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CONJUGATES OF STRONGLY EQUIVARIANT MAPS

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Let $\tau\colon X_1 \to X_2$ be a strongly equivariant holomorphic embedding of one bounded symmetric domain into another. We show that if σ is an automorphism of C, then $\tau^\sigma : X_1^\sigma \to X_2^\sigma$ is also strongly equivariant.

1. Introduction. A semisimple algebraic group, G, over $\mathbf{Q}$, is said to be of hermitian type if $G(\mathbf{R})^0/K$ is a bounded symmetric domain for a maximal compact subgroup K of $G(\mathbf{R})^0$. Let X_1 and X_2 be bounded symmetric domains associated with algebraic groups G_1 and G_2 respectively. A holomorphic embedding $\tau\colon X_1 \to X_2$ is called weakly equivariant if there exists a homomorphism of algebraic groups $\rho\colon G_1 \to G_2$ defined over $\mathbf{Q}$, such that

(1.1) $\rho(g) \cdot \tau(x) = \tau(g \cdot x)$ for all $g \in G_1(\mathbf{R})^0$ and all $x \in X_1$.

τ is called strongly equivariant if, in addition, the image of X_1 is totally geodesic in X_2. It is not known, at least to me, whether every weakly equivariant holomorphic map of bounded symmetric domains is strongly equivariant.

Strongly equivariant maps form the central theme of Satake's book [**21**]. They have (at least) two important applications to number theory. If G_2 is a symplectic group, then X_2 parametrizes a universal family of abelian varieties. Pulling back the universal family to X_1, and taking the quotient by an arithmetic group, gives a family of abelian varieties called a Kuga fiber variety. These are defined in [**10**], where Kuga calls τ a (generalized) Eichler map. For the second application, to compactification of arithmetic varieties, see [**20**] and also [**4**], where strongly equivariant maps are called symmetric maps.

Since the quotient of a symmetric domain by an arithmetic group is an algebraic variety, it is possible to define the conjugate $\tau^\sigma : X_1^\sigma \to X_2^\sigma$ for an automorphism σ of $\mathbf{C}$. Conjugates of equivariant maps of symmetric domains were studied by Min Ho Lee in his thesis [**13**] which has been published in a series of papers [**14**, **15**, **16**]. He proved that any conjugate of a weakly equivariant map is weakly equivariant [**13**, **14**]. He also proved that any conjugate of an H_2-equivariant map (i.e., a strongly equivariant map such that ρ takes any symmetry of X_1 to a symmetry of X_2) is again H_2-equivariant, assuming certain additional hypotheses [**13**, **16**]. This led him to conjecture [**13**, p. 26] that conjugates of strongly equivariant maps are strongly equivariant. We prove this conjecture in this paper.

We shall now describe the ideas involved in the proof. A pair (G, X) as above defines a connected Shimura variety. The proof of Langlands' conjectures on conjugation of Shimura varieties by Borovoi [**7**] and Milne [**17**] shows how an automorphism of $\mathbf{C}$ acts on the special points of X (see 3.2 for the definition of a special point). This implies our theorem for a strongly equivariant map which takes special points to special points. Now, as Kuga observed [**11**, **12**], Satake's classification of strongly equivariant maps shows that the set of all holomorphic embeddings $\tau: X_1 \to X_2$ which are strongly equivariant with a given homomorphism $\rho: G_1 \to G_2$ is itself a bounded symmetric domain, which we call X_ρ. We construct a map $X_1 \times X_\rho \to X_2$ which preserves special points, and use it to show that τ^σ is strongly equivariant.

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2. Equivariant maps.

2.1 Symmetric domains. By a *hermitian pair*, we shall mean a pair (G, X), where G is a semisimple algebraic group over $\mathbf{Q}$ of hermitian type such that $G(\mathbf{R})$ has no compact factors defined over $\mathbf{Q}$, and X is the symmetric domain associated to G. Borovoi [**7**] calls (G, X) a *Mumford manifold.* A hermitian pair (G, X) determines a connected Shimura variety $\mathrm{Sh}(G, X)$; this is a proalgebraic variety over $\mathbf{C}$ such that

$$\mathrm{Sh}(G, X)(\mathbf{C}) = \operatorname{proj}\lim \Gamma\backslash X$$

where the projective limit is taken over all congruence subgroups Γ of G [**7**, 2.1].

Let $x \in X$. The stabilizer, K_x, of x in $G(\mathbf{R})^0$ is a maximal compact subgroup of $G(\mathbf{R})^0$. Let G and K_x be the Lie algebras of $G(\mathbf{R})$ and K_x, respectively. We then have a Cartan decomposition

$$\mathfrak{g} = \mathfrak{k}_x \oplus \mathfrak{p}_x \tag{2.1.1}$$

where $\mathfrak{p}_x$ is the orthogonal complement of $\mathfrak{k}_x$ with respect to the Killing form. Let $v\colon G(\mathbf{R})^0 \to X$ be the map $g \mapsto g \cdot x$. Then dv induces an isomorphism of $\mathfrak{p}_x$ with $T_x(X)$, the tangent space of X at x. There exists a unique $H_x \in Z(\mathfrak{k}_x)$, called the *$H$-element* at x, such that ad $H_x|\mathfrak{p}_x$ is the complex structure on $\mathfrak{p}_x = T_x(X)$ [**21**, p. 54].

It will also be convenient to say that a reductive group G is of hermitian type if the center of $G(\mathbf{R})$ is compact and the derived group, G^{der}, is of hermitian type. We still have a Cartan decomposition (2.1.1); however the H-element is only determined modulo the center of $\mathfrak{g}$.

2.2 Strongly equivariant maps. Let (G_1, X_1) and (G_2, X_2) be hermitian pairs. A *weakly equivariant* map from (G_1, X_1) to (G_2, X_2) consists of a pair (ρ, τ), where $\rho\colon G_1 \to G_2$ is a homomorphism of algebraic groups defined over $\mathbf{Q}$, and $\tau\colon X_1 \to X_2$ is a holomorphic embedding satisfying (1.1). (ρ, τ) is called *strongly equivariant* if it satisfies the additional condition

$$[H_{\tau(x)} - d\rho(H_x), d\rho(g)] = 0 \text{ for all } g \in \mathfrak{g}_1 \ , \tag{H$_1$}$$

where H_x and $H_{\tau(x)}$ are the H-elements at $x \in X_1$ and $\tau(x) \in X_2$, respectively. This condition is independent of the choice of the point $x \in X_1$ [**21**, Lemma II.2.2, p. 48]. Also (H_1) is equivalent to requiring that the image of X_1 be totally geodesic in X_2 [**21**, p. 49].

2.3 The space of equivariant maps. Let $(\rho, \tau)\colon (G_1, X_1) \to (G_2, X_2)$ be strongly equivariant. The homomorphism ρ is uniquely determined by τ [**4**, p. 173]; we wish to describe the set of all holomorphic embeddings $X_1 \to X_2$ which are strongly equivariant with ρ. Choose a base point $x \in X_1$. Since $\tau(g \cdot x) = \rho(g) \cdot \tau(x)$, τ is uniquely determined by its value at x. Let

(2.3.1)

$$X_\rho := \{z \in X_2 | \tau_z \text{ is well-defined and strongly } \rho\text{-equivariant}\}$$

where

$$\tau_z(g \cdot x) := \rho(g) \cdot z \ .$$

PROPOSITION 2.3.2. [**21**, Proposition IV.4.1, p. 180]. *Let $(\rho, \tau)\colon (G_1, X_1) \to (G_2, X_2)$ be strongly equivariant. Choose a base point $x \in X_1$; let H_x and $H_{\tau(x)}$ be the H-elements at x and $\tau(x)$ respectively. Let G_ρ be the Zariski-connected component of the centralizer of $\rho(G_1)$ in G_2. Let X_ρ be the set given by (2.3.1). Then G_ρ is a reductive group of hermitian type with symmetric domain X_ρ. The H-element of G_ρ at $\tau(x)$ is*

$$H_\rho := H_{\tau(x)} - d\rho(H_x) \ . \tag{2.3.3}$$

The inclusions $G_\rho^{der} \to G_2$ and $X_\rho \to X_2$ form a strongly equivariant pair.

The group $G_\rho^{\mathrm{der}}(\mathbf{R})$ may have compact factors defined over $\mathbf{Q}$; hence $(G_\rho^{\mathrm{der}}, X_\rho)$ is not necessarily a hermitian pair as defined in (2.1). Let G_ρ^1 be the product of those simple factors of G_ρ^{der} which are not compact over $\mathbf{R}$. Then (G_ρ^1, X_ρ) is a hermitian pair. It defines a connected Shimura variety which parametrizes strongly equivariant maps from (G_1, X_1) to (G_2, X_2). ρ is called *rigid* if there is a unique τ such that (ρ, τ) is strongly equivariant, i.e., if X_ρ reduces to a point, or, if $G_\rho(\mathbf{R})$ is compact.

Since $\rho(G_1)$ is semisimple and G_ρ centralizes $\rho(G_1)$, the product $G :=\ \rho(G_1) \cdot G_\rho$ is an almost direct product. G is a reductive group of hermitian type. Its symmetric domain is $X_0 := X_1 \times X_\rho$. The H-element at the point $(x, \tau(x))$ is $H_{\tau(x)}$. We have a strongly equivariant pair

$$(\rho', \tau')\colon (G_0, X_0) \to (G_2, X_2) \tag{2.3.4}$$

where $G_0 := \rho(G_1) \cdot G_\rho^1$, ρ' is the inclusion, and $\tau'(x, y) = \tau_y(x)$.

LEMMA 2.3.5. $(\rho', \tau')\colon (G_0, X_0) \to (G_2, X_2)$ *is rigid.*

Proof. The connected component of the centralizer of $\rho'(G_0)$ in G_2 is contained in G_ρ, and centralizes G_ρ^1; therefore it is compact over $\mathbf{R}$. □

2.4 An example. To illuminate the above discussion, we give an example to show how embeddings of a Shimura curve into a Siegel modular variety are parametrized by another Shimura curve. This is a special case of [**1**, Example 4.2, p. 342]. Let k be a totally real quadratic number field, with $\{\alpha, \beta\}$ the set of embeddings of k into $\mathbf{R}$. Let B be a quaternion algebra over k such that $B \otimes_\alpha \mathbf{R} \cong M_2(\mathbf{R})$ and $B \otimes_\beta \mathbf{R} \cong \mathbf{H}$, where $\mathbf{H}$ denotes the Hamilton quaternion algebra. Let G' be the group of norm one units of B, and G the restriction of G' from k to $\mathbf{Q}$. Then $G(\mathbf{R}) \cong SL_2(\mathbf{R}) \times SU_2(\mathbf{R})$ is of hermitian type and acts on the upper half plane $\mathfrak{h}$ through the first factor. The quotient of $\mathfrak{h}$ by a torsion-free arithmetic subgroup of G is a smooth and complete algebraic curve, C, called a Shimura curve. G' acts on B by left multiplication. Take the direct sum of two copies of this representation, and restrict scalars from k to $\mathbf{Q}$; this gives a representation $\rho\colon G \to GL(F)$ which is defined over $\mathbf{Q}$. Addington's classification [**3**] shows that the image is actually contained in a symplectic group $\mathrm{Sp}(F, A)$, and there exists a strongly equivariant holomorphic map $\tau\colon \mathfrak{h} \to \mathfrak{S}(F, A)$, where $\mathfrak{S}(F, A)$ denotes the Siegel space associated with $\mathrm{Sp}(F, A)$. After a choice of a suitable lattice in F, each such holomorphic embedding gives a family of 8-dimensional abelian varieties over C.

Let σ be the nontrivial automorphism of k, $B^\sigma := B \otimes_\sigma k$, and let G^σ be the restriction from k to $\mathbf{Q}$ of the group of norm one units of B^σ. We have $B \otimes_k B^\sigma \cong M_2(B_0)$ for a totally definite quaternion algebra B_0 over k. This gives a representation, ρ', of $G \times G^\sigma$ on $F := B_0^2$. Identifying $G \times G^\sigma(\mathbf{C})$ with $SL_2(\mathbf{C})^4$, ρ' is equivalent over $\mathbf{C}$ to the sum of two copies of the representation

$$(\alpha, \beta, \alpha', \beta') \mapsto (\alpha \otimes \beta') \oplus (\alpha' \otimes \beta) .$$

It follows from [**1**, Theorem 4.1, p. 341] that ρ' is rigid, and there exists a unique $\tau'\colon \mathfrak{h} \times \mathfrak{h} \to \mathfrak{S}(F, A)$ which is strongly ρ'-equivariant. The restriction of ρ' to the first factor is equivalent to ρ. For each $z \in \mathfrak{h}$ we get a map $\tau_z\colon \mathfrak{h} \to \mathfrak{S}(F, A)$ which is strongly equivariant with ρ. Thus the hermitian pair $(G_\rho, \mathfrak{h}_\rho)$ in this example is isomorphic to $(G^\sigma, \mathfrak{h})$. For more examples, see [**11**, §5] and [**1**, §4].

3. Morphisms of Hodge type.

3.1 Hodge group. Let (G, X) be a hermitian pair. Let H_x be the H-element at a point $x \in X$. H_x defines a 1-parameter subgroup of

$G(\mathbf{R})$ by $h_x(t) := \exp(tH_x)$. The *Hodge group* at x, denoted $\mathrm{Hg}(x)$, is the smallest $\mathbf{Q}$-subgroup of G which contains the image of h_x. It is a connected, reductive group [**7**, Remark 1.14, p. 12]. The derived group of $\mathrm{Hg}(x)$ is of hermitian type, because $\mathrm{Ad}h_x(\pi/2)$ defines a complex structure.

3.2 Special points. We say that x is *special*, or a *CM-point*, if its Hodge group is abelian. The motivation for this definition is a theorem of Mumford that an abelian variety is of CM-type if and only if its Hodge group is abelian [**19**, p. 347].

A strongly equivariant pair (ρ, τ) is said to be of *Hodge type* if τ takes special points to special points. Borovoi refers to such maps as morphisms of Mumford manifolds — note that property (iii) of [**7**, 1.11, p. 11] implies strong equivariance.

The following is a generalization of Mumford's theorem that any Kuga fiber variety which contains an abelian variety of CM-type is of Hodge type [**19**, p. 348].

PROPOSITION 3.3. *Let $(\rho, \tau)\colon (G_1, X_1) \to (G_2, X_2)$ be strongly equivariant. If $\tau(x)$ is a CM-point in X_2, then x is a CM-point in X_1 and furthermore, (ρ, τ) is of Hodge type.*

Proof. Let the notation be as in Proposition 2.3.2. Let M be the smallest $\mathbf{Q}$-subgroup of G_ρ which contains $\exp(t \cdot H_\rho)$ for all $t \in \mathbf{R}$. Since

$$\exp(t \cdot d\rho(H_x)) = \exp(tH_{\tau(x)}) \exp(-tH_\rho) \in \mathrm{Hg}(\tau(x))M \ ,$$

we have $\rho(\mathrm{Hg}(x))$ contained in $\mathrm{Hg}(\tau(x))M$. Similarly, M is contained in $\mathrm{Hg}(x)\,\mathrm{Hg}(\tau(x))$.

Let K be the maximal compact subgroup of $G_2(\mathbf{R})$ at $\tau(x)$. Then $\mathrm{Hg}(\tau(x)) \subset K$. Since M centralizes $\mathrm{Hg}(x)$, it follows that $\mathrm{Hg}(x) \subset K$. Therefore $\mathrm{Hg}(x)(\mathbf{R})$ is compact. Since $\mathrm{Hg}(x)^{der}(\mathbf{R})$ has no compact factors defined over $\mathbf{Q}$ [**7**, Remark 1.14, p. 12], this implies that $\mathrm{Hg}(x)$ is abelian, i.e., x is a CM-point. Since τ takes the CM-point x to a CM-point, τ takes every CM-point to a CM-point [**7**, 1.11, p. 11]. Therefore (ρ, τ) is of Hodge type. □

The proof of the following theorem is inspired by the proof of Mumford's theorem that any Hodge family contains a fiber of CM-type [**19**, p. 348]. In the abelian case (i.e., when G_2 is a symplectic group), a sketch of the proof was given in [**2**, Proposition 1.5.1, p. 228].

THEOREM 3.4. *Any rigid strongly equivariant map is of Hodge type.*

Proof. Let $(\rho, \tau): (G_1, X_1) \to (G_2, X_2)$ be rigid. Then G_ρ, the Zariski-connected component of the centralizer of $\rho(G_1)$ in G_2, is compact. Let $G := \rho(G_1) \cdot G_\rho$. Choose a base point $x \in X$ and let

$$T := \{\exp(tH_{\tau(x)}) | t \in \mathbf{R}\} .$$

T is a 1-dimensional torus in G defined over $\mathbf{R}$. The centralizer, K, of T in G is an $\mathbf{R}$-subgroup of G, and hence contains a maximal torus T_1 defined over $\mathbf{R}$ [**6**, Proposition 7.10, p. 480]. T_1 contains T because T is contained in the center of K. If T_1' is any torus in G containing T_1, then T_1' will centralize T, so $T_1' \subset K$ and $T_1 = T_1'$. Thus T_1 is a maximal torus in G. By [**5**, Proposition 2.5, p. 465] there exists $g \in G(\mathbf{R})^0$ such that $T_2 = gT_1g^{-1}$ is defined over $\mathbf{Q}$. Then $T_2 \supset \mathrm{Hg}(g \cdot \tau(x))$, so $\mathrm{Hg}(g \cdot \tau(x))$ is abelian, and $g \cdot \tau(x)$ is a CM-point. We have shown that $\tau(X)$ has a CM-point; by Proposition 3.3, (ρ, τ) is of Hodge type. □

REMARKS 3.5. Suppose that $G_1(\mathbf{R})$ has no compact factors, and G_2 is a symplectic group. Then [**21**, Proposition IV.4.3, p. 183] shows that (ρ, τ) is rigid. Theorem 3.4 then implies that (ρ, τ) is of Hodge type. This is Proposition 3 of [**18**].

Suppose (ρ, τ) is not rigid. Then the set of strongly ρ-equivariant maps is parametrized by the bounded symmetric domain X_ρ. Lemma 2.3.5 shows that there exists $z \in X_\rho$ such that (ρ, τ_z) is of Hodge type. However, since there are only countably many special points, (ρ, τ_z) is not of Hodge type for a "general" $z \in X_\rho$.

4. Conjugates of strongly equivariant maps. Let (G, X) be a hermitian pair, and Γ an arithmetic subgroup of G. The locally symmetric space $V := \Gamma \backslash X$ is a quasiprojective variety [**5**], called an *arithmetic variety*. Let σ be an automorphism of $\mathbf{C}$. Kazhdan [**8, 9**] proved that any conjugate of an arithmetic variety is again an arithmetic variety. Therefore, there exists a hermitian pair (G^σ, X^σ) and an arithmetic subgroup Γ^σ of G^σ such that $V^\sigma \cong \Gamma^\sigma \setminus X^\sigma$. (Kazhdan's proof shows that X and X^σ are biholomorphically equivalent; however, we will not identify them, to avoid confusion.) Note that the group G^σ is not uniquely determined; however, it is unique up to finite coverings [**2**, Proposition 1.3.1, p. 227].

Let $(\rho, \tau)\colon (G_1, X_1) \to (G_2, X_2)$ be a strongly equivariant map. Let Γ_1 and Γ_2 be arithmetic subgroups of G_1 and G_2 respectively, such that $\rho(\Gamma_1) \subset \Gamma_2$. Let $V_i := \Gamma_i \setminus X_i$ $(i = 1, 2)$. A theorem of Satake [**20**, p. 231] implies that the map $f\colon V_1 \to V_2$ induced by τ is a morphism of algebraic varieties. Let σ be an automorphism of $\mathbf{C}$. Then $f^\sigma\colon V_1^\sigma \to V_2^\sigma$ is again a map of algebraic varieties. Lifting $f^\sigma\colon V_1^\sigma \to V_2^\sigma$ to the universal covering spaces gives a holomorphic map $X_1^\sigma \to X_2^\sigma$ which we define to be the conjugate of τ by σ, and denote by τ^σ. Clearly, τ^σ does not depend on the choice of Γ_1 and Γ_2. It is known [**14**] that τ^σ is weakly equivariant.

THEOREM 4.1. *If $(\rho, \tau)\colon (G_1, X_1) \to (G_2, X_2)$ is strongly equivariant, and σ is an automorphism of $\mathbf{C}$, then there exists a homomorphism $\rho^\sigma\colon G_1^\sigma \to G_2^\sigma$ such that $(\rho^\sigma, \tau^\sigma)\colon (G_1^\sigma, X_1^\sigma) \to (G_2^\sigma, X_2^\sigma)$ is strongly equivariant.*

Proof. We can factor (ρ, τ) as a composition of two strongly equivariant maps

$$(G_1, X_1) \to (G_0, X_0) \to (G_2, X_2)\ .$$

The hermitian pair (G_0, X_0) and the map

$$(\rho', \tau')\colon (G_0, X_0) \to (G_2, X_2)$$

are as in (2.3.4). Recall that $G_0 = \rho(G_1) \cdot G_\rho^1$ and $X_0 = X_1 \times X_\rho$. The map

$$(\rho_0, \tau_0)\colon (G_1, X_1) \to (G_0, X_0)$$

is given by $g \mapsto \rho(g)$ and $z \mapsto (z, \tau(x))$, where x is our fixed base point in X_1.

(ρ', τ') is rigid (Lemma 2.3.5), and therefore of Hodge type (Theorem 3.4). Langlands' conjectures on conjugation of Shimura varieties, which are proved in [**7**] and [**17**], show that any automorphism of $\mathbf{C}$ takes a CM-point to a CM-point. Hence $(\rho'^\sigma, \tau'^\sigma)$ is also of

Hodge type; in particular, it is strongly equivariant (see [7], 1.11(iii), p. 11]).

We have $X_0^\sigma = X_1^\sigma \times X_\rho^\sigma$ and hence $G_0^\sigma = G_1^\sigma \times G_\rho^\sigma$. Since the projection of τ_0 to the second factor is the constant $\tau(x)$, there exists $\tau(x)^\sigma \in X_\rho^\sigma$ such that the projection of τ_0^σ to the second factor is the constant $\tau(x)^\sigma$. We then have a strongly equivariant pair

$$(\rho_0^\sigma, \tau_0^\sigma) \colon (G_1^\sigma, X_1^\sigma) \to (G_0^\sigma, X_0^\sigma)$$

given by $g \mapsto (g,1)$ and $z \mapsto (z, \tau(x)^\sigma)$. Observe that $\tau^\sigma = \tau_0^\sigma \circ \tau'^\sigma$ Let $\rho^\sigma := \rho_0^\sigma \circ \rho'^\sigma$. Then $(\rho^\sigma, \tau^\sigma)$ is strongly equivariant. □

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