_g\rangle_\tau \\ &\le 2 - 2\,\mathrm{Re}\,\tau(u_g^*\Phi_\iota(u_g)) = 2 - 2\,\mathrm{Re}\,\phi_\iota(g). \end{aligned}$$

We prove now that $T_{\Phi_\iota} \in \mathscr{K}_{\mathscr{L}(G_0)}(\mathscr{L}(G))$, $\iota \in I$. Fix $\iota \in I$ and denote $\phi = \phi_\iota$, $\Phi = \Phi_\iota$. Let S be a complete system of representants for G/G_0 in G and let X_iG_0, $i \geq 1$ be the orbits of G/G_0 under the action of G_0, with X_i finite subsets of S, $i \geq 1$, and $\bigcup_{i\geq 1} X_i = S$. Note that $X_iG_0 = g_0X_iG_0$, $g_0 \in G_0$, $i \geq 1$.

For each orbit XG_0 of G/G_0 under the action of G_0, we check that $\sum_{g\in X} u_g e_{\mathscr{L}(G_0)} u_g^* \in \mathscr{L}(G_0)'$.

Indeed, for any $g_0 \in G_0$, $x = \sum'_{w\in G} a_w u_w$, $a_w \in \mathbb{C}$, we have

$$\begin{aligned}
&\sum_{g\in X} u_g e_{\mathscr{L}(G_0)} u_g^* u_{g_0} x_\tau \\
&\quad = \sum_{g\in X}\sum_{w\in G}{}' a_w u_g E_{\mathscr{L}(G_0)}(u_{g^{-1}g_0w}) = \sum_{g\in X}\sum_{w\in G_0}{}' a_{g_0^{-1}gw} u_{gw} \\
&\quad = \sum_{g\in XG_0}{}' a_{g_0^{-1}g} u_g = \sum_{g\in g_0XG_0}{}' a_{g_0^{-1}g} u_g = \sum_{g\in XG_0}{}' a_g u_{g_0g} \\
&\quad = \sum_{g\in X}\sum_{w\in G_0}{}' a_{gw} u_{g_0gw} = \sum_{g\in X}\sum_{w\in G}{}' a_w u_{g_0g} E_{\mathscr{L}(G_0)}(u_{g^{-1}w}) \\
&\quad = u_{g_0} \sum_{g\in X} u_g e_{\mathscr{L}(G_0)} u_g^* x_\tau .
\end{aligned}$$

Since

$$u_g e_{\mathscr{L}(G_0)} u_g^* \left(\sum_{w\in G}{}' a_w u_w\right)_\tau = \sum_{w\in G_0}{}' a_{gw} u_{gw} ,$$

we get

$$\sum_{g\in X} u_g e_{\mathscr{L}(G_0)} u_g^* = P^{l^2(G)}_{l^2(XG_0)} .$$

Let $S_n = \bigcup_{i=1}^n X_i \subset S$ and define $T_n = \sum_{g\in S_n} \phi(g) u_g e_{\mathscr{L}(G_0)} u_g^* = \sum_{g\in S_n} \phi(g) P^{l^2(G)}_{l^2(gG)}$. Note that, since ϕ is G_0-bivariant, it is constant on each orbit X_i, and hence $T_n \in \mathscr{L}(G_0)'$. Since ϕ is positive defined and $\phi(e) = 1$, one easily checks that $|\phi(g)| \leq 1$, $g \in G$, and hence $\|T_n\| \leq 1$.

Since $\tilde{\phi}$ vanishes at infinity on G/G_0, there exists a subsequence $\{k_n\}_{n\geq 1}$ such that $\sup_{g\in S\setminus S_{k_n}} |\phi(g)| \leq \frac{1}{n}$.

Let $x \in \mathscr{L}(G)$. Since $\{u_g\}_{g\in S}$ is a left orthonormal basis for $\mathscr{L}(G)$ over $\mathscr{L}(G_0)$, it follows that for any $\varepsilon > 0$, there exists $k(\varepsilon) \geq 1$ such that $\|x - \sum_{g\in S_k} u_g E_{\mathscr{L}(G_0)}(u_g^* x)\|_2 \leq \varepsilon \|x\|_2$, for all $k \geq k(\varepsilon)$.

Then, pick an $n \geq 1$ and assume that $k(\varepsilon) \geq k_n$. We obtain

$$\begin{aligned}
\|T_\Phi x_\tau - T_{k_n} x_\tau\|_2 &= \|\Phi(x)_\tau - T_{k_n} x_\tau\|_2 \\
&\leq \varepsilon + \left\| \sum_{g \in S_{k(\varepsilon)}} \Phi(u_g E_{\mathscr{L}(G_0)}(u_g^* x)) - \sum_{g \in S_{k_n}} \phi(g) u_g E_{\mathscr{L}(G_0)}(u_g^* x) \right\|_2 \\
&= \varepsilon + \sum_{g \in S_{k(\varepsilon)} \setminus S_{k_n}} \|\phi(g) u_g E_{\mathscr{L}(G_0)}(u_g^* x)\|_2 \\
&= \varepsilon + \left(\sum_{g \in S_{k(\varepsilon)} \setminus S_{k_n}} \|\phi(g) u_g E_{\mathscr{L}(G_0)}(u_g^* x)\|_2^2 \right)^{1/2} \\
&\leq \varepsilon + \left(\sum_{g \in S \setminus S_{k_n}} |\phi(g)|^2 \|E_{\mathscr{L}(G_0)}(u_g^* x)\|_2^2 \right)^{1/2} \\
&\leq \varepsilon + \frac{1}{n} \left(\sum_{g \in S \setminus S_{k_n}} \|E_{\mathscr{L}(G_0)}(u_g^* x)\|_2^2 \right)^{1/2} \leq \left(\varepsilon + \frac{1}{n} \right) \|x\|_2 .
\end{aligned}$$

Consequently $\|T_\Phi - T_{k_n}\| \leq \frac{1}{n}$ and $T_\Phi \in \mathscr{K}_{\mathscr{L}(G_0)}(\mathscr{L}(G))$. □

COROLLARY 3.20. *If G_0 is a normal subgroup of the discrete countable group G, then $\mathscr{L}(G_0) \subset \mathscr{L}(G)$ has the Haagerup property if and only if there exists a net $(\phi_\iota)_{\iota \in I}$ of unital positive defined functions on the quotient group G/G_0 that vanish at infinity on G/G_0 and such that $\lim_{\iota \in I} \phi_\iota(g) = 1$, for all $g \in G/G_0$.*

The Propositions 3.18 and 3.19 were proved in **[Ch]** for $G_0 = \{e\}$.

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