ON DEDEKIND'S FUNCTION $\eta(\tau)$

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1. Introduction. A transformation of the form

$$\tau' = \frac{a\tau + b}{c\tau + d} , \tag{1.1}$$

where a, b, c, d are rational integers satisfying

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb = 1 , \tag{1.2}$$

is called a *modular transformation.* Without loss of generality we may assume $c \geq 0$. A function $f(\tau)$, analytic in the upper halfplane $\mathfrak{J}(\tau) > 0$, and satisfying the functional equation

$$f(\tau) = (c\tau + d)^k \, f\left(\frac{a\tau + b}{c\tau + d}\right), \tag{1.3}$$

is called a *modular form of dimension k.* An example of a modular form is the discriminant

$$\Delta(\tau) = \exp\{2\pi i\tau\} \prod_{m=1}^{\omega} (1 - \exp\{2\pi i m\tau\})^{24} , \tag{1.4}$$

which is of dimension -12; that is, it satisfies the equation*

$$\Delta(\tau') = (c\tau + d)^{12}\Delta(\tau) . \tag{1.5}$$

An important role in the theory of modular functions is played by the function

$$\eta(\tau) = \exp\left\{\frac{\pi i \tau}{12}\right\} \prod_{m=1}^{\infty} (1 - \exp\{2\pi i m\tau\}) , \tag{1.6}$$

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*Cf. Hurwitz [6] ; however, he gives this formula only in homogeneous coordinates.

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which is the 24th root of $\Delta(\tau)$. The transformation formula for this function may be obtained from (1.5) and is conveniently written as:

$$\eta(\tau') = \eta\left(\frac{a\tau + b}{c\tau + d}\right) = \epsilon\sqrt{-i(c\tau + d)}\ \eta(\tau)\,. \tag{1.7}$$

Since we have assumed $c \geq 0$ and $\Im(\tau) > 0$, the radicand has a nonnegative real part. By the square root we always mean the principal branch; that is, $\Re(\sqrt{\ }) > 0$. The ϵ appearing in (1.7) is a 24th root of unity. The purpose of the present paper is to determine this ϵ completely.

Investigations concerning this root of unity were carried out first by Dedekind [2] and later by Tannery and Molk [10] and Rademacher [8; 9]. However, they use the theory of $\log \eta(\tau)$, which requires much more than is needed for this purpose. Hurwitz discusses only $[\Delta(\tau)]^{1/12}$ and remarks that the transformation formula of $\eta(\tau)$ can be obtained by means of θ-functions. The investigations of Hermite [5] are likewise not sufficient for our purpose, because he discusses only $\eta^3(\tau)$, and therefore a third root of unity remains still undetermined.

In the following, we shall approach the determination of ϵ directly by investigations of the function $\eta(\tau)$, which, by a well-known formula due to Euler, can be written as the following sum:

$$\eta(\tau) = \exp\left\{\frac{\pi i \tau}{12}\right\} \sum_{\lambda=-\infty}^{+\infty} (-1)^\lambda \exp\{\pi i \tau\lambda(3\lambda - 1)\} \tag{1.8}$$

$$= \sum_{\lambda=-\infty}^{+\infty} (-1)^\lambda \exp\left\{3\pi i \tau\left(\lambda - \frac{1}{6}\right)^2\right\}.$$

Our starting point is formula (1.8); our principal tools are a Poisson transformation formula and Gaussian sums.

2. Application of a Poisson formula. We introduce a new variable z with $\Re(z) > 0$ by the substitution*

$$\tau' = \frac{iz}{c} + \frac{a}{c}\,, \qquad c > 0;\ (a, c) = 1\,, \tag{2.1}$$

*This requires $c \neq 0$, but the case $c = 0$ is trivial.

and obtain, from (1.8)

$$\eta\left(\frac{a}{c}+\frac{iz}{c}\right)=\sum_{\lambda=-\infty}^{+\infty}(-1)^{\lambda}\exp\left\{\frac{3\pi i}{c}(a+iz)\left(\lambda-\frac{1}{6}\right)^{2}\right\} \tag{2.2}$$
$$=\sum_{j \bmod 2c}\exp\pi i\left\{j+\frac{3a}{c}\left(j-\frac{1}{6}\right)^{2}\right\}$$
$$\times\sum_{q=-\infty}^{+\infty}\exp\left\{-\frac{3\pi z}{c}\left(2cq+j-\frac{1}{6}\right)^{2}\right\}.$$

To the inner sum,

$$F_c(z)=\sum_{q=-\infty}^{+\infty}\exp\left\{-12\pi cz\left(q+\frac{6j-1}{12c}\right)^{2}\right\},$$

we apply Poisson's formula (cf. [11]),

$$\sum_{m=-\infty}^{+\infty}\exp\{-\pi(m+\alpha)^{2}t\}=\frac{1}{\sqrt{t}}\sum_{m=-\infty}^{+\infty}\exp\left\{2\pi im\alpha-\frac{\pi m^{2}}{t}\right\},\qquad \Re(t)>0,$$

and obtain

$$F_c(z)=\frac{1}{2\sqrt{3cz}}\sum_{q=-\infty}^{+\infty}\exp\left\{2\pi iq\,\frac{6j-1}{12c}-\frac{\pi q^{2}}{12cz}\right\}.$$

Putting this in (2.2), we get:

$$\eta\left(\frac{a}{c}+\frac{iz}{c}\right)=\frac{1}{\sqrt{3cz}}\sum_{q=-\infty}^{+\infty}\exp\left\{\frac{-\pi q^{2}}{12cz}\right\}T_q(c), \tag{2.3}$$

where

$$T_q(c)=\frac{1}{2}\sum_{j \bmod 2c}\exp\pi i\left\{j+\frac{3a}{c}\left(j-\frac{1}{6}\right)^{2}+q\,\frac{6j-1}{6c}\right\}$$
$$=\frac{1}{2}\exp\pi i\left\{\frac{a-2q}{12c}\right\}[1+\exp\pi i\{3ac+c-a+q\}]$$
$$\times\sum_{j=1}^{c}\exp\left\{\frac{\pi i}{c}\left[3aj^{2}+j(c-a+q)\right]\right\}.$$

But, a and c being coprime, and thus

$$3ac + c - a \equiv 1 \pmod 2,$$

only the T_q with odd subscripts actually appear so that we have

$$(2.4)\quad T_{2r+1}(c) = \exp \pi i \left\{\frac{a - 4r - 2}{12c}\right\} \sum_{j=1}^{c} \exp\left\{\frac{\pi i}{c}\left[3aj^2 + j(c - a + 1 + 2r)\right]\right\}.$$

In order to have a complete square in the exponent we multiply each term of the sum by

$$\exp \pi i \left\{j\, \frac{ad - 1}{c}(c - 1 + 2r)\right\} = \exp \pi i\,\{jb(c + 1)\}.$$

As we do not wish to change T_{2r+1} by this multiplication, we have to assume that, for c even, b also is even. Using the abbreviation

$$(2.5)\qquad \beta = cd + d - 1,$$

we obtain from (2.4):

$$(2.6)\quad T_{2r+1}(c) = \exp \pi i \left\{\frac{a - 4r - 2}{12c}\right\} \sum_{j=1}^{c} \exp\left\{\frac{\pi i a}{12c}\left[36j^2 + 12j(cd + d - 1 + 2rd)\right]\right\}$$

$$= \exp \pi i \left\{\frac{a - a\beta^2 - 2}{12c} - \frac{r}{3c}(ad^2 r + ad\beta + 1)\right\}$$

$$\times \sum_{j=1}^{c} \exp\left\{\frac{\pi i a}{12c}(6j + \beta + 2rd)^2\right\}.$$

In the sum appearing here, j can be taken as running over any full residue system mod c, because $\beta \equiv c \pmod 2$ and therefore the sum remains unchanged if j is replaced by $j + c$. Consequently, β can be chosen arbitrarily, mod 6, and $T_{2r+1}(c)$ can be simplified by the substitution $r = 3\mu + \nu$. We note that

$$\exp \pi i \left\{\frac{-r}{3c}(ad^2 r + ad\beta + 1)\right\}$$

$$= \exp \pi i \left\{\frac{-\mu}{c}(3\mu d + 3\mu bcd + 2d\nu + bc\beta + cd + d)\right.$$

$$\left. - \frac{\nu}{3c}(d\nu + bcd\nu + bc\beta + cd + d)\right\};$$

and considering

$$\exp\{-\pi i\mu(b\beta+d+3\mu bd)\} = \exp\{-\pi i\mu(bcd-b+d)\} = \exp\{\pi i\mu\},$$

we obtain

$$T_{6\mu+2\nu+1}(c) = \exp\pi i\left\{\frac{a-a\beta^2-2}{12c} - \frac{\nu}{3}\left[bd\nu+d+b\beta+\frac{d}{c}(\nu+1)\right] - \frac{\mu}{c}\left[3\mu d+d(1+2\nu)+c\right]\right\} H_{a,c}(\beta+2\nu d)$$

with the abbreviation

$$(2.7)\qquad H_{a,c}(\beta) = \sum_{j \bmod c} \exp\left\{\frac{\pi i a}{12c}(6j+\beta)^2\right\}, \qquad \beta \equiv c \pmod 2.$$

Looking back to (2.3), we see that the result we have obtained so far may be written as:

$$(2.8)\quad \eta\left(\frac{a}{c}+\frac{iz}{c}\right) = \frac{1}{\sqrt{3cz}}\exp\pi i\left\{\frac{a-a\beta^2-2}{12c}\right\} \times \sum_{\nu=0}^{2}\exp\left\{\frac{-\pi i\nu}{3}\left[bd\nu+d+b\beta+\frac{d}{c}(\nu+1)\right]\right\} U_\nu(z) H_{a,c}(\beta+2d\nu),$$

with

$$U_\nu(z) = \sum_{\mu=-\infty}^{+\infty}\exp\left\{\pi i\left[\mu-\frac{3d}{c}\mu^2-\frac{d}{c}\mu(2\nu+1)\right]-\frac{3\pi}{cz}\left(\mu+\frac{2\nu+1}{6}\right)^2\right\}.$$

These expressions are easy to sum, since, according to (1.8), we have

$$U_0(z) = \sum_{\mu=-\infty}^{+\infty}\exp\left\{\pi i\left[\mu-\frac{3d}{c}\left(\mu^2+\frac{\mu}{3}\right)\right]-\frac{3\pi}{cz}\left(\mu+\frac{1}{6}\right)^2\right\} = \exp\left\{\frac{\pi i d}{12c}\right\}\eta\left(-\frac{d}{c}+\frac{i}{cz}\right);$$

and, replacing μ by $-\mu-1$, we see that

$$U_1(z) = -U_1(z), \quad \text{or} \quad U_1(z) = 0\ ,$$

$$U_2(z) = -\exp\left\{\pi i\ \frac{2d}{c}\right\}\ U_0(z)\ .$$

Now, by the meaning of z in (2.1), we get

$$-\frac{d}{c} + \frac{i}{cz} = \frac{-d\tau' + b}{c\tau' - a} = \tau\ ,$$

and have therefore:

$$\text{(2.9)} \qquad \eta(\tau') = \frac{1}{\sqrt{3c}}\ \exp \pi i \left\{\frac{a(1-\beta^2)-2+d}{12c}\right\}$$
$$\times\left[H_{a,c}(\beta) - \exp\left\{\frac{-2\pi i}{3}\ (d+2bd+b\beta)\right\}\right.$$
$$\left.\times H_{a,c}(\beta+4d)\right]\sqrt{-i(c\tau+d)}\ \ \eta(\tau)\ .$$

Comparing this with (1.7), we see that we have obtained so far:

$$\text{(2.91)} \quad \epsilon = \frac{1}{\sqrt{3c}}\ \exp \pi i \left\{\frac{a(1-\beta^2)-2+d}{12c}\right\}$$
$$\times\left[H_{a,c}(\beta) - \exp\left\{\frac{-2\pi i}{3}\ (d+2bd+b\beta)\right\}\ H_{a,c}(\beta+4d)\right]$$
$$= \frac{1}{\sqrt{3c}}\ \exp \pi i \left\{\frac{bd(1-c^2)-cd}{12} + \frac{(1-d)(b+ad)}{6}\right\}$$
$$\times\left[H_{a,c}(\beta) - \exp\left\{\frac{-2\pi i}{3}\ (d+2bd+b\beta)\right\} H_{a,c}(\beta+4d)\right]$$

and it remains to be shown that this is a root of unity.

3. Reduction to Gaussian sums. The sums $H_{a,c}(\beta)$ which appear in (2.91) are defined in (2.7) only for $\beta \equiv c \pmod 2$. In this section, however, it will be more convenient to consider the more general sums*

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*We have used the letters h and k instead of a and c in order to indicate that the investigations of this section are independent from our previous results.

(3.1) $$H_{h,k}(\gamma)=\frac{1}{2}\sum_{j \bmod 2k}\exp\left\{\frac{\pi i h}{12k}(6j+\gamma)^2\right\},$$

with no restriction on γ. These sums can be expressed in terms of Gaussian sums

(3.2) $$G(h,k)=\sum_{j \bmod k}\exp\left\{\frac{2\pi i h}{k}j^2\right\}.$$

Comparing the definitions (3.1) and (3.2) one finds immediately that:

$$H_{h,k}(0)+H_{h,k}(1)+H_{h,k}(2)+H_{h,k}(3)+H_{h,k}(4)+H_{h,k}(5)=\frac{1}{4}G(h,24k),$$

$$H_{h,k}(0)+H_{h,k}(2)+H_{h,k}(4)=\frac{1}{2}G(h,6k),$$

$$H_{h,k}(0)+H_{h,k}(3)=\frac{1}{4}G(3h,8k).$$

If we consider that

$$H_{h,k}(-\gamma)=H_{h,k}(\gamma)=H_{h,k}(\gamma+6n),$$

we get the following relations:

(3.31) $$H_{h,k}(0)=\frac{1}{2}G(3h,2k),$$

(3.32) $$H_{h,k}(3)=\frac{1}{4}G(3h,8k)-\frac{1}{2}G(3h,2k),$$

(3.33) $$H_{h,k}(2)=\frac{1}{4}G(h,6k)-\frac{1}{4}G(3h,2k),$$

(3.34) $$H_{h,k}(1)=\frac{1}{8}G(h,24k)-\frac{1}{8}G(3h,8k)-\frac{1}{4}G(h,6k)+\frac{1}{4}G(3h,2k).$$

In order to obtain the sums $H_{h,k}(\gamma)$ explicitly, the following rules concerning Gaussian sums may be useful.*

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* For the formulas (3.41)–(3.47) see [1] or [3]; (3.46) may also be found in [7].

As elementary consequences of the definition (3.2) we have:

(3.41) $G(mh, mk) = mG(h, k)$ $\qquad m > 0$

(3.42) $G(h, k_1k_2) = G(hk_1, k_2)\ G(hk_2, k_1)$ $\qquad (k_1, k_2) = 1$

(3.43) $G(m^2h, k) = G(h, k)$ $\qquad (m, k) = 1$

(3.44) $G(h, m^2k) = mG(h, k)$ $\qquad (m, h) = 1;\quad m > 0$ and odd.

The following results, due to Gauss [4], are a little deeper:

(3.45) $$G(h_1h_2, k) = \left(\frac{h_1}{k}\right) G(h_2, k) \qquad (h_1h_2, k) = 1,\quad k \text{ odd}$$

(3.46) $$G(1, k) = \sqrt{k}\ i^{[(k-1)/2]^2} \qquad k \text{ odd}$$

(3.47) $$G(h, 2^{\alpha}) = \begin{cases} 0 & h \text{ odd},\quad \alpha = 1 \\ 2^{(\alpha+1)/2} \left(\frac{2}{h}\right)^{\alpha+1} e^{\pi i h/4} & h \text{ odd},\quad \alpha \geq 2. \end{cases}$$

The symbol $\left(\frac{h}{k}\right)$ is the Jacobi symbol.

The following discussion may be restricted to the case $\gamma \equiv k \pmod 2$, which will be sufficient for our purpose. Furthermore, we put* throughout $k = 2^{\lambda}k_1$ (k_1 being odd), and have then to distinguish whether 3 does or does not divide k_1.

Assume first $3 \mid k_1$. Then we find, using (3.41) and (3.44), that

(3.51) $$H_{h,k}(1) = 0, \qquad H_{h,k}(2) = 0;$$

and, applying (3.41), (3.42), (3.44), (3.45), and (3.47), we obtain:

(3.52) $$H_{h,k}(0) = 2^{\lambda/2}\left(\frac{2}{h}\right)^{\lambda} \exp\left\{\frac{3}{4}\pi i h k_1\right\} G(2h, 3k_1),$$

(3.53) $$H_{h,k}(3) = \exp\left\{\frac{3}{4}\pi i h k\right\}\ G(2h, 3k).$$

*We do this in order to avoid the reciprocity law for Gaussian sums which would require additional distinctions concerning the sign of h.

As a consequence of (3.46) we have:

$$G(1, 3k) = \sqrt{3k} \exp\left\{\frac{\pi i}{8} (3k-1)^2\right\} = -\sqrt{3} \exp\left\{\frac{-\pi i k}{2}\right\} G(1,k),$$

and therefore, according to (3.45),

$$G(2h,3k) = \left(\frac{2h}{3k}\right) G(1,3k) = -\left(\frac{2h}{3}\right) \sqrt{3} \exp\left\{\frac{-\pi i k}{2}\right\} G(2h,k).$$

This formula enables us to express (3.52) and (3.53) in the single formula:

$$(3.6) \quad H_{h,k}(k) = \sqrt{3}\, 2^{\lambda/2} \left(\frac{h}{3}\right) \exp \pi i \left\{\frac{k_1(h-1)}{2} + \frac{hk_1}{4} + \lambda\frac{h^2-1}{8}\right\} G(2h, k_1).$$

In case $3 \nmid k_1$, by use of (3.42) and (3.43) we can express the more complicated sums $H_{h,k}(1)$ and $H_{h,k}(2)$ by $H_{h,k}(3)$ and $H_{h,k}(0)$, respectively:

$$(3.71) \qquad H_{h,k}(1) = \exp\left\{\frac{4}{3} \pi i hk\right\} H_{h,k}(3),$$

$$(3.72) \qquad H_{h,k}(2) = \exp\left\{\frac{4}{3} \pi i hk\right\} H_{h,k}(0).$$

More generally, the following recursion formula holds:

$$(3.73) \qquad H_{h,k}(\gamma + 2n) = \exp\left\{\frac{\pi i}{3} (\gamma + n)\, nhk\right\} H_{h,k}(\gamma).$$

In order to compute $H_{h,k}(0)$ and $H_{h,k}(3)$, we apply (3.42), (3.43), (3.45), and (3.47) to obtain:

$$H_{h,k}(3) = \left(\frac{k}{3}\right) \exp \pi i \left\{\frac{k-1}{2} + \frac{3hk}{4}\right\} G(2h, k),$$

$$H_{h,k}(0) = \left(\frac{k}{3}\right) 2^{\lambda/2}\left(\frac{2}{h}\right)^{\lambda} \exp \pi i \left\{\frac{k_1-1}{2} + \frac{3hk_1}{4}\right\} G(2h, k_1).$$

Applying this on (3.71) and (3.72), and considering

$$\exp \pi i \left\{\frac{4}{3} hk + \frac{3}{4} hk_1\right\} = \exp \pi i \left\{\frac{hk}{12} + \frac{3}{4} h(k_1 - k)\right\},$$

we can combine (3.71) and (3.72) into:

$$(3.8)\quad H_{h,k}(k) = 2^{\lambda/2}\left(\frac{k}{3}\right)$$
$$\times \exp \pi i \left\{\frac{hk}{12} + \frac{3}{4} h(k_1 - k) + \frac{k_1 - 1}{2} + \lambda \frac{h^2 - 1}{8}\right\} G(2h, k_1) .$$

4. Determination of the root of unity. Now we go back to our result (2.9) and consider the following expression:

$$(4.1)\qquad \rho = \frac{1}{\sqrt{3c}} \exp\left\{\frac{\pi i}{6} (1 - d)(b + ad)\right\}$$
$$\times \left[H_{a,c}(\beta) - \exp\left\{\frac{-2\pi i}{3} (d + 2bd + b\beta)\right\} H_{a,c}(\beta + 4d)\right] .$$

According to the results of the preceding section, we have to distinguish whether c is divisible by 3 or not and to keep in mind that $c = 2^\lambda c_1$, c_1 odd.

Let us assume first $3 \mid c$; according to (3.51) we know that:

$$H_{a,c}(\beta) = H_{a,c}(dc + d - 1) = 0 \qquad \text{if } d \equiv -1 \pmod 3 ,$$
$$H_{a,c}(\beta + 4d) = H_{a,c}(dc + 5d - 1) = 0 \qquad \text{if } d \equiv +1 \pmod 3 .$$

Therefore we have:

$$(4.2)\qquad \rho = \left(\frac{d}{3}\right) \frac{1}{\sqrt{3c}} \exp \pi i \left\{\frac{1}{6} (1 - d)(b + ad) + \frac{2}{3} (d - 1)(1 + b)\right\} H_{a,c}(c)$$
$$= \left(\frac{a}{3}\right) \frac{1}{\sqrt{3c}} \exp\left\{\frac{\pi i}{2} (d - 1)(b + ad)\right\} H_{a,c}(c) .$$

Considering that

$$\exp\left\{\frac{\pi i}{2} [(d - 1)(b + ad + c) + (a - 1)(c_1 - c)]\right\} = 1 ,$$

and therefore that

$$\exp \pi i \left\{ \frac{1}{2} (d-1)(b+ad) + \frac{1}{2} (a-1) c_1 \right\}$$

$$= \exp\left\{ \frac{\pi i}{2} \left[(d-1)(b+ad+c) + (a-1)(c_1-c) - c(d-a) \right] \right\}$$

$$= \exp\left\{ \frac{\pi i}{6} c(d-a) \right\} ,$$

we get from (4.2) and (3.6):

$$\rho = \frac{1}{\sqrt{c_1}} \exp \pi i \left\{ \frac{a}{4} (c_1 - c) + \frac{cd}{6} + \frac{ac}{12} + \lambda \frac{a^2-1}{8} \right\} G(2a, c_1) . \tag{4.3}$$

In case $3 \nmid c$, we can apply (3.73), which gives us

$$H_{a,c}(\beta + 4d) = \exp\left\{ \frac{2\pi i}{3} (\beta + 2d) \, acd \right\} H_{a,c}(\beta)$$

$$= \exp\left\{ \frac{2\pi i}{3} (b\beta + 2bd + d - c) \right\} H_{a,c}(\beta) ,$$

and obtain from (4.1):

$$\rho = \frac{1}{\sqrt{3c}} \exp\left\{ \frac{\pi i}{6} (1-d)(b+ad) \right\} \left[1 - \exp\left\{ \frac{-2\pi i c}{3} \right\} \right] H_{a,c}(\beta)$$

$$= \frac{1}{\sqrt{c}} \left(\frac{c}{3} \right) \exp \pi i \left\{ \frac{1}{6} (1-d)(b+ad) - \frac{1}{2} + \frac{2c}{3} \right\} H_{a,c}(\beta) .$$

Now we apply (3.37) once more, putting

$$H_{a,c}(\beta) = H_{a,c}(c + \beta - c) = \exp\left\{ \frac{\pi i}{3} \left(c + \frac{\beta - c}{2} \right) \frac{\beta - c}{2} \, ac \right\} H_{a,c}(c)$$

$$= \exp\left\{ \frac{\pi i}{12} (\beta^2 - c^2) \, ac \right\} H_{a,c}(c) .$$

Using (3.8) and considering

$$\exp\left\{\frac{\pi i}{12}(\beta^2-c^2)\,ac\right\}$$

$$= \exp\left\{\frac{\pi i}{12}\left[ac(c^2-1)(d^2-1)+2ac(d-1)(cd+d)\right]\right\}$$

$$= \exp\left\{\frac{\pi i}{6}(d-1)(bc+c+b+c^2)\right\}$$

$$= \exp\pi i\left\{\frac{1}{6}(d-1)(b+ad)-\frac{1}{2}(d-1)(c^2-1)+\frac{c}{6}(d-1)\right\},$$

$$\exp\left\{\frac{\pi i}{2}\left[(a-1)(c_1-c)-(d-1)(c^2-1)\right]\right\}=1,$$

we see that the expression for ρ becomes again (4.3). Therefore, we have in all cases:

$$(4.4)\qquad \epsilon = \exp\pi i\left\{\frac{1}{12}\left[bd(1-c^2)+c(a+d)\right]+a\,\frac{c_1-c}{4}+\lambda\frac{a^2-1}{8}\right\}\times\frac{1}{\sqrt{c_1}}G(2a,c_1),$$

with the only restriction that, for even c, b also has to be even.

In order to show that our formula (4.4) holds even if this condition is not satisfied, we put

$$\tau' = \frac{a\tau+b}{c\tau+d},\qquad c\text{ even},\quad b\text{ odd},$$

$$\tau^* = \frac{(a+c)\,\tau+(b+d)}{c\tau+d} = \tau'+1.$$

Then, for τ^*, formula (4.4) holds; considering

$$\eta(\tau+1) = \exp\left\{\frac{-\pi i}{12}\right\}\eta(\tau),$$

which is an immediate consequence of (1.6), we find:

$$\eta(\tau^*) = \epsilon^*\eta(\tau) = \exp\left\{\frac{-\pi i}{12}\right\} \eta(\tau') = \exp\left\{\frac{-\pi i}{12}\right\} \epsilon\eta(\tau)$$

$$\epsilon = \exp\left\{\frac{\pi i}{12}\right\} \epsilon^* . \tag{4.5}$$

Now, if we compute ϵ^* by means of (4.4), and then ϵ, using (4.5), the result will be exactly the same as we get computing ϵ directly by means of (4.4).

Finally, we can omit the Gaussian sums in (4.3) and, using (3.45) and (3.46), obtain:

$$\epsilon = \left(\frac{a}{c_1}\right) \times \exp \pi i \left\{\frac{1}{12}\left[bd(1-c^2)+c(a+d)\right] + \frac{1-c_1}{4} + a\,\frac{c-c_1}{4} + \lambda\frac{a^2-1}{8}\right\} . \tag{4.6}$$

This formula agrees with the one given by Tannery and Molk [10, p. 112] .

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