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TWO-POINT DISTORTION THEOREMS FOR UNIVALENT FUNCTIONS

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We establish a one-parameter family of symmetric, linearly invariant two-point distortion theorems for univalent functions defined on the unit disk. The weakest theorem in the family is a symmetric, linearly invariant form of a classical distortion theorem of Koebe, while another special case is a distortion theorem of Blatter. All of these distortion theorems are necessary and sufficient for univalence. Each of these distortion theorems can be expressed as a two-point comparison theorem between euclidean and hyperbolic geometry on a simply connected region; however, none of these comparison theorems characterize simply connected regions. We obtain analogous results for convex univalent functions and convex regions, except that in this context the two-point comparison theorems do characterize convex regions.

1. Introduction. We begin by recalling some basic information about the hyperbolic metric and related material. The hyperbolic metric on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ is given by

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{|dz|}{1-|z|^2}.$$

It is normalized to have constant Gaussian curvature -4. A region Ω in the complex plane $\mathbb{C}$ is called hyperbolic if $\mathbb{C}\backslash\Omega$ contains at least two points. The density of the hyperbolic metric on a hyperbolic region Ω is obtained from

$$\lambda_\Omega(f(z))|f'(z)| = \lambda_{\mathbb{D}}(z),$$

where $f : \mathbb{D} \to \Omega$ is any holomorphic universal covering projection of $\mathbb{D}$ onto Ω. The density is independent of the choice of the covering projection of $\mathbb{D}$ onto Ω. The hyperbolic metric on Ω induces the hyperbolic distance function d_Ω as follows:

$$d_\Omega(a, b) = \inf \int_\gamma \lambda_\Omega(w)|dw|,$$

where the infimum is taken over all paths γ in Ω joining a and b. The infimum is actually a minimum; there always exists a path δ in

Ω connecting a and b such that

$$d_\Omega(a, b) = \int_\delta \lambda_\Omega(w)|dw|.$$

Any such path δ is called a hyperbolic geodesic joining a and b. There may be more than one hyperbolic geodesic joining a and b when Ω is not simply connected. Recall that

$$d_{\mathbb{D}}(a, b) = \operatorname{artanh}\left|\frac{b-a}{1-\bar{a}b}\right|.$$

Both the hyperbolic metric and the hyperbolic distance are conformally invariant.

Blatter **[1]** commented that a classical distortion theorem of Koebe for normalized univalent functions $g(z) = z + a_2z^2 + a_3z^3 + \cdots$, namely,

$$|g(z)| \geq \frac{|z|}{(1+|z|)^2}, \qquad z \in \mathbb{D},$$

was necessary, but not sufficient, for univalence. Recall that equality holds at $z \neq 0$ if and only if g is a rotation of the Koebe function $k(z) = z/(1-z)^2$ **[3**, p. 33**]**. Koebe's distortion theorem is a consequence of the coefficient bound $|a_2| \leq 2$ for normalized univalent functions. Blatter inquired whether there were distortion theorems for univalent functions that were also sufficient for univalence. He established the following two-point distortion theorem which is both necessary and sufficient for univalence **[1]**. There is no normalization on the univalent function.

BLATTER'S DISTORTION THEOREM. *Suppose f is univalent in $\mathbb{D}$ and $a, b \in \mathbb{D}$. Then*

$$|f(a) - f(b)|^2 \geq \frac{\sinh^2(2d_{\mathbb{D}}(a, b))}{8\cosh(4d_{\mathbb{D}}(a, b))}([(1-|a|^2)|f'(a)|]^2 + [(1-|b|^2)|f'(b)|]^2).$$

Equality holds for distinct points $a, b \in \mathbb{D}$ if and only if $f = S \circ k \circ T$, where S is a conformal automorphism of $\mathbb{C}$, k is the Koebe function and T is a conformal automorphism of $\mathbb{D}$, and a and b lie on the axis of symmetry of f. Conversely, if a nonconstant holomorphic function f satisfies this inequality, then f is univalent on $\mathbb{D}$.

The square on the term $\sinh^2(2d_{\mathbb{D}}(a, b))$ is missing in the statement, but not in the proof, of this result in Blatter's paper. The proof of Blatter's distortion theorem is more sophisticated than the proof of Koebe's distortion theorem; it requires three coefficient inequalities for normalized univalent functions: $|a_2| \leq 2$, $|a_3| \leq 3$, and $|a_3 - a_2^2| \leq 1$. Blatter's distortion theorem is symmetric in a and b and linearly invariant. In this context, linear invariance means that if f is replaced in the inequality by $\tilde{f} = S \circ f \circ T$, where S is a conformal automorphism of $\mathbb{C}$ and T is a conformal automorphism of $\mathbb{D}$, then the new inequality has exactly the same form, except that f is replaced by $\tilde{f}$. This is closely related to the notion of linear invariance introduced by Pommerenke [**13**]. We shall establish a one-parameter family of symmetric, linearly invariant two-point distortion theorems for univalent functions; each of these distortion theorems characterizes univalence. The method of proof is an extension of Blatter's technique. The weakest two-point distortion theorem in the family is a symmetric, linearly invariant version of Koebe's distortion theorem. Blatter's distortion theorem is stronger than the symmetric, linearly invariant version of Koebe's distortion theorem, but is not the strongest one in the family.

Blatter's distortion theorem can easily be formulated as a two-point comparison theorem between euclidean and hyperbolic geometry on a simply connected region. It relates the euclidean distance between two points to their hyperbolic distance and the density of the hyperbolic metric at the points. This formulation asserts that if Ω is a simply connected hyperbolic region in $\mathbb{C}$ and $A, B \in \Omega$, then

$$|A - B|^2 \geq \frac{\sinh^2(2d_\Omega(A, B))}{8\cosh(4d_\Omega(A, B))}\left(\frac{1}{\lambda_\Omega^2(A)} + \frac{1}{\lambda_\Omega^2(B)}\right).$$

Equality holds if and only if Ω is a slit plane and A and B lie on the extension of the slit into Ω. This two-point comparison theorem can be viewed as an extension of the inequality $\lambda_\Omega \geq 1/(4\delta_\Omega)$ for simply connected regions [**6**, p. 45], where $\delta_\Omega(z)$ is the euclidean distance from z to $\partial\Omega$, since this inequality is a limiting case. Because Blatter's distortion theorem characterizes univalence, it is natural to inquire whether this comparison inequality characterizes simply connected regions. The answer is negative. In fact, there is a one-parameter family of similar two-point comparison theorems and not even the strongest comparison theorem in the family characterizes simple connectivity. Narrow annuli also satisfy these comparison inequalities.

Finally, we consider analogs of these results for both convex univalent functions and convex regions. The case of convex univalent functions parallels the univalent function situation. There is a one-parameter family of two-point distortion theorems for convex univalent functions, the weakest of which is the symmetric, linearly invariant version of a classical distortion theorem. These distortion theorems all characterize convex univalent functions. There is an associated one-parameter family of two-point comparison theorems for euclidean and hyperbolic geometry on convex regions. These comparison theorems characterize convex regions and are refinements of the inequality $\lambda_\Omega \geq 1/(2\delta_\Omega)$ **[10]** for convex regions.

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2. Preliminaries. We first recall some results from Blatter's paper **[1]**. Some of these are reformulated in invariant terms here, while others are stated in more generality. We do not prove these generalizations if the proofs given in **[1]** immediately extend.

Minimum Principle. Suppose that a function $u\colon [-L, L] \to \mathbb{R}$ satisfies the following two conditions:

(i) $|u'| \leq q$,
(ii) $u'' \leq p(q^2 - (u')^2)$,

where p and q are positive constants. If v is the solution of the inequality $|y'| \leq q$ and the differential equation $y'' = p(q^2 - (y')^2)$ which satisfies the boundary conditions $v(L) = u(L)$ and $v(-L) = u(-L)$, then $u(s) \geq v(s)$ for all $s \in [-L, L]$. Moreover, if strict inequality holds in both (i) and (ii), then $u(s) > v(s)$ for all $s \in (-L, L)$.

The solution v can be expressed in elementary form:

$$v(s) = \frac{1}{p}\log[\cosh(pqs) + \tau\sinh(pqs)] + \log C,$$

where the constants $\tau \in [-1, 1]$ and $C > 0$ are determined by the boundary conditions. In fact,

$$C = \left(\frac{\exp(pu(L)) + \exp(pu(-L))}{2\cosh(pqL)}\right)^{1/p}.$$

LEMMA 1. *For $p > 1$, $q > 0$ and $\tau \in [-1, 1]$ let*

$$B(\tau) = \int_{-L}^{L} (\cosh(pqs) + \tau\sinh(pqs))^{1/p}\, ds.$$

Then for $\tau \in (-1, 1)$

$$B(\tau) > B(\pm 1) = \frac{2}{q}\sinh(qL).$$

Proof. Now,

$$B'(\tau) = \frac{1}{p}\int_{-L}^{L} \sinh(pqs)(\cosh(pqs) + \tau\sinh(pqs))^{(1-p)/p}\,ds$$

and

$$B''(\tau) = \frac{1-p}{p^2}\int_{-L}^{L} \sinh^2(pqs)\,(\cosh(pqs) + \tau\sinh(pqs))^{(1-2p)/p}\,ds.$$

Thus, $B''(\tau) < 0$ since $p > 1$, so $B(\tau)$ is strictly concave on $[-1, 1]$. This implies that the minimum value of $B(\tau)$ is either $B(1)$ or $B(-1)$. Because

$$B(1) = B(-1) = \frac{2}{q}\sinh(qL),$$

the proof is complete.

REMARKS. (i) When $p = 1$ the function $B(\tau)$ is the constant $\frac{2}{q}\sinh(qL)$.

(ii) If u and v are as in the statement of the minimum principle, then

$$\int_{-L}^{L} \exp(u(s))\,ds \geq \int_{-L}^{L} \exp(v(s))\,ds = CB(\tau) \geq C\frac{2}{q}\sinh(qL),$$

with equality if and only if $\exp u(s) = C\exp(\pm qs)$.

Next, we want to recall some differential geometric formulas relating to locally schlicht holomorphic functions. Before stating these formulas, it is convenient to introduce several invariant differential operators which were also considered in **[3]** and **[8]**. For a holomorphic function f defined on $\mathbb{D}$, let

$$\begin{aligned} D_1 f(z) &= (1-|z|^2)f'(z), \\ D_2 f(z) &= (1-|z|^2)^2 f''(z) - 2\bar{z}(1-|z|^2)f'(z), \\ D_3 f(z) &= (1-|z|^2)^3 f'''(z) - 6\bar{z}(1-|z|^2)^2 f''(z) \\ &\quad + 6\bar{z}^2(1-|z|^2)f'(z). \end{aligned}$$

If $T(z) = (z+a)/(1+\bar{a}z)$, then $D_j f(a) = (f\circ T)^{(j)}(0)$ for $j = 1, 2, 3$. In particular, $D_j f(0)$ is just the ordinary jth derivative at the origin. These differential operators are invariant in the sense that

$$|D_j(S\circ f\circ T)| = |D_j(f)|\circ T \qquad (j = 1, 2, 3),$$

where T is any conformal automorphism of $\mathbb{D}$ and S is any euclidean motion of $\mathbb{C}$ **[8]**. Observe that for a locally schlicht function f

$$\frac{D_3 f(z)}{D_1 f(z)} - \frac{3}{2}\left(\frac{D_2 f(z)}{D_1 f(z)}\right)^2 = (1-|z|^2)^2 S_f(z),$$

where

$$S_f(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

denotes the Schwarzian derivative of f. For a locally schlicht holomorphic function f defined on the unit disk it is useful to introduce the abbreviation

$$Q_f(z) = \frac{D_2 f(z)}{D_1 f(z)} = (1-|z|)^2 \frac{f''(z)}{f'(z)} - 2\bar{z}.$$

Now, we establish some notation that will be in force for the remainder of the paper. Suppose f is a locally schlicht holomorphic function defined on the unit disk $\mathbb{D}$. We assume that there is a Jordan arc γ in $\mathbb{D}$ with finite hyperbolic length $2L$ joining a and b such that f maps γ injectively onto the euclidean segment $[f(a), f(b)] = [A, B]$. Suppose the arc γ is parametrized by hyperbolic arc length, say $\gamma\colon z = z(s)$, $s \in [-L, L]$. This implies $z'(s) = (1-|z(s)|^2)e^{i\theta(s)}$, where $\theta(s) = \arg z'(s)$. The hyperbolic curvature of γ is

$$\begin{aligned}\kappa_h(z(s), \gamma) &= (1-|z(s)|^2)\kappa_e(z(s), \gamma) + \operatorname{Im}\left\{\frac{2\overline{z(s)}z'(s)}{|z'(s)|}\right\} \\ &= (1-|z(s)|^2)\kappa_e(z(s), \gamma) + \operatorname{Im}\{2\overline{z(s)}e^{i\theta(s)}\}.\end{aligned}$$

Here $\kappa_e(z(s), \gamma)$ is the euclidean curvature of γ at $z(s)$; explicitly,

$$\kappa_e(z(s), \gamma) = \frac{1}{|z'(s)|}\operatorname{Im}\left\{\frac{z''(s)}{z'(s)}\right\}.$$

The formula which relates the euclidean curvature of $f\circ\gamma$ to the hyperbolic curvature of γ is

$$\kappa_e(f(z(s)), f\circ\gamma)|D_1 f(z(s))| = \kappa_h(z(s), \gamma) + \operatorname{Im}\left\{Q_f(z(s))\frac{z'(s)}{|z'(s)|}\right\}.$$

When $f\circ\gamma$ is a euclidean line segment, this simplifies to

$$\kappa_h(z(s), \gamma) = -\operatorname{Im}\left\{Q_f(z(s))\frac{z'(s)}{|z'(s)|}\right\}.$$

The rate of change of the euclidean curvature of $f \circ \gamma$ is related to the rate of change of the hyperbolic curvature of γ by

$$\begin{aligned}\frac{d\kappa_e(f(z(s)), f\circ\gamma)}{ds}&|D_1 f(z(s))| \\ &= \frac{d\kappa_h(z(s), \gamma)}{ds} + \operatorname{Im}\left\{(1-|z(s)|^2)^2 S_f(z(s))\left(\frac{z'(s)}{|z'(s)|}\right)^2\right\}.\end{aligned}$$

When $f \circ \gamma$ is a euclidean line segment, this becomes

$$\frac{d\kappa_h(z(s), \gamma)}{ds} = -\operatorname{Im}\left\{(1-|z(s)|^2)^2 S_f(z(s))\left(\frac{z'(s)}{|z'(s)|}\right)^2\right\}.$$

Set

$$u(s) = \log|D_1 f(z(s))|.$$

Then

$$u'(s) = \operatorname{Re}\{Q_f(z(s))e^{i\theta(s)}\},$$

so that

$$|u'(s)| \le |Q_f(z(s))|$$

and

$$(u')^2(s) = \frac{1}{2}\operatorname{Re}\{(Q_f(z(s)))^2 e^{2i\theta(s)}\} + \frac{1}{2}|Q_f(z(s))|^2.$$

Also,

$$u''(s) = \operatorname{Re}\{(1-|z(s)|^2)^2 S_f(z(s))e^{2i\theta(s)}\} + \frac{1}{2}|Q_f(z(s))|^2 - 2.$$

By making use of some of these formulas, we obtain the identity

$$\begin{aligned}u''(s) &+ p(u')^2(s) \\ &= \operatorname{Re}\left\{\left[(1-|z(s)|^2)^2 S_f(z(s)) + \frac{p}{2}(Q_f(z(s)))^2\right] e^{2i\theta(s)}\right\} \\ &\quad + \frac{p+1}{2}|Q_f(z(s))|^2 - 2,\end{aligned}$$

and so the differential inequality

$$\begin{aligned}u''(s) + p(u')^2(s) \le &\left|(1-|z(s)|^2)^2 S_f(z(s)) + \frac{p}{2}\,(Q_f(z(s)))^2\right| \\ &+ \frac{p+1}{2}|Q_f(z(s))|^2 - 2.\end{aligned}$$

3. Univalent functions and simply connected regions. We establish symmetric, linearly invariant, two-point distortion theorems for univalent functions and consider the associated two-point comparison theorems between euclidean and hyperbolic geometry on simply connected regions.

INVARIANT KOEBE DISTORTION THEOREM. *Suppose f is univalent on $\mathbb{D}$. Then for all a, $b \in \mathbb{D}$,*

$$|f(a) - f(b)| \geq \frac{\sinh(2d_{\mathbb{D}}(a, b))}{2\exp(2d_{\mathbb{D}}(a, b))} \max\{|D_1 f(a)|, |D_1 f(b)|\}.$$

Equality holds for distinct points a, $b \in \mathbb{D}$ if and only if $f = S \circ k \circ T$, where S is a conformal automorphism of $\mathbb{C}$, k is the Koebe function and T is a conformal automorphism of $\mathbb{D}$, and a and b lie on the axis of symmetry of f. Conversely, if a nonconstant holomorhpic function f satisfies this inequality, then f is univalent on $\mathbb{D}$.

Proof. First, note that Koebe's classical distortion theorem can be written in the form

$$|g(z)| \geq \frac{|z|}{(1 + |z|)^2} = \frac{\sinh(2d_{\mathbb{D}}(0, z))}{2\exp(2d_{\mathbb{D}}(0, z))}.$$

Here g is a normalized univalent function.

Now, assume f is univalent (not necessarily normalized) in $\mathbb{D}$ and a, $b \in \mathbb{D}$. Set $T(z) = (z + a)/(1 + \bar{a}z)$; T is a conformal automorphism of $\mathbb{D}$ which sends 0 to a. Then

$$g(z) = [f \circ T(z) - f \circ T(0)]/(f \circ T)'(0)$$

is a normalized univalent function. If we apply the classical Koebe distortion theorem to g and use the fact that hyperbolic distance is conformally invariant, then we obtain

$$|f(a) - f(b)| \geq \frac{\sinh(2d_{\mathbb{D}}(a, b))}{2\exp(2d_{\mathbb{D}}(a, b))}|D_1 f(a)|.$$

We obtain a similar inequality when we interchange the roles of a and b. The final formula is obtained by taking the maximum value of these two lower bounds on $|f(a) - f(b)|$. The necessary and sufficient conditions for equality follow from the conditions for equality in the classical Koebe distortion theorem.

The fact that the condition is sufficient for univalence is elementary, but we give the details here and then omit them in subsequent related theorems. Suppose f is a nonconstant holomorphic function defined on $\mathbb{D}$ which satisfies the inequality. Assume $f(a) = f(b)$ for distinct points a, $b \in \mathbb{D}$. The inequality implies that $f'(a) = f'(b) = 0$. Then f is not univalent in any neighborhood of a (or b), so there exist two sequences $\{a_n\}$ and $\{b_n\}$ of distinct points such that $a_n \to a$, $b_n \to a$ and $f(a_n) = f(b_n)$ for all n. This gives $f'(a_n) = 0$ for all n which contradicts the fact that f is nonconstant since this implies f' must have an isolated zero at a. Hence, f is univalent on $\mathbb{D}$.

Thus, the invariant form of Koebe's distortion theorem is sufficient for univalence, so it provides an elementary answer to the question raised by Blatter. Theorem 2 will provide a connection between the invariant form of Koebe's distortion theorem and Blatter's distortion theorem. But first we need to establish a result for normalized univalent functions.

THEOREM 1. *If* $g(z) = z+a_2z^2+a_3z^3+\cdots$ *is a normalized univalent function on* $\mathbb{D}$, *then*

$$\left|a_3 - \left(\frac{3-p}{3}\right)a_2^2\right| + \frac{p}{3}|a_2|^2 \leq \begin{cases} 1+2\exp\left(\frac{2p-3}{p}\right), & 0<p<\frac{3}{2}, \\ \frac{8p-3}{3}, & \frac{3}{2}\leq p. \end{cases}$$

This inequality is sharp for all $p>0$. *For* $p\geq 3/2$, *equality holds if and only if* g *is a rotation of the Koebe function.*

Proof. It suffices to obtain the sharp upper bound on the functional

$$\begin{aligned} L_p(g) &= \operatorname{Re}\left\{a_3 - \left(\frac{3-p}{3}\right)a_2^2\right\} + \frac{p}{3}|a_2|^2 \\ &= \operatorname{Re}\{a_3\} - \left(\frac{3-2p}{3}\right)(\operatorname{Re}a_2)^2 + (\operatorname{Im}a_2)^2 \end{aligned}$$

over the family of normalized univalent functions. Because replacing $g(z)$ by $-g(-z)$ does not change the value of $L_p(g)$, we may assume that $\operatorname{Re}\{a_2\}\geq 0$ without loss of generality. Since $0\leq \operatorname{Re}\{a_2\}\leq 2$, there is a unique $\lambda\in[0, 2]$ with $\operatorname{Re}\{a_2\} = \lambda(1+\log\frac{2}{\lambda})$.

Jenkins [5] obtained the sharp relationship between the second and third coefficients of a normalized univalent function. We shall use the version of this result from [14, p. 120]; specifically, we need inequality (12) of this reference which states

$$\operatorname{Re}\{a_3\} \leq (\operatorname{Re}a_2)^2 - (\operatorname{Im}a_2)^2 - 2\lambda\operatorname{Re}a_2 + \lambda^2\log\frac{2}{\lambda} + \frac{3}{2}\lambda^2 + 1.$$

From this inequality we obtain

$$\begin{aligned} L_p(g) &\le \frac{2p}{3}(\operatorname{Re} a_2)^2 - 2\lambda(\operatorname{Re} a_2) + \lambda^2 \log\frac{2}{\lambda} + \frac{3}{2}\lambda^2 + 1 \\ &= \left(\frac{4p-3}{6}\right)\lambda^2 + \left(\frac{4p-3}{3}\right)\lambda^2 \log\frac{2}{\lambda} \\ &\quad + \frac{2p}{3}\lambda^2\left(\log\frac{2}{\lambda}\right)^2 + 1 = H(\lambda). \end{aligned}$$

Note that $H(0) = 1$, $H(2) = (8p-3)/3$ and

$$H'(\lambda) = \frac{2\lambda}{3}\log\left(\frac{2}{\lambda}\right)\left[2p - 3 + 2p\log\left(\frac{2}{\lambda}\right)\right].$$

For $p \ge 3/2$, $H'(\lambda)$ has no roots in $(0, 2)$, so $H(\lambda)$ is strictly increasing in this case with maximum value $(8p-3)/3$ attained uniquely at $\lambda = 2$. This produces the sharp upper bound on $L_p(g)$ when $p \ge 3/2$, and implies that equality holds only if g is a rotation of the Koebe function. It is trivial that equality holds for a rotation of the Koebe function. When $0 < p < 3/2$, $H'(\lambda)$ has a root at $\lambda_0 = 2\exp((2p-3)/(2p)) \in (0, 2)$ and $H(\lambda)$ is increasing on $(0, \lambda_0)$ and decreasing on $(\lambda_0, 1)$. Thus, $H(\lambda)$ has maximum value $H(\lambda_0) = 1 + 2\exp((2p-3)/p)$ when $0 < p < 3/2$. The sharpness of the inequality in this case follows from the work of Jenkins; note that the Koebe function is not extremal.

COROLLARY. *If $g(z) = z + a_2 z^2 + a_3 z^3 + \cdots$ is a normalized univalent function on $\mathbb{D}$, then*

$$\left|a_3 - \frac{1}{2}a_2^2\right| + \frac{5}{6}|a_2|^2 \le \frac{13}{3}$$

with equality if and only if f is a rotation of the Koebe function.

Proof. By making use of the theorem with $p = 3/2$ and $|a_2| \le 2$, we get

$$\left|a_3 - \frac{1}{2}a_2^2\right| + \frac{5}{6}|a_2|^2 \le \left|a_3 - \frac{1}{2}a_2^2\right| + \frac{1}{2}|a_2|^2 + \frac{1}{3}|a_2|^2 \le 3 + \frac{4}{3} = \frac{13}{3}.$$

THEOREM 2. *Suppose f is univalent in $\mathbb{D}$. There is a constant $P \in (1, 3/2]$ such that for any $p \ge P$ and all $a, b \in \mathbb{D}$,*

$$|f(a) - f(b)| \ge \frac{\sinh(2d_{\mathbb{D}}(a, b))}{2\left[2\cosh(2p d_{\mathbb{D}}(a, b))\right]^{1/p}}\left(|D_1 f(a)|^p + |D_1 f(b)|^p\right)^{1/p}.$$

Equality holds for distinct points $a, b \in \mathbb{D}$ if and only if $f = S \circ k \circ T$, where S is a conformal automorphism of $\mathbb{C}$, k is the Koebe function and T is a conformal automorphism of $\mathbb{D}$, and a and b lie on the axis of symmetry of f. Conversely, if a nonconstant holomorphic function f satisfies this inequality, then f is univalent on $\mathbb{D}$.

Proof. The sufficiency for univalence follows exactly as in the proof of the invariant form of the Koebe distortion theorem.

For the necessity, we make use of the notation established in §2. Because f is univalent, we know that $|u'(s)| \leq 4$; this is the invariant version of the sharp classical coefficient bound $|a_2| \leq 2$ for normalized univalent functions [2, p. 32]. We will make use of some of the results from §2 with $q = 4$. Suppose $p \geq 1$ is any number such that

$$(1) \qquad \left|(1-|z|^2)^2 S_f(z) + \frac{p}{2}\left(Q_f(z)\right)^2\right| + \frac{p+1}{2}|Q_f(z)|^2 - 2 \leq 16p$$

for every univalent function f defined on $\mathbb{D}$ and all $z \in \mathbb{D}$. Then the results of §2 with $q = 4$ give

$$u''(s) + p(u')^2(s) \leq 16p.$$

Therefore, we get

$$\begin{aligned} |f(a) - f(b)| &= \int_{-L}^{L} |f'(z(s))||dz(s)| \\ &= \int_{-L}^{L} (1-|z(s)|^2)|f'(z(s))| \frac{|dz(s)|}{1-|z(s)|^2} \\ &= \int_{-L}^{L} \exp u(s)\, ds \geq \int_{-L}^{L} \exp v(s)\, ds \geq \frac{C \sinh(4L)}{2}, \end{aligned}$$

with equality if and only if $\exp u(s) = C \exp(\pm 4s)$, where

$$C = \left(\frac{|D_1 f(a)|^p + |D_1 f(b)|^p}{2\cosh(4pL)}\right)^{1/p}.$$

Thus,

$$|f(a) - f(b)| \geq \frac{\sinh(4L)}{2[2\cosh(4pL)]^{1/p}} \left(|D_1 f(a)|^p + |D_1 f(b)|^p\right)^{1/p}.$$

Since the function $h(t) = \sinh(t)/[2\cosh(pt)]^{1/p}$ is increasing and $2d_{\mathbb{D}}(a, b) \leq 4L$, we obtain

$$|f(a) - f(b)| \geq \frac{\sinh(2d_{\mathbb{D}}(a, b))}{2[2\cosh(2pd_{\mathbb{D}}(a, b))]^{1/p}} \left(|D_1 f(a)|^p + |D_1 f(b)|^p\right)^{1/p}.$$

This establishes the lower bound when $[f(a), f(b)]$ is contained in $f(\mathbb{D})$. If equality holds, then $d_{\mathbb{D}}(a, b) = 2L$ and so γ must be a hyperbolic geodesic.

We require a limiting form of this inequality. Set $\Omega = f(\mathbb{D})$. Suppose $\alpha \in \partial\Omega$ and $[f(a), \alpha) \subset \Omega$. Then for any $b \in \mathbb{D}$ with $f(b) \in [f(a), \alpha)$, the preceding inequality gives

$$|f(a) - f(b)| \geq \frac{\sinh(2d_{\mathbb{D}}(a, b))}{2[2\cosh(2pd_{\mathbb{D}}(a, b))]^{1/p}}|D_1 f(a)|.$$

When $f(b) \to \partial\Omega$ along the segment $[f(a), \alpha)$, then $b \to \partial\mathbb{D}$ and so $d_{\mathbb{D}}(a, b) \to \infty$. Since $h(\infty) = 1/2$, we get

$$|f(a) - \alpha| \geq \frac{1}{4}|D_1 f(a)|.$$

This is just an invariant form of the Koebe 1/4-theorem.

Now, suppose $[f(a), f(b)]$ does not lie in Ω. Then there exist points $\alpha, \beta \in \partial\Omega$ such that the half-open intervals $[f(a), \alpha)$ and $(\beta, f(b)]$ are disjoint, lie in Ω and their union is contained in $[f(a), f(b)]$. The preceding inequality implies that

$$|f(a) - \alpha| \geq \frac{1}{4}|D_1 f(a)| \quad \text{and} \quad |f(b) - \beta| \geq \frac{1}{4}|D_1 f(b)|.$$

Hence,

$$\begin{aligned} |f(a) - f(b)| &\geq |f(a) - \alpha| + |f(b) - \beta| \geq \frac{1}{4}\left(|D_1 f(a)| + |D_1 f(b)|\right) \\ &\geq \frac{1}{4}\left(|D_1 f(a)|^p + |D_1 f(b)|^p\right)^{1/p}. \end{aligned}$$

Since $h(\infty) = 1/2$ and h is strictly increasing, we obtain

$$|f(a) - f(b)| > \frac{\sinh(2d_{\mathbb{D}}(a, b))}{2[2\cosh(2pd_{\mathbb{D}}(a, b))]^{1/p}}\left(|D_1 f(a)|^p + |D_1 f(b)|^p\right)^{1/p}.$$

This establishes the lower bound in all cases.

Next, we determine necessary and sufficient conditions for equality. If equality holds, then γ must be a hyperbolic geodesic in $\mathbb{D}$. By performing a conformal automorphism of $\mathbb{D}$ if necessary, we may assume that $\gamma \subset (-1, 1)$ and is symmetric about the origin. There is no harm in assuming $[f(a), f(b)] \subset \mathbb{R}$ and is symmetric about the origin with $f(a) < 0$ and $f(b) = -f(a)$; if this were not true just compose f with a conformal automorphism of $\mathbb{C}$. Then the hyperbolic arc length parametrization of γ is $z(s) = \tanh(s)$ and $f'(z(s)) > 0$ for $s \in [-L, L]$. Symmetry implies $f(0) = 0$. Equality forces $\exp(u) = C\exp(\pm 4s)$. We consider the plus sign; the case of the minus sign is similar. We have

$$(1 - z(s)^2)f'(z(s)) = C\exp(4s).$$

Since

$$s = \operatorname{artanh} z = \frac{1}{2}\log\frac{1+z}{1-z}$$

holds on γ, we obtain

$$(1 - z^2)f'(z) = C\left(\frac{1+z}{1-z}\right)^2$$

or

$$f'(z) = C\frac{1+z}{(1-z)^3}$$

for z on γ. The identity theorem implies that this holds for all z in $\mathbb{D}$. Since $f(0) = 0$, we get $f(z) = Ck(z)$. This demonstrates that if equality holds then $f = S \circ k \circ T$, where S is a conformal automorphism of $\mathbb{C}$, k is the Koebe function and T is a conformal automorphism of $\mathbb{D}$, and a and b lie on the axis of symmetry of f. Conversely, if f has this form, then it is straightforward to show that equality holds for all points on the axis of symmetry of f, or equivalently, equality holds for all pairs of points on $(-1, 1)$ for the Koebe function itself.

Finally, we show that inequality (1) holds for all $p \geq P$, where P is some constant in $(1, 3/2]$. It is elementary to verify that if inequality (1) holds for one value of $p \geq 1$, then it also holds for all larger values of p. Let P be the minimum of all $p \geq 1$ such that inequality (1) holds for all univalent functions f defined on $\mathbb{D}$. Since the class of univalent functions is linearly invariant, it suffices to establish inequality (1) for $z = 0$ and normalized univalent functions. Thus, we want to find the smallest value of p such that

$$\left|a_3 - \left(\frac{3-p}{3}\right)a_2^2\right| + \left(\frac{p+1}{3}\right)|a_2|^2 - \frac{1}{3} \leq \frac{8p}{3}.$$

The corollary to Theorem 1 shows that this inequality is valid for $p = 3/2$. It might seem plausible that $P = 1$; this is equivalent to the coefficient inequality

$$\left|a_3 - \frac{2}{3}a_2^2\right| \leq 3 - \frac{2}{3}|a_2|^2$$

for a normalized univalent function. However, Ruscheweyh **[15]**, with the use of a computer, has shown that this inequality is false for the full class S of normalized univalent functions and that the best result for the class S is about

$$\left|a_3 - \frac{2}{3}a_2^2\right| + \frac{2}{3}|a_2|^2 < 3.0031896592.$$

Thus, $P > 1$.

REMARKS. (i) What is the best value of P in Theorem 2?

(ii) The right-hand side of the inequality in Theorem 2 is a decreasing function of p for $p \geq 1$. Consequently, the weakest necessary condition for univalence that Theorem 2 yields is the case $p = \infty$, or more precisely, $p \to \infty$. This is the invariant version of Koebe's distortion theorem. The case $p = 2$ is Blatter's distortion theorem, but it is not the strongest two-point distortion theorem contained in Theorem 2.

COROLLARY. *Let Ω be a simply connected hyperbolic region in $\mathbb{C}$. Then for any $p \geq P$ and all $A, B \in \Omega$,*

$$|A - B| \geq \frac{\sinh(2d_\Omega(A, B))}{2[2\cosh(2pd_\Omega(A, B))]^{1/p}} \left(\frac{1}{\lambda_\Omega^p(A)} + \frac{1}{\lambda_\Omega^p(B)}\right)^{1/p}.$$

Equality holds if and only if Ω is a slit plane A and B lie on the extension of the slit into Ω.

Proof. Apply the theorem to a conformal map f of $\mathbb{D}$ onto Ω and make use of the facts that f is an isometry from the hyperbolic metric on $\mathbb{D}$ to the hyperbolic metric on Ω and $|D_1 f(z)| = 1/\lambda_\Omega(f(z))$.

REMARK. Suppose Ω is any region which satisfies the inequality in the corollary for some $p \geq P$. Fix $A \in \Omega$. Select $\alpha \in \partial\Omega$ so that $|A - \alpha| = \delta_\Omega(A)$. Let $B \in \Omega$ tend to α along the half-open segment $[A, \alpha)$. Then $d_\Omega(A, B) \to \infty$ since the hyperbolic distance is complete and $\lambda_\Omega(B) \to \infty$ **[12]** so the inequality in the corollary yields $\lambda_\Omega \geq 1/(4\delta_\Omega)$. For simply connected regions this inequality is equivalent to the Koebe 1/4-theorem for univalent functions **[6**, p. 45].

EXAMPLE. Let $\Omega = \Omega(\delta) = \{z\colon \exp(-\pi\delta/2) < |z| < \exp(\pi\delta/2)\}$ for $\delta > 0$. We shall show that if $\delta > 0$ is sufficiently small, then for $A, B \in \Omega$

$$|A - B| \geq \frac{1}{4}\tanh[2d_\Omega(A, B)]\left(\frac{1}{\lambda_\Omega(A)} + \frac{1}{\lambda_\Omega(B)}\right).$$

This inequality corresponds to the case $p = 1$; it is the strongest possible lower bound in the corollary and shows that no comparison theorem in the corollary can characterize simply connected regions.

A holomorphic universal covering projection of $\mathbb{D}$ onto Ω is $f(z) = [(1+z)/(1-z)]^{i\delta}$. Then [13, p. 128]

$$\sup\{|Q_f(z)| : z \in \mathbb{D}\} = 2\sqrt{1+\delta^2}$$

and [11]

$$\sup\{(1-|z|^2)^2|S_f(z)| : z \in \mathbb{D}\} = 2(1+\delta^2).$$

We shall show that

$$|u'(s)| \le 4$$

and

$$u''(s) + (u')^2(s) \le 16$$

for δ sufficiently small. This is the case $p = 1$ and $q = 4$ in §2. Note that

$$|u'(s)| \le |Q_f(z(s))| \le 2\sqrt{1+\delta^2},$$

so the desired bound on $|u'(s)|$ will hold when $\delta \le \sqrt{3}$. The other differential inequality will hold if

$$\left|(1-|z|^2)^2 S_f(z) + \frac{1}{2}\left(Q_f(z)\right)^2\right| + \left|Q_f(z)\right|^2 \le 18,$$

which is weaker than

$$(1-|z|^2)^2|S_f(z)| + \frac{3}{2}|Q_f(z)|^2 \le 18.$$

The preceding bounds show that this inequality will hold if $8(1+\delta^2) \le 18$, that is, provided $\delta \le \sqrt{5}/2$. Thus, both needed inequalities hold when $\delta \le \sqrt{5}/2$.

The proof of Theorem 2 shows that if $[f(a), f(b)] \subset \Omega$, then

$$|f(a) - f(b)| \ge \frac{1}{4}(\tanh(4L))(|D_1 f(a)| + |D_1 f(b)|).$$

Since $\tanh(t)$ is an increasing function and $d_\Omega(f(a), f(b)) \le 2L$, this gives

$$|f(a) - f(b)| \ge \frac{1}{4}(\tanh(2d_\Omega(f(a), f(b))))(|D_1 f(a)| + |D_1 f(b)|),$$

or equivalently,

$$|f(a) - f(b)| \ge \frac{1}{4}(\tanh(2d_\Omega(f(a), f(b))))\left(\frac{1}{\lambda_\Omega(f(a))} + \frac{1}{\lambda_\Omega(f(b))}\right).$$

This is the desired result when $[A, B] = [f(a), f(b)] \subset \Omega$. Then, just as in the proof of Theorem 2, this inequality holds even if $[f(a), f(b)]$ does not lie entirely in Ω. In fact, strict inequality holds in this case.

REMARK. If $g(z) = z + a_2z^2 + a_3z^3 + \cdots$ is a normalized close-to-convex function on $\mathbb{D}$, then Wancang Ma [7] has shown

$$\left|a_3 - \frac{2}{3}a_2^2\right| \leq 3 - \frac{2}{3}|a_2|^2$$

with equality if and only if f is a rotation of the Koebe function. Thus, if f is a close-to-convex univalent function, then the inequality in Theorem 2 holds for all $p \geq 1$. Does the inequality in Theorem 2 for $p = 1$ characterize close-to-convex univalent functions? Similarly, the inequality in the corollary to Theorem 2 holds for $p \geq 1$ if the region Ω is close-to-convex.

4. Convex univalent functions and convex regions. We now turn our attention to convex hyperbolic regions and convex univalent functions.

THEOREM 3. *Suppose Ω is a convex hyperbolic region. Then for any $p \geq 1$ and all $A, B \in \Omega$,*

$$|A - B| \geq \frac{\sinh(d_\Omega(A, B))}{[2\cosh(pd_\Omega(A, B))]^{1/p}} \left(\frac{1}{\lambda_\Omega^p(A)} + \frac{1}{\lambda_\Omega^p(B)}\right)^{1/p}.$$

Equality holds if and only if Ω is a half-plane and A and B lie on a line perpendicular to the edge of the half-plane. Conversely, if Ω is a hyperbolic region in $\mathbb{C}$ and the preceding inequality holds for some $p \geq 1$ and all $A, B \in \Omega$, then Ω is convex.

Proof. We first show that a hyperbolic region which satisfies the inequality must be convex. Fix $A \in \Omega$. As in the remark after the corollary to Theorem 2, select $\alpha \in \partial\Omega$ so that $|A - \alpha| = \delta_\Omega(A)$. Let $B \in \Omega$ tend to α along the half-open segment $[A, \alpha)$. Then $d_\Omega(A, B) \to \infty$ and $\lambda_\Omega(B) \to \infty$, so the inequality in the theorem yields $\lambda_\Omega \geq 1/(2\delta_\Omega)$. This inequality characterizes convex regions ([4], [9]).

Now, we turn to the proof of the inequality when Ω is convex. The proof is very similar to that of Theorem 2. If f is a conformal mapping of $\mathbb{D}$ onto Ω, then $|u'(s)| \leq 2$ is the invariant form of the coefficient bound $|a_2| \leq 1$ for a normalized convex univalent function [2, p. 45]. Therefore, we want to use the results from §2 with $q = 2$, so we wish to determine all $p \geq 1$ such that

$$\left|(1-|z|^2)^2 S_f(z) + \frac{p}{2}(Q_f(z))^2\right| + \frac{p+1}{2}|Q_f(z)|^2 - 2 \leq 4p$$

for any convex univalent function f defined on $\mathbb{D}$ and all $z \in \mathbb{D}$. It is easy to verify that if this inequality holds for some value of p, then it also holds for all larger values of p. We shall establish it when $p = 1$:

$$\left|(1-|z|^2)^2 S_f(z) + \frac{1}{2}(Q_f(z))^2\right| + |Q_f(z)|^2 \leq 6. \tag{2}$$

Trimble **[16]** established the following inequality for convex functions when $z = 0$; this was rediscovered and established in invariant form by Harmelin **[3]**:

$$(1-|z|^2)^2 |S_f(z)| + \frac{1}{2}|Q_f(z)|^2 \leq 2.$$

It is now clear that (2) holds.

Then from §2 with $q = 2$, we have

$$u''(s) + p(u')^2(s) \leq 4p.$$

Given $A, B \in \Omega$, select $a, b \in \mathbb{D}$ with $f(a) = A$ and $f(b) = B$. Since Ω is convex, the straight line segment $[f(a), f(b)]$ always lies in Ω. Then we get

$$|f(a) - f(b)| = \int_{-L}^{L} \exp u(s)\, ds \geq \int_{-L}^{L} \exp v(s)\, ds \geq C \sinh(2L),$$

where

$$C = \left(\frac{|D_1 f(a)|^p + |D_1 f(b)|^p}{2\cosh(2pL)}\right)^{1/p}.$$

Thus,

$$|f(a) - f(b)| \geq \frac{\sinh(2L)}{[2\cosh(2pL)]^{1/p}} \left(|D_1 f(a)|^p + |D_1 f(b)|^p\right)^{1/p},$$

or

$$|A - B| \geq \frac{\sinh(2L)}{[2\cosh(2pL)]^{1/p}} \left(\frac{1}{\lambda_\Omega^p(A)} + \frac{1}{\lambda_\Omega^p(B)}\right)^{1/p}.$$

Recall that $2L$ denotes the hyperbolic length (relative to Ω) of the segment $[A, B]$. Since the function $h(t) = \sinh(t)/[2\cosh(pt)]^{1/p}$ is increasing and $d_\Omega(A, B) \leq 2L$, we obtain

$$|A - B| \geq \frac{\sinh(d_\Omega(A, B))}{[2\cosh(p d_\Omega(A, B))]^{1/p}} \left(\frac{1}{\lambda_\Omega^p(A)} + \frac{1}{\lambda_\Omega^p(B)}\right)^{1/p}.$$

This establishes the lower bound.

Finally, we determine when equality holds. First, suppose $p > 1$. If equality holds, then $[A, B]$ must be a hyperbolic geodesic. There is no harm in assuming $[A, B] \subset \mathbb{R}$ and is symmetric about the origin with $A < 0$ and $B = -A$; if this were not true, apply a conformal automorphism of $\mathbb{C}$ to Ω. Now, γ is a hyperbolic geodesic in $\mathbb{D}$; by performing a conformal automorphism of $\mathbb{D}$ if necessary, we may assume that $\gamma \subset (-1, 1)$ and is symmetric about the origin. The hyperbolic arclength parametrization of the path γ is $z(s) = \tanh(s)$ and $f'(z(s)) > 0$ for $s \in [-L, L]$. Symmetry implies $f(0) = 0$. Equality forces $\exp(u) = C\exp(\pm 2s)$. We consider the plus sign; the case of the minus sign is similar. As in the proof of Theorem 2, we obtain

$$f'(z) = \frac{C}{(1-z)^2}.$$

Since $f(0) = 0$, $f(z) = CK(z)$, where $K(z) = z/(1-z)$. In this situation $\Omega = f(\mathbb{D})$ is a half-plane and the segment $[A, B]$ is orthogonal to the edge of the half-plane. Conversely, if Ω is a half-plane, it is straightforward to show that equality holds whenever $[A, B]$ is orthogonal to the edge of the half-plane. It is sufficient to verify this for the special case of the upper half-plane $\mathbb{H} = \{z\colon \operatorname{Im} z > 0\}$. In this case,

$$d_{\mathbb{H}}(A, B) = \operatorname{artanh}\left|\frac{A-B}{A-\overline{B}}\right| \quad \text{and} \quad \lambda_{\mathbb{H}}(z) = \frac{1}{2\operatorname{Im}(z)}.$$

We omit the details.

It remains to consider the case of equality when $p = 1$. In this situation Lemma 1 does not apply, so we use a different method. If Ω is not a half-plane, then $|u'(s)| < 2$ and $u''(s) + (u')^2(s) < 4$. These strict inequalities imply that equality cannot hold in this case. Thus, we need only determine necessary and sufficient conditions for equality when Ω is a half-plane. Because of the invariance of the inequality under conformal automorphisms of $\mathbb{C}$, we may assume Ω is the upper half-plane $\mathbb{H} = \{z\colon \operatorname{Im} z > 0\}$. We need to determine when equality holds in

$$(3) \qquad |A-B| \geq \frac{1}{2}\tanh(d_{\mathbb{H}}(A, B))\left(\frac{1}{\lambda_{\mathbb{H}}(A)} + \frac{1}{\lambda_{\mathbb{H}}(B)}\right).$$

Inequality (3) is equivalent to

$$|A - \overline{B}| \geq \operatorname{Im}(A) + \operatorname{Im}(B).$$

But trivially

$$|A - \overline{B}| \geq \operatorname{Im}(A - \overline{B}) = \operatorname{Im}(A) + \operatorname{Im}(B)$$

with equality if and only if $\operatorname{Re}(A - \overline{B}) = 0$, that is, $\operatorname{Re} A = \operatorname{Re} \overline{B} = \operatorname{Re} B$. In geometric terms this necessary and sufficient condition for equality is that $[A, B]$ be orthogonal to the real axis, the edge of $\mathbb{H}$.

COROLLARY. *Suppose f is univalent in $\mathbb{D}$ and $f(\mathbb{D})$ is a convex region. Then for $p \geq 1$ and all $a, b \in \mathbb{D}$,*

(4)

$$|f(a) - f(b)| \geq \frac{\sinh(d_{\mathbb{D}}(a, b))}{[2\cosh(p d_{\mathbb{D}}(a, b))]^{1/p}} \left(|D_1 f(a)|^p + |D_1 f(b)|^p\right)^{1/p}.$$

Equality holds for distinct $a, b \in \mathbb{D}$ if and only if $f = S \circ K \circ T$, where S is a conformal automorphism of $\mathbb{C}$, $K(z) = z/(1 - z)$ and T is a conformal automorphism of $\mathbb{D}$, and a and b lie on any axis of symmetry of f. Conversely, if a nonconstant holomorphic function f defined on $\mathbb{D}$ satisfies this inequality for some $p \geq 1$, then f is univalent on $\mathbb{D}$ and $f(\mathbb{D})$ is a convex region.

Proof. Suppose f is convex univalent in $\mathbb{D}$. Set $\Omega = f(\mathbb{D})$. Then the inequality and the necessary and sufficient conditions for equality follow from applying Theorem 3 to Ω and the points $A = f(a)$ and $B = f(b)$.

Conversely, suppose f is a nonconstant holomorphic function defined on $\mathbb{D}$ which satisfies the inequality. As in the proof of the invariant form of the Koebe distortion theorem, we conclude that f is univalent on $\mathbb{D}$. Set $\Omega = f(\mathbb{D})$. Since f is a conformal map of $\mathbb{D}$ onto Ω and hyperbolic distance is preserved, inequality (4) implies that the inequality in the theorem holds. Hence, Ω is convex, so f is convex univalent.

REMARK. The right-hand side of the inequality in the corollary is a decreasing function of p for $p \geq 1$. Therefore, the strongest necessary condition for a convex univalent function that the corollary produces is the case $p = 1$:

$$|f(a) - f(b)| \geq \frac{1}{2} \tanh(d_{\mathbb{D}}(a, b)) \left(|D_1 f(a)| + |D_1 f(b)|\right).$$

The weakest sufficient condition for convex univalence that the corollary yields is $p = \infty$ (or more precisely, the limit as $p \to \infty$):

$$|f(a) - f(b)| \geq \frac{\sinh(d_{\mathbb{D}}(a, b))}{\exp(d_{\mathbb{D}}(a, b))} \max\{|D_1 f(a)|, |D_1 f(b)|\}.$$

This is the symmetric, linearly invariant form of the classical distortion theorem

$$|g(z)| \geq \frac{|z|}{1+|z|}, \qquad z \in \mathbb{D},$$

for a normalized convex univalent function g [**2**, p. 70].

5. Comments. The method of Blatter that we have employed in this paper uses certain differential geometric ideas in conjunction with coefficient bounds for univalent functions to produce symmetric, linearly invariant two-point distortion theorems for (convex) univalent functions which characterize (convex) univalence. Can these results be established in a purely differential geometric fashion without using coefficient bounds? In the convex case our results characterize convex regions so it is plausible that, at least in this setting, a purely differential geometric proof might be available.

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