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KLEINIAN GROUPS WITH AN INVARIANT JORDAN CURVE: J-GROUPS

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We study some topological and analytical properties of a Kleinian group G for which there is an invariant Jordan curve by the action of G. These groups may be infinitely generated.

1. Preliminaries. A Kleinian group G is a group of Möbius transformations acting discontinuously on some region of the Riemann sphere $\hat{\mathbb{C}}$. The (open) set of points at which G acts discontinuously is called the region of discontinuity of G and it is denoted by $\Omega(G)$. The complement (on the Riemann sphere) of $\Omega(G)$ is called the limit set of G and it is denoted by $\Lambda(G)$. We will denote the upper and lower half-planes by $U = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ and $L = \{z \in \mathbb{C} : \mathrm{Im}(z) < 0\}$, respectively. If G is a group of Möbius transformations and D is a subset of the Riemann sphere, then the stabilizer of D is defined by $G_D = \{g \in G : g(D) = D\}$.

J-GROUPS. A Kleinian group G is called a J-group if there exists a Jordan curve γ so that $G_\gamma = G$. We say that γ is a $G-$invariant Jordan curve. In this case, we have necessarily that $\Lambda(G) \subset \gamma$. If $\Lambda(G) = \gamma$, then we say that G is a J-group of the first kind; otherwise, we say that it is of the second kind.

If G is a J-group and γ is a $G-$invariant Jordan curve, then we can associate to G a 3-tuple (γ, D_1, D_2), where D_1 and D_2 are the two topological discs bounded by γ. If the J-group G is of the first kind, then such a 3-tuple is unique (modulo permutation of the two topological discs). In any case, the stabilizers G_{D_1} and G_{D_2} coincide. We have that G_{D_1} is either G or it is a subgroup of index two in G. Classical examples of J-groups are given by Fuchsian groups, $\mathbb{Z}_2-$extension of Fuchsian groups, Schottky groups and QuasiFuchsian groups.

Let G be a Kleinian group and let D be a simply connected component of $\Omega(G)$. A parabolic transformation $k \in G_D$ is accidental if there is a Riemann map $w : U \to D$ so that $w^{-1} \circ k \circ w$ is hyperbolic. This definition is independent of the Riemann map $w : U \to D$.

We need the following extension theorem for Riemann maps due to Carathéodory. A proof can be found in the book of C. Pommerenke [**10**].

THEOREM 1.1 (Carathéodory Extension Theorem [**3**]). *Let D be a simply connected plane domain, conformally equivalent to the upper half plane U, and so that the euclidean boundary of D is a Jordan curve. If $w : U \to D$ is a Riemann map, then it has an extension to the closure of U as a homeomorphism onto the closure of D.*

PROPOSITION 1.2. *A J-group G of the first kind contains no accidental parabolic transformations.*

Proof. Let (γ, D_1, D_2) be the 3-tuple associated to the J-group of the first kind G. Let $w : U \to D_1$ be a Riemann map. The group $F = w^{-1} \circ G_{D_1} \circ w$ is a Fuchsian group keeping U invariant. As a consequence of Theorem 1.1., we can extend w to a homeomorphism from the closure of U onto the closure of D_1, that is, $D_1 \cup \gamma$. Denote by $\bar{w} : \bar{U} \to D_1 \cup \gamma$ such an extension. Let $f \in F$ be a hyperbolic transformation with fixed points x and y (necessarily, $x \neq y$). We must show that $w \circ f \circ w^{-1}$ is not parabolic. The injectivity of $\bar{w}$ implies that $\bar{w}(x) \neq \bar{w}(y)$. Since the points $\bar{w}(x)$ and $\bar{w}(y)$ are fixed points of $w \circ f \circ w^{-1}$, the transformation $w \circ f \circ w^{-1}$ is necessarily loxodromic (it cannot be an elliptic transformation since G is a discrete group). □

Let G be a J-group of the first kind with associated 3-tuple (γ, D_1, D_2). Each component D_i $(i = 1, 2)$ has a "natural" hyperbolic metric as follows. Set $U_1 = U$ and $U_2 = L$. For each $i \in \{1, 2\}$, let $w_i : U_i \to D_i$ be a Riemann map. We transfer, under the map w_i, the hyperbolic metric on U_i onto the disc D_i. Since, any Riemann map $\tilde{w}_i : U_i \to D_i$ has the form $\tilde{w}_i = w_i \circ A$, where A

is a conformal automorphism of U_i, the hyperbolic metric induced on D_i is independent of the chosen Riemann map. We can define axes for loxodromic elements of G as follows. Let g be a loxodromic element in G_{D_1} $(= G_{D_2})$. Let $A(g) \subset D_1$ and $B(g) \subset D_2$ be the images of the axes of $w_1^{-1} \circ g \circ w_1$ and $w_2^{-1} \circ g \circ w_2$ under w_1 and w_2, respectively. We call the sets $A(g)$ and $B(g)$ the axes of g. These axes have common end points, these are the fixed points of g. If $g \in G - G_{D_1}$ is loxodromic, then we define the axes of g, $A(g)$ and $B(g)$, to be the axes $A(g^2)$ and $B(g^2)$, respectively. The simple loop $L(g) = A(g) \cup B(g) \cup \{\text{fixed points of } g\}$ is called the *total axis* of the transformation $g \in G$.

A sequence of simple loops $L_n \subset \hat{\mathbb{C}}$ is nested if the loop L_n separates L_{n-1} from L_{n+1}. We say that the above sequence nests about a point $x \in \hat{\mathbb{C}}$ if each L_n separates x from L_{n-1} and, for every point $z_n \in L_n$, the sequence z_n converges to x.

Let G be a J-group of the first kind with associated 3-tuple (γ, D_1, D_2), and consider a sequence of loxodromic elements $g_n \in G$. We say that the sequence of axes $A(g_n)$ (resp., $B(g_n)$) is nested (respectively, nests about x) if the sequence of total axes $L(g_n)$ is nested (respectively, nests about x).

PROPOSITION 1.3. *Let G be a J-group of the first kind, and let $h_n \in G$ be a sequence of loxodromic elements such that $L(h_n)$ is nested. Then the sequence $L(h_n)$ nests about $x \in \Lambda(G)$ if and only if both fixed points of the transformations h_n converge to x.*

Proof. Let (γ, D_1, D_2) be the 3-tuple associated to G. From our definition of the total axes $L(h_n)$, we may assume $h_n \in G_{D_1}$. Let $w : U \to D_1$ be a Riemann map. Extend it as a homeomorphism from the closure of U onto the closure of D_1 (Theorem 1.1). Clearly the sequence of total axes of the transformations $w^{-1} \circ h_n \circ w$ is also a nested sequence. The sequence $L(h_n)$ nests about x if and only if the sequence of total axes of $w^{-1} \circ h_n \circ w$ nests about the point $w^{-1}(x)$ (consequence of the continuity of the extension of w). The last sequence nests about $w^{-1}(x)$ if and only if the fixed points of $w^{-1} \circ h_n \circ w$ converge to $w^{-1}(x)$. This is equivalent to the convergence of the fixed points of h_n to x (the extension of w is a homeomorphism). □

PROPOSITION 1.4. *Let G_1 and G_2 be J-groups of the first kind, and let $h_n \in G_1$ be a sequence of loxodromic elements so that the total axes $L(h_n)$ nest about some point $x \in \Lambda(G_1)$. Let $f : \Omega(G_1) \to \Omega(G_2)$ be a homeomorphism defining an isomorphism $\phi : G_1 \to G_2$, where $\phi(g) = f \circ g \circ f^{-1}$. Then the total axes $L(f \circ h_n \circ f^{-1})$ nest about some point $y \in \Lambda(G_2)$.*

REMARK. If the above homeomorphism $f : \Omega(G_1) \to \Omega(G_2)$ is the restriction of a homeomorphism $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, then the continuity of the map F implies $y = F(x)$.

Proof. Let (γ_i, D_1^i, D_2^i) be the 3-tuple associated to G_i, for $i = 1, 2$. The isomorphism $\phi : G_1 \to G_2$ is type preserving, that is, $\phi(h) \in G_2$ is loxodromic if and only if $h \in G_1$ is loxodromic. In fact, assume there is a loxodromic element $h \in G_1$ so that $\phi(h) \in G_2$ is parabolic with fixed point $q \in \gamma_2$. The set $f(A(h)) \cup \{q\}$ is a simple loop which is invariant under $\phi(h)$. Denote by a and r the attracting and repelling fixed points of h. Then $\gamma_1 - \{a, r\}$ consists of two arcs, say l_1 and l_2. We can find loxodromic elements k and t in G_1 so that both fixed points of k are in l_1 and both fixed points of t are in l_2 (see Proposition E.5 of Chapter V in [**8**]). In particular, k and t cannot commute with h (see Chapter I in [**8**]). The image $f(A(k))$ (resp., $f(A(t))$) is a simple path which is invariant under the action of $\phi(k)$ (resp., $\phi(t)$). It follows that the end points of $f(A(k))$ (resp., $f(A(t))$) are the fixed points of $\phi(k)$ (resp., $\phi(t)$). Now, either $\phi(k)$ or $\phi(t)$ must be a parabolic element with fixed point q. Without lost of generality, let us assume $\phi(t)$ is parabolic. In particular, $\phi(h)$ and $\phi(t)$ commute. This is a contradiction to the facts that t and h do not commute and ϕ is an isomorphism.

As a consequence, we can extend f to the set of fixed points as follows. If q is the fixed point of a parabolic element $h \in G_1$, then define $f(q)$ as the fixed point of the parabolic element $\phi(h) \in G_2$. If a and r are the attracting and repelling fixed points, respectively, of a loxodromic element $k \in G_1$, then we define $f(a)$ and $f(r)$ to be the attracting and repelling fixed points, respectively, of $\phi(k) \in G_2$. This extension is one-to-one. In fact, if there are two non-elliptic transformations k_1 and k_2 in G_1 so that the images of the fixed points intersect, then the discreteness of G_2 implies that $\phi(k_1)$ and $\phi(k_2)$ have the same fixed points (see Chapter II in [**8**]). It follows that $\phi(k_1)$ and $\phi(k_2)$ commute. Since ϕ is an isomorphism, we must have that k_1 and k_2 commute and, in particular, they have the same fixed points.

We denote by $\tilde{f}$ the above one-to-one extension of f. Let us now consider a sequence of loxodromic elements $h_n \in G_1$ so that the sequence $L(h_n)$ nests about some $x \in \gamma_1$. Proposition 1.3 asserts that the sequence $L(h_n)$ nests about x if and only if the sequence of fixed points of h_n converges to x. The image under f of the axes $A(h_n)$ and $B(h_n)$ are invariant under the transformation $f \circ h_n \circ f^{-1}$. It follows that the end points of these axes are exactly the fixed points of $f \circ h_n \circ f^{-1}$. The sequence of images $\tilde{f}(L(h_n))$ is again nested. The fact that the end points of the axis $A(f \circ h_n \circ f^{-1})$ are exactly the end points of the image under $\tilde{f}$ of the axis $A(h_n)$ implies that the sequence of total axes $L(f \circ h_n \circ f^{-1})$ is also nested. We must show that this sequence in fact nests about some point. Let us assume this is not the case. In this situation, there exists a geodesic L in D_1^2, with different end points u and v, to which the axes $A(f \circ h_n \circ f^{-1})$ converge (see Proposition 1.3). Let (u, v) be the arc of γ_2 determined by the points u and v and not containing the fixed points of $f \circ h_n \circ f^{-1}$. It is possible to find two different loxodromic elements s and t in G_2 both of them with fixed points in (u, v) and not commuting. Let k be either s or t. In this case the fixed points of $f^{-1} \circ k \circ f$ are contained in the arc of γ_1 containing x and bounded by the fixed points of h_n, for all n. In particular, $f^{-1} \circ k \circ f$ must be a parabolic element in G_1 with x as its fixed point. The discreteness of the group G_1 implies that $f^{-1} \circ s \circ f$ and $f^{-1} \circ t \circ f$ commute, obtaining a contradiction with the fact that ϕ is an isomorphism. □

It is known (see [**8**] and [**9**]) that a finitely generated Kleinian group G with exactly two components is an extended quasi-Fuchsian group, that is, there is a finitely generated extended Fuchsian group

F (either Fuchsian or $\mathbb{Z}_2-$extension of a Fuchsian group) and a quasiconformal homeomorphism $w : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $wFw^{-1} = G$. In particular, G is a J-group. In general, an infinitely generated Kleinian group with exactly two components is not necessarily a J-group as it is shown by an example in [**6**]. In the same paper can be found necessary and sufficient conditions for a Kleinian group with exactly two components to be a J-group. We prove that certain infinitely generated J-groups (for example, of the first kind with invariant components) are topological deformations of Fuchsian groups (see Theorem 2.4.).

CONJECTURE 1. If G is a J-group, then there exist a possible extended Fuchsian group F and a homeomorphism $W : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ so that $G = W \circ F \circ W^{-1}$.

2. Main Results. In this section we present some topological and analytical properties satisfied by J-groups. These are extensions of some properties satisfied by finitely generated Kleinian groups. Theorems 2.1 and 2.3 are generalizations of the main result of Maskit in [**7**]. Using a result of Tukia (see Theorem 3.3), we prove that certain infinitely generated J-groups are topological deformations of Fuchsian groups (see Theorem 2.4). Theorems 2.5 and 2.6 are extensions of two theorems in [**4**].

THEOREM 2.1. *Let G be a J-group of the first kind and let $f : \Omega(G) \to \Omega(G)$ be a (conformal, anticonformal) homeomorphism such that $f \circ g \circ f^{-1} = g$, for all $g \in G$. Then there exists a (conformal, anticonformal) homeomorphism $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, such that $F/_{\Omega(G)} = f$ and $F(x) = x$, for all $x \in \Lambda(G)$. Moreover, if f is anticonformal, then $F^2 = I$ and G is either a Fuchsian group or a $\mathbb{Z}_2-$extension of a Fuchsian group.*

COROLLARY 2.2. *Let G be a J-group of the first kind and let $f : \Omega(G) \to \Omega(G)$ be an anticonformal homeomorphism such that $f \circ g \circ f^{-1} = g$, for all $g \in G$. Then f is the restriction of an anticonformal involutory fractional linear transformation F (that is, $F(z) = (a\bar{z} + b)/(c\bar{z} + d)$, $F^2 = I$), and G is either a Fuchsian group or a $\mathbb{Z}_2-$extension of a Fuchsian group. Further, the mapping F with the above properties is unique.*

The following result is a natural extension of Theorem 2.1.

THEOREM 2.3. *Let G_1 and G_2 be J-groups of the first kind and let $f : \Omega(G_1) \to \Omega(G_2)$ be a homeomorphism inducing an isomorphism $f_* : G_1 \to G_2$, defined by $f_*(g) = f \circ g \circ f^{-1}$. Then there exists a homeomorphism $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $F/_{\Omega(G_1)} = f$.*

THEOREM 2.4. *Let G be a J-group with a 3-tuple (γ, D_1, D_2) so that $G_{D_1} = G$. Then there exist a Fuchsian group F and a homeomorphism $W : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that $W \circ F \circ W^{-1} = G$.*

Examples 36 and 37 in [**5**] show that we cannot expect to have a quasiconformal conjugation of a J-group G with a Fuchsian group (as in the situation of finitely generated J-groups [**9**]). It is possible to construct two Fuchsian groups either of the first or second kind (necessarily infinitely generated) which are topologically conjugated but not quasiconformally conjugated (see [**1**] and [**2**]).

THEOREM 2.5. *Let F be a Fuchsian group operating on $U_1 = U$ and $U_2 = L$. For each $i \in \{1, 2\}$, let $f_i : U_i \to \hat{\mathbb{C}}$ be a one-to-one analytic map defining an isomorphism $(f_i)_* : F \to G$ onto a Kleinian group G, where $(f_i)_*(k) = f_i \circ k \circ f_i^{-1}$, for k in F. If f_1 and f_2 can be extended continuously to $\mathbb{R} \cup \{\infty\}$ so that $f_1 = f_2$ on $\Omega(F) \cap (\mathbb{R} \cup \{\infty\})$ and $(f_1)_* = (f_2)_*$, then f_1 and f_2 are restrictions of the same fractional linear transformation. In particular, G is a Fuchsian group of the same kind as F.*

THEOREM 2.6. *Let F be a Fuchsian group of the first kind acting on $U_1 = U$ and $U_2 = L$. For each $i \in \{1, 2\}$, let $f_i : U_i \to f_i(U_i)$ be a holomorphic cover mapping defining a surjective homomorphism $(f_i)_* : F \to G$ onto a Kleinian group G, defined by $f_i \circ k = (f_i)_*(k) \circ f_i$, for $k \in F$. If we can extend f_1 and f_2 continuously to $\mathbb{R} \cup \{\infty\}$ and $(f_1)_* = (f_2)_*$, then G is a Fuchsian group.*

REMARK. We observe that Theorem 2.5, for Fuchsian groups of the first kind, is a special case of Theorem 2.6.

3. Proof of Theorems. We proceed to prove the theorems of the last section. First, we need some lemmas.

LEMMA 3.1. *Let G be a J-group of the first kind with associated 3-tuple (γ, D_1, D_2). If x belongs to γ, then there exists a sequence of loxodromic elements g_i in G_{D_1} such that the total axes $L(g_i)$ nest about x.*

Proof. Choose an orientation for the loop γ. Let z_{-1} and z_{+1} be limit points of G, so that z_{-1} and z_{+1} are on the left and right sides of x, respectively. Since the limit set of G and G_{D_1} are the same, we can find a loxodromic element g_1 in G_{D_1} with attracting fixed point x_{-1} lying between x and z_{-1}, and repelling fixed point x_{+1} lying between x and z_{+1} (see Chapter V, Proposition E.5. in [**8**]). In a similar way, we can find a loxodromic element g_2 in G_{D_1} with attracting fixed point x_{-2} between x and x_{-1}, and repelling fixed point x_{+2} between x and x_{+1}. We proceed inductively to obtain a sequence of loxodromic elements $g_i \in G_{D_1}$, with attracting and repelling fixed points x_{-i} and x_{+i}, respectively, both of them converging to the point x, such that x_{-i} lies between x and $x_{-(i-1)}$, and x_{+i} lies between x and $x_{+(i-1)}$. As a consequence of Proposition 1.3., the sequence of axes $L(g_i)$ nests about x. □

LEMMA 3.2. *Assume the hypotheses of Theorem 2.3. Let $\{g_i\}$ and $\{h_j\}$ be two sequences of loxodromic elements in G_1, with total axes $L(g_i)$ and $L(h_j)$, respectively. If both sequences nest about a point $z \in \gamma_1$, then the sequences $L(f_*(g_i))$ and $L(f_*(h_j))$ nest about the same point in γ_2.*

Proof. We denote by g^* the transformation $f_*(g)$, for all $g \in G_1$. As a consequence of Proposition 1.4, the sequences of total axes $L(g_i^*)$ and $L(h_j^*)$ each nests about some some point. Assume that $L(g_i^*)$ and $L(h_j^*)$ nest about different points, say z_1 and z_2, respectively ($z_1 \neq z_2$). Denote by $D(g_i^*)$ and $D(h_j^*)$ the closed topological discs bounded by the total axes $L(g_i^*)$ and $L(h_j^*)$, respectively, so that z_1 is contained in $D(g_i^*)$ and z_2 is contained in $D(h_j^*)$. For sufficiently large value of i, the sets $M(g_i^*)$ and $M(h_i^*)$ are disjoint. We may assume that this holds for all values of i. The (only) four possibilities for the configuration of the total axes of g_i and h_i, denoted by cases (1), (2), (3) and (4), are shown in Figure 1. In cases 3 and 4, $L(g_i)$ and $L(h_i)$ have common points. In this case, since the end points of $A(g_i^*)$ (respectively, $A(h_i^*)$) are the same as for $f(A(g_i))$ (respectively, $f(A(h_i)))$, we must have that $L(g_i^*)$ and $L(h_i^*)$ have also common points, a contradiction to our assumption. Let us now assume we are in case 2 (similar to case 1). There are only two possibilities (see Figure 2):

(1) Given i, there exists n_i such that $L(g_i)$ separates $L(h_{n_i})$ from $L(h_{n_i+1})$; or

(2) Given j, there exists m_j such that $L(h_j)$ separates $L(g_{m_j})$ from $L(g_{m_j+1})$.

The sequence of simple loops

$$f(A(g_i)) \cup f(B(g_i)) \cup \{\text{the fixed points of } g_i^*\}$$

is necessarily nested. We claim that it nests about z_1. If this is not the case, then there exists a sequence of points $w_i \in f(A(g_i))$ converging to some point $w \in \Omega(G_1)$. Let $t_n \in A(g_i)$ be such that $f(t_i) = w_i$ and let $t \in \Omega(G_1)$ be such that $f(t) = w$. The fact that $f : \Omega(G_1) \to \Omega(G_2)$ is a homeomorphism implies that there exists a subsequence of t_i converging to t. This is a contradiction to the fact that t_i converges to $z \in \gamma_1 = \Lambda(G_1)$. Since the end points of $f(A(g_i))$ are the same as for $A(g_i^*)$, we must have that the sequence $L(g_i^*)$ nests about z_1 (Proposition 1.3). Now, the above possibilities ensure that the sequence $L(h_i^*)$ also nests about z_1. This is a contradiction to the facts that $L(h_i^*)$ nests about z_2 and $z_1 \neq z_2$. □

To prove theorem 2.4, we need a result due to Tukia (see [**12**]). Suppose that K and H are Kleinian groups, each one preserving a domain A and B, respectively. Suppose also there is an isomorphism $\phi : K \to H$. We say that ϕ is geometric, if there exists a homeomorphism $f : A \to B$ such that

$$\phi(k)(z) = f \circ k \circ f^{-1}(z), \text{ for all } k \in K \text{ and } z \in B.$$

In the case $A = B = U$, U the upper half-plane, we say that the isomorphism ϕ preserves the relation of being crossed if the following holds:

(1) k in K is hyperbolic if and only if $\phi(k)$ is hyperbolic; and

(2) the axes of hyperbolic elements k and t in K intersect if and only if the axes of $\phi(k)$ and $\phi(t)$ intersect.

THEOREM 3.3 [**12**]. *Let K and H be Fuchsian groups with infinitely many limit points. Then an isomorphism $\phi : K \to H$ is geometric if and only if it preserves the relation of being crossed.*

LEMMA 3.4. *Assume the hypotheses of Theorem 2.5. Then f_1 and f_2 coincide on the limit set of F.*

Proof. We consider two cases.

CASE 1. Assume $\Omega(F) \cap \mathbb{R} \cup \{\infty\} \neq \phi$. We have by hypotheses that $f_1 = f_2$ on a dense subset of $\mathbb{R} \cup \{\infty\}$. Since f_1 and f_2 are continuous on $\mathbb{R} \cup \{\infty\}$, both agree on all of $\mathbb{R} \cup \{\infty\}$.

CASE 2. Assume $\Omega(F) \cap \mathbb{R} \cup \{\infty\} = \phi$. The group F is a Fuchsian group of the first kind. Since $\Lambda(F)$ is the closure of the fixed points of the hyperbolic elements of F, we only need to check the equality of both functions at these points. Let k be a hyperbolic element of F. Denote by p its repelling fixed point and by q its attracting fixed point. Let g be the transformation in G given by $f_1 \circ k \circ f_1^{-1} = g = f_2 \circ k \circ f_2^{-1}$. In this way, the points $f_1(p)$, $f_1(q)$, $f_2(p)$ and $f_2(q)$ are fixed points of the transformation g. Since the map $k \to f_i \circ k \circ f_i^{-1}$ is an isomorphism, for $i = 1, 2$, the transformation g is different from the identity. In particular, g has at most two fixed points.

SUBCASE 2.1. Let us assume $f_1(p) = f_1(q)$. Since $f_1(U_1)$ is necessarily G–invariant, its boundary must contain all of the limit set of G. If g has another fixed point, say x, different from $f_1(p)$, then x is in the boundary of $f_1(U_1)$. Let $\{z_n\}$ be a sequence of different points in $f_1(U_1)$ converging to x. The property that f_1 is a one-to-one map on U_1 implies that there are unique points $w_n \in U_1$ such that $z_n = f_1(w_n)$. The sequence w_n has a subsequence converging to some point w in the closure of U_1. We denote such a subsequence again by w_n. The continuity of f_1 implies $f_1(w) = x$, and w must be a fixed point of k. Since x is different from the image of p and q under f_1, we have that x is different from p and q (f_1 is one-to-one on the closure of U_1). Hence k has three fixed points and it must be the identity transformation, a contradiction. We have shown that in this case g must be a parabolic transformation with $f_1(p)$ as fixed point. In particular, f_1 and f_2 have the same value at p and q.

SUBCASE 2.2. Assume $f_1(p) \neq f_1(q)$. Using the same kind of arguments as in the previous case, we obtain the following two possibilities.

(1) $f_1(p) = f_2(p)$ and $f_1(q) = f_2(q)$; or
(2) $f_1(p) = f_2(q)$ and $f_1(q) = f_2(p)$.

Since $f_1(p)$ and $f_2(p)$ are the attracting fixed points of g, the second case cannot happen. □

Proof. (Theorem 2.1.) First, we proceed to show the topological part of the theorem. Let G be a J-group of the first kind with associated 3-tuple (γ, D_1, D_2). Define the transformation $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ by

$$F(z) = \begin{cases} f(z), \text{ if } z \in \Omega(G) = D_1 \cup D_2; \\ z, \quad\ \text{ if } z \in \Lambda(G) = \gamma \end{cases}$$

The function $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a bijection and it is continuous on $\Omega(G) = D_1 \cup D_2$. If we prove the continuity of F on γ, then we obtain that F is a continuous bijection and, since $\hat{\mathbb{C}}$ is compact, the desired homeomorphism. Now we proceed to check the continuity of F at a point $x \in \gamma$. Consider a sequence of different points x_n converging to x. If we show the existence of a subsequence x_{n_i} such that $F(x_{n_i})$ converges to x, then we obtain the continuity of F at x.

We consider three cases:
(1) x_n belongs to γ, for all n.
(2) x_n belongs to $D_1 \cup D_2$, for all n.
(3) x_n may belong to any of the above sets for different values of n.

Case (1) is trivial by definition of F. Case (3) can be reduced to one of the cases (1) or (2) by choosing a suitable subsequence. From now on, we assume that x_n belongs to $D_1 \cup D_2$, for all n.

As a consequence of Lemma 3.1., we have the existence of a sequence of loxodromic elements $g_i \in G_{D_1}$ so that their total axes $L(g_i)$ nest about x. Denote by L_i the axis $L(g_i)$. We claim that the sequence of simple loops $F(L_i)$ nests about x. In fact, denote by x_{-i} and x_{+i} the attracting and repelling fixed points of g_i. Set $A_i = A(g_i)$ and $B_i = B(g_i)$. Since $f \circ g_i = g_i \circ f$ and f is one-to-one, we must have that $f(A_i)$ and $f(B_i)$ are g_i invariant and, in particular, the end points of $f(A_i)$ and $f(B_i)$ are the fixed points of g_i. The fact that the sequence of axes L_i is a nested sequence implies that the sequence of simple loops $F(L_i)$ is again nested. Let us assume the sequence $F(L_i)$ does not nest about x. Denote by K_i the compact topological disc bounded by $F(L_i)$ containing x. If we have $T = \cap K_i \neq \{x\}$, then the convergence of x_{-i} and x_{+i} to x implies the existence of some point y in $D_1 \cap T$. It follows that we can find a sequence of points $z_i \in L_i$ (in particular, z_i converges to x) such that $f(z_i)$ converges to y. Let us consider $z = f^{-1}(y)$, which belongs either to D_1 or D_2. Denote by D the disc D_1 or D_2 containing z. Choose a compact disc X_1 in D with center z and a compact disc X_2 in D_1 with center y, so that $f(X_1)$ is contained in X_2 (continuity of f on $\Omega(G)$). Since f is one-to-one, we must have that $z_i = f^{-1}(f(z_i))$ belongs to X_1, for large i. It follows that x must also belongs to X_1. This is a contradiction to the fact that x belongs to γ.

We choose a subsequence x_{n_i} from the sequence x_n such that x_{n_i} belongs to the disc bounded by L_i and containing the point x in its interior. The image $f(x_{n_i})$ also belongs to the interior of the topological disc bounded by $f(L_i)$ containing x. The fact that the sequence $F(L_i)$ nests about x implies that $f(x_{n_i})$ necessarily converges to x. At this point, we have proved the topological part of our theorem.

Now we proceed to prove the conformal part. We must prove that our topological extension is conformal provided its restriction to $D_1 \cup D_2$ is. We have two situations, either $F(D_1) = D_1$ or $F(D_1) = D_2$. Let us recall that the map F restricted to the Jordan curve γ is the identity. In particular, basic topological arguments (and the fact that F preserves orientation on $D_1 \cup D_2$) imply that the last situation cannot occur and we must have that $F(D_i) = D_i$, for $i = 1, 2$. Consider a Riemann map $w : U \to D_1$. By Theorem 1.1., we can extend this Riemann map to a homeomorphism $\bar{w} : \bar{U} \to D_1 \cup \gamma$. Denote by F_1 the restriction of F to $D_1 \cup \gamma$. Then $h = \bar{w}^{-1} \circ F_1 \circ \bar{w}$ is a homeomorphism of $\bar{U}$ which is a conformal automorphism on U and acts as the identity on the boundary of U. That can only happen if h is the identity. In particular, F_1 is the identity. In the same way one shows that F restricted to D_2 is the identity. Now, $F = I$ and it is trivially a conformal automorphism of the Riemann sphere.

Let us assume now that the homeomorphism f is anticonformal. We have again two possibilities; either $F(D_1) = D_1$ or $F(D_1) = D_2$. If we have $F(D_1) = D_1$, then we can use a Riemann map and argue similarly as in the conformal case to show that in this case $F = I$ on D_1 which is a contradiction. In particular, we must have $F(D_{3-i}) = D_i$, for $i = 1, 2$. Observe that F^2 is conformal and keeps invariant both discs. It follows from the conformal part above that $F^2 = I$, that is, F is an involution. Choose a Riemann map $w : U_1 \to D_1$, where $U_1 = U$. Set $U_2 = L$ and $j(z) = \bar{z}$. Define a map $L : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as follows:

$$L(z) = \begin{cases} w(z), & \text{if } z \in \bar{U_1}; \\ F \circ w \circ j(z), & \text{if } z \in \bar{U_2} \end{cases}$$

Observe that L is a conformal automorphism of the Riemann sphere and it satisfies $F = L \circ j \circ L^{-1}$. In particular, F is an anticonformal involution. Since the set of fixed points of an anticonformal involution of the Riemann sphere is a round circle, we have that γ is a circle and G is either a Fuchsian group or a $\mathbb{Z}_2-$extension of a Fuchsian group of the first kind. □

Observe that in the above proof, in the anticonformal part, we have obtained Corollary 2.2.

Proof. (Theorem 2.3) Let (γ_j, D_1^j, D_2^j) be the 3-tuple associated to G_j, for $j = 1, 2$. Define $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as follows.
(1) $F(z) = f(z)$, if z belongs to $\Omega(G_1)$.
(2) If z belongs to γ_1, then we consider a sequence of axes L_i of loxodromics elements $g_i \in G_1$, nesting about z (the existence of such a sequence is given by Lemma 3.1.). Let w be the point in γ_2 to where the sequence of axes $L(f \circ g_i \circ f^{-1}) = L(f_*(g_i)) = L(g_i^*)$ nests about (see Proposition 1.4.), and define $F(z) = w$.

Lemma 3.2 asserts that F is well defined. The above function satisfies the following properties:
(1) $F(\Omega(G_1)) = \Omega(G_2)$.
(2) $F(\Lambda(G_1)) = \Lambda(G_2)$.
(3) $F/_{\Omega(G_1)} = f$.
(4) F is a bijection.

Properties (1), (2) and (3) are direct consequence of the definition of F. We only need to check (4). Lemmas 3.1 and 3.2 applied to the function $f^{-1} : \Omega(G_2) \to \Omega(G_1)$ permit the construction of an inverse function for $F : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$.

The continuity of F on $\Omega(G_1)$ is trivial by hypotheses. We proceed to check the continuity of F on γ_1. Let $x \in \gamma_1$, and choose a sequence of loxodromic elements g_i in G_1 with axes $L(g_i)$ nesting about x (Lemma 3.1). The axes $L(g_i^*)$ of the transformations $g_i^* \in G_2$ nest about $F(x)$ as a consequence of Lemma 3.2. and the definition of F. Since the end points of $A(g_i^*)$ are the same as for $f(A(g_i))$, it follows that the sequence of simple loops $F(L(g_i))$ also nests about $F(x)$. Now we can proceed, in a similar way as in the proof of Theorem 2.1, to show the continuity of F at x and, by compactness of the Riemann sphere, that F is the desired homeomorphism. □

Proof. (Theorem 2.4) CASE 1. Let G be a J-group of the first kind with associated 3-tuple (γ, D_1, D_2), so that $G_{D_1} = G$. Let $w : U \to D_1$ and $v : L \to D_2$ be Riemann maps. Denote by $w : \bar{U} \to \bar{U}$

the homeomorphic extension of w given by Theorem 1.1., and by F and H the Fuchsian groups, necessarily of the first kind, given by $w^{-1} \circ G \circ w$ and $v^{-1} \circ G \circ v$, respectively. Denote by $\psi_w : F \to G$ and $\phi_v : H \to G$ the isomorphisms defined by $\phi_w(k) = w \circ k \circ w^{-1}$ and $\phi_v(t) = v \circ t \circ v^{-1}$, respectively. Consider the isomorphism $\Phi : F \to H$ given by $\Phi(k) = \phi_v^{-1}(\phi_w(k))$. This isomorphism preserves the relation of being crossed and, by Theorem 3.3., there exists a homeomorphism $T : U \to U$ such that $T \circ k \circ T^{-1} = \Phi(k)$, for all k in F. The map $W : \Omega(F) = U \cup L \to \Omega(G) = D_1 \cup D_2$ defined by:

(1) $W(x) = w(x)$, for $x \in U$,

(2) $W(x) = v \circ j \circ T \circ j(x)$, for $x \in L$,

is a homeomorphism inducing the isomorphism $\psi : F \to G$. Now the result, in this case, follows from Theorem 2.3.

CASE 2. Let G be a J-group of the second kind with a 3-tuple (γ, D_1, D_2) so that $g(D_1) = D_1$, for all $g \in G$. If we can construct a J-group K of the first kind with invariant components, so that G is a subgroup of K, then the result will follow from the above case. The existence of such a J-group K is as follows. Let γ_1, γ_2,...., be a maximal set of non-equivalent (under G) maximal arcs in $\gamma \cap \Omega(G)$. Denote by u_i and v_i the ends of the arc γ_i. We consider on D_i the complete hyperbolic metric induced from U via a Riemann map. Let $\alpha_i \subset D_1$ and $\beta_i \subset D_2$ be the unique geodesics with end points u_i and v_i. Denote by Q_i the interior of the topological disc bounded by $\alpha_i \cup \beta_i$ and contained in $\Omega(G)$. In the disc Q_i we can construct an infinite sequence of circles C_j^i, $j \in \mathbb{Z}$, satisfying the following properties (see Figure 3):

(a) C_j^i is tangent to C_{j-1}^i and C_{j+1}^i.

(b) $C_n^i \cap C_m^i = \emptyset$, for $n \notin \{m-1, m, m+1\}$.

(c) The euclidean centers of C_j^i approach u_i and v_i as j approach ∞ and $-\infty$, respectively.

Now we consider a sequence of parabolic elements P_j^i such that $P_j^i(C_{2j-1}^i) = P_{2j}^i$ and sends tangencies to tangencies, that is, $P_j^i(C_{2j-2}^i \cap C_{2j-1}^i) = C_{2j} \cap C_{2j+1}^i$. The circles may have different radii, but necessarily converging to zero as j approaches either $+\infty$ or $-\infty$.

Let K be the group generated by G and all the parabolic transformations P_j^i. By construction, there is a Jordan curve invariant under the group K. Each of the topological discs bounded for such a curve is also invariant under the group K. □

Proof. (Theorem 2.5.) Let us denote by $\phi : F \to G$ the isomorphism $(f_i)_*$. Lemma 3.4 and the fact that $f_1/_{\Omega(F)\cap\mathbb{R}} = f_2/_{\Omega(F)\cap\mathbb{R}}$ imply that f_1 and f_2 coincide in $\mathbb{R}\cup\{\infty\}$. Define a map as follows:

$$L(z) = \begin{cases} f_1(z), & if\ z \in \bar{U_1}; \\ f_2(z), & if\ z \in \bar{U_2}, \end{cases}$$

where the bar represents the Euclidean closure of sets.

The map L is clearly analytic on the Riemann sphere minus the extended real line and continuous on all the Riemann sphere. It follows that L is analytic on all of the Riemann sphere. One can also observe that necessarily the image of a limit point of F is again a limit point of G. In fact, if $x \in \Lambda(F)$, then we can find a point $u \in U_1$ and a sequence of different elements γ_n in F so that $\gamma_n(u)$ converges to x. Since $\phi(\gamma_n)(f_1(u)) = f_1(\gamma_n(u))$, ϕ is isomorphism and, by hypothesis, f_1 extends continuously to the extended real line, we have that $f_1(x)$ is a limit point of G. The group G is non-elementary as can be seen from the following argument. Since the group F is non-elementary, we can find three different hyperbolic elements in F, say γ_1, γ_2 and γ_3, no two of them commuting. The fact that $\phi : F \to G$ is an isomorphism, asserts that $g_1 = \phi(\gamma_1)$, $g_2 = \phi(\gamma_2)$ and $g_3 = \phi(\gamma_3)$ are three different non-elliptic transformations no two of them commuting. The set of fixed points of these thre transformations are disjoint and they are contained in the limit set of G.

If we have $f_1(U_1) \cap f_2(U_2) = \emptyset$, then the map L must be one-to-one on the extended real line. In fact, if there are two different points on $\hat{\mathbb{R}}$, say x and y, so that $L(x) = L(y)$, then the same holds for f_1 and f_2. The image of both arcs determined by x and y are common boundary points of both $f_1(U_1)$ and $f_2(U_2)$. It follows that one of these arcs must project onto the point $L(x)$. This is impossible for a non-constant analytic map as it is the case of L. As a consequence, the map L is one-to-one on all the Riemann sphere and, in particular, is a fractional linear transformation and G is a Fuchsian group.

Let us assume $f_1(U_1) \cap f_2(U_2) \neq \emptyset$. Denote by Δ the connected (invariant) component of the region of discontinuity of G containing $f_i(U_i)$. First, we proceed to show that $f_1(U_1) = f_2(U_2)$. If F is of the first kind, then the boundary points of $f_i(U_i)$ are necessary limit points (the boundary points are images of boundary points of U_i and, as shown above, the image of limit points are again limit points). As a consequence, $f_i(U_i) = \Delta$ and the desired equality follows. If F is a Fuchsian group of the second kind and $f_1(U_1) \neq f_2(U_2)$, then the path-connectivity of $f_i(U_i)$, for $i = 1, 2$, implies the existence of a point x either in both $f_1(U_1)$ and the boundary of $f_2(U_2)$ or in both $f_2(U_2)$ and the boundary of $f_1(U_1)$. Without lost of generality, we may assume the existence of a point x in both $f_1(U_1)$ and the boundary of $f_2(U_2)$. It follows that there is a point y in the extended real line so that $f_2(y) = x$. Since x is a regular point for G, the point y is also a regular point for F. Let $z \in U_1$ be so that $f_1(z) = x$. We can find open neighborhoods R of z and T of x with $f_1(R) = T$ and such that there is a neighborhood N of y which does not intersect R. The continuity property of the extension of f_1 to the extended real line implies the existence of a point $w \in N$ so that $f_1(w) \in T$. This gives a contradiction to the injectivity of f_1 on U_1.

Since we have the equality $f_1(U_1) = f_2(U_2)$, we are able to define a transformation $K : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ as follows:

$$K(z) = \begin{cases} f_1^{-1} \circ f_2(z), & if\ z \in \bar{U_2}; \\ f_2^{-1} \circ f_1(z), & if\ z \in \bar{U_1}, \end{cases}$$

The transformation K is a conformal map acting as the identity on the real line and, as a consequence, the identity map. We obtain a contradiction to the fact that K must permute both U_1 and U_2. □

Proof. (Theorem 2.6.) The images $f_1(U_1)$ and $f_2(U_2)$ are invariant sets for the Kleinian group G. Since f_i is continuous at the boundary of U_i, the boundary of $f_i(U_i)$ is contained in the image under f_i of

$\mathbb{R} \cup \{\infty\} = \hat{\mathbb{R}}$. On the other side, if we denote by N the kernel of the homomorphism induced by f_i, then we have either N is trivial or N is a non-trivial normal subgroup of F. Assume that N is non-trivial. In this case N is a Fuchsian group of the first kind. Now, for each point y in the boundary of U_i we can find a point $u \in U_i$ and a sequence of different elements of N, say g_n, so that $g_n(u)$ converges to y. The equation $f_i = (f_i)_*(g_n) \circ f_i = f_i \circ g_n$ implies that $f_i(u) = f_i(g_n(u))$. Now the continuous extension property of f_i to the boundary of U_i implies that $f_i(u) = f_i(y)$. In particular, the image of the boundary points of U_i are interior points of $f_i(U_i)$. The above implies that $f_i(U_i)$ is a hyperbolic surface on the Riemann sphere without boundary points, a contradiction. As a consequence, the group N must be trivial and the maps f_i are necessarily injective. Now we are in the hypothesis of Theorem 2.5. □

REMARK. The continuity extension hypothesis of Theorems 2.5 and 2.6 can be removed for the class of Fuchsian group of divergence type (see [**11**]).

REFERENCES

[1] A. Basmajian, *Hyperbolic invariants for infinitely generated fuchsian groups*, Thesis Ph.D. in Mathematic, S.U.N.Y. at Stony Brook

[2] ______, *Hyperbolic structures for surfaces of infinite type*, Trans. Amer. Math. Soc., **336** (1993), 421-444.

[3] C. Carathéodory, *Über die gegenseitige Beziehung der Ränder bei der Abbildung des Innern einer Jordanschen Kurve auf einen Kreis*, Math. Ann., **73** (1913) 305-320.

[4] I. Kra and B. Maskit, *Involutions on Kleinian groups*, Bull. Amer. Math. Soc., **78** (1972), 801-805.

[5] S.L. Krushkal', B.N. Apanasov and N.A. Gusevskii, *Kleinian Groups and Uniformization in Examples and Problems*, Translations of Mathematical monographs, Amer. Math. Soc., vol. 62 (1981).

[6] L. Kurittu, *Jordan curves as the limit point set of Kleinian groups*, Ann. Acad. Sci. Fenn. Ser. A I, Mathematica Dissertations 78.

[7] B. Maskit, *Self-maps on Kleinian groups*, Amer. J. of Math., **XCIII** (1971), 840-856.

[8] ______, *Kleinian Groups*, Springer-Verlag, Berlin-Heidelberg (1988).

[9] ______, *On extended quasifuchsian groups*, Ann. Acad. Sci. Fenn. Ser. A I Math., **15** (1990), 53-64.

[10] C. Pommerenke, *Boundary Behavior of Conformak Maps*, Grundlehren der mathematischen Wissenschaften 299, Springer-Verlag (1991).

[11] C. Pommerenke and R. Hidalgo, *On simultaneous homomorphisms onto Kleinian groups*, Preprint.

[12] P. Tukia, *On discrete groups of the unit disc and their isomorphisms*, Ann. Acad. Sci. Fenn. Ser. A I Math., **504** (1972).

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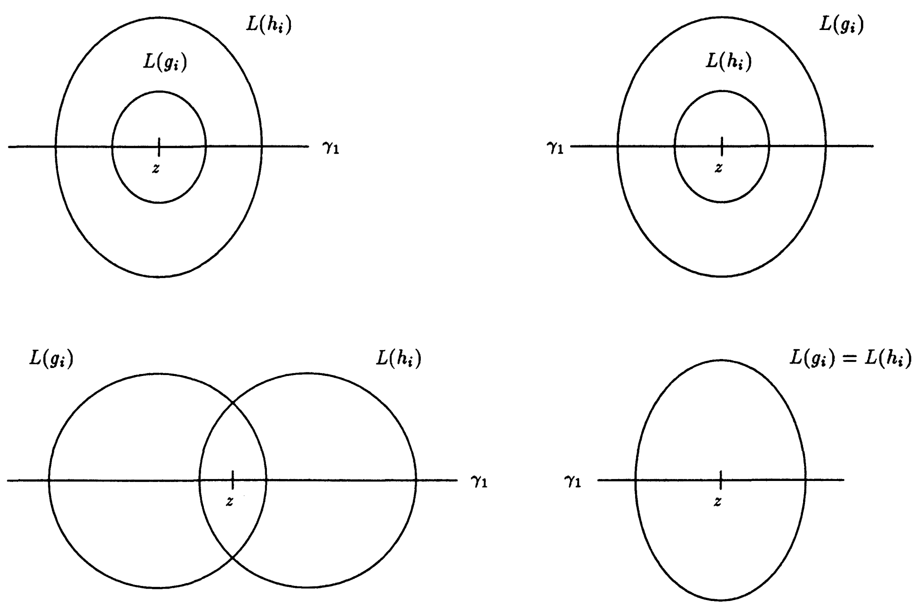

Figure 1.

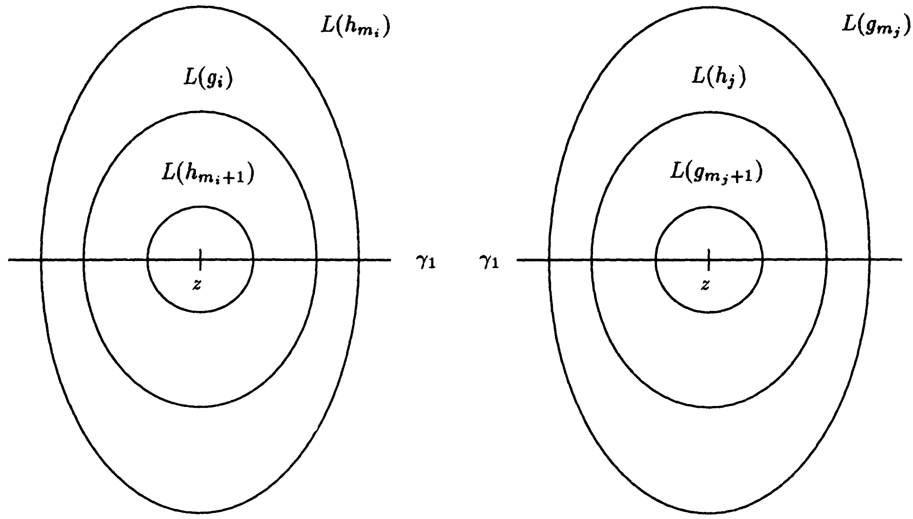

FIGURE 2.

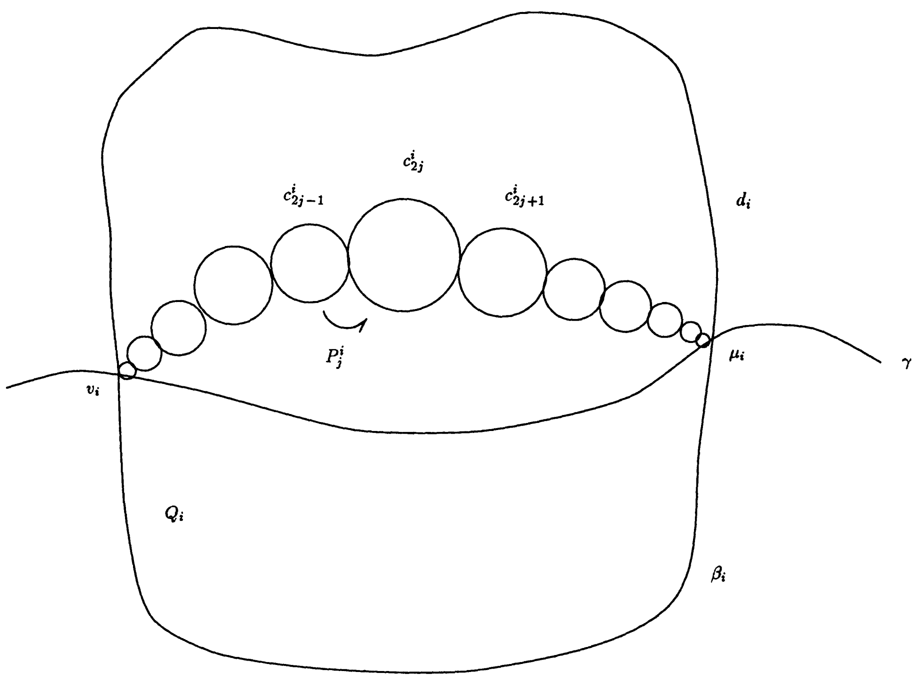

FIGURE 3.