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THE CORESTRICTION OF VALUED DIVISION ALGEBRAS OVER HENSELIAN FIELDS II

YOON SUNG HWANG

When L/F is a tame extension of Henselian fields (i.e. $\operatorname{char}(\overline{F}) \nmid [L:F]$), we analyze the underlying division algebra ${}^cD$ of the corestriction $\operatorname{cor}_{L/F}(D)$ of a tame division algebra D over L with respect to the unique valuations of ${}^cD$ and D extending the valuations on F and L. We show that the value group of ${}^cD$ lies in the value group of D and for the center of residue division algebra, $Z(\overline{{}^cD}) \subseteq \mathcal{N}(Z(\overline{D})\,/\,\overline{F})^{1/k}$, where $\mathcal{N}(Z(\overline{D})\,/\,\overline{F})$ is the normal closure of $Z(\overline{D})$ over $\overline{F}$ and k is an integer depending on which roots of unity lie in F and L.

Introduction.

This paper is a continuation of [$\mathbf{H}_2$], where we analyzed the corestriction $\operatorname{cor}_{L/F}(D)$ of a tame division algebra D over L when L/F is an inertial (unramified) extension of Henselian valued fields. We will follow terminology and notations in that paper. We will here concentrate on the cases when $\overline{L} = \overline{F}$, when L/F is a totally ramified of radical type (TRRT) extension (see below for definition) and when L/F is tame, where L/F is a finite separable extension of Henselian fields. We will consider only division algebras *finite-dimensional* over their centers.

Here is an overview of the paper: After a preliminary section, in section 2 we will analyze the underlying division algebra ${}^cD$ of the corestriction $\operatorname{cor}_{L/F}(D)$ of inertially split division algebras D over L when $\overline{L} = \overline{F}$. In sections 3 and 4, we will consider the corestriction of tame division algebras when L/F is TRRT and when L/F is tame, respectively.

The following definition of a TRRT extension was given in [**JW**, Sec. 4]. For a finite extension L of a valued field (F, v), we say that L is a *totally ramified* extension of F of *radical type* with respect to v ($TRRT$) if v extends to a valuation w on L such that L is totally ramified over F and there is a subgroup $\mathcal{A}$ of L^*/F^* which maps via w isomorphically onto Γ_L/Γ_F.

Our basic results are summarized in the following table. Here Γ_D is the value group of the valuation on D and $\overline{D}$ is the residue division ring of the valuation ring of D. Also $\mathcal{N}(Z(\overline{D})\,/\,\overline{F})$ denotes the normal closure of $Z(\overline{D})$ over $\overline{F}$, D^n is the underlying division algebra of the n-fold product

$D \otimes_L \cdots \otimes_L D$, and θ_D is the map of (1) below, so $\ker(\theta_D)$ is a subgroup of Γ_D/Γ_L.

	D inertially split	D tame
$\overline{L} = \overline{F}$ (Th. 8)	$\Gamma_{^cD} = [L\colon F]\cdot\Gamma_D + \Gamma_F$ $Z(\overline{D^n}) \subseteq Z(\overline{^cD}) \subseteq Z(\overline{D})$	
L/F TRRT (Th. 15)	"	$[L\colon F]\Gamma_D \subseteq \Gamma_{^cD} \subseteq \Gamma_D$ $Z(\overline{^cD}) \subseteq Z(\overline{D})$
L/F tame (Th. 17, 18)	$\Gamma_{^cD} \subseteq \lvert\Gamma_L\colon\Gamma_F\rvert\cdot\Gamma_D + \Gamma_F$ $Z(\overline{^cD}) \subseteq \mathcal{N}(Z(\overline{D}\,/\,\overline{F}))$	$\Gamma_{^cD} \subseteq \Gamma_D$ $Z(\overline{^cD}) \subseteq \mathcal{N}(Z(\overline{D})\,/\,\overline{F})^{1/k}$ $k \mid \exp(\ker\theta_D)$

The integer k in the table above depends not only on Γ_D/Γ_L and $[\overline{L}\colon\overline{F}]$ but also on which roots of unity lie in F. One of the interesting results of this investigation is to see how heavily the corestriction depends on the roots of unity in F and L.

1. Preliminaries.

Let (D, v) be a valued division algebra, that is, a division ring D with valuation v. Associated to v, we have its value group $\Gamma_D = v(D^*)$; the valuation ring $V_D = \{d \in D^* \mid v(d) \geq 0\} \cup \{0\}$; the unique maximal left (and right) ideal M_D of V_D, $M_D = \{d \in D^* \mid v(d) > 0\} \cup \{0\}$; the group of v-units of D^*, $U_D = V_D - M_D = V_D^*$; the residue division ring $\overline{D} = V_D/M_D$. If F is the center $Z(D)$ of D, there is a well-defined epimorphism

$$\theta_D\colon \Gamma_D/\Gamma_F \to \operatorname{Gal}(Z(\overline{D})\,/\,\overline{F}), \tag{1}$$

induced by $\alpha\colon D^* \to \operatorname{Gal}(Z(\overline{D})\,/\,\overline{F})$ which is given by $d \mapsto \overline{c_d}$ where $\overline{c_d}$ is the map induced by conjugation by d. (cf. [**JW**, 1.6]).

We recall two propositions which will be particularly useful for this paper.

Proposition 2 [**M**, Th. 1]. *Let D and E be division algebras over a field F with $[D\colon F] < \infty$. Suppose D has a valuation v and E has a valuation w with $v|_F = w|_F$. Suppose further*

(i) *D is defectless over F relative to v;*

(ii) $\overline{D} \otimes_{\overline{F}} \overline{E}$ *is a division ring;*
(iii) $\Gamma_D \cap \Gamma_E = \Gamma_F$.
Then $D \otimes_F E$ *is a division ring with a unique valuation* u *such that* $u|_D = v$, *and* $u|_E = w$. *Furthermore,* $\overline{D \otimes_F E} \cong \overline{D} \otimes_{\overline{F}} \overline{E}$ *and* $\Gamma_{D \otimes_F E} = \Gamma_D + \Gamma_E$.

Proposition 3 [**JW**, Lemma 6.2, Th. 6.3]. *If* D *is a tame division algebra over a Henselian field* F, *then there exist* $S \in \mathcal{D}_{is}(F)$ *and* $T \in \mathcal{D}_{ttr}(F)$ *such that* $D \sim S \otimes_F T$ *in* $\operatorname{Br}(F)$. (*Such* S *and* T *are not unique.*) *Furthermore, if* $D \sim S \otimes_F T$ *is such a decomposition,* $Z(\overline{D}) = \mathcal{F}(\theta_S((\Gamma_S \cap \Gamma_T) / \Gamma_F)) \subseteq Z(\overline{S})$, $\Gamma_D = \Gamma_S + \Gamma_T$ *and* $\ker(\theta_D) = \Gamma_T / \Gamma_F$.

2. The case when $\overline{L} = \overline{F}$.

In this section, we assume that $(L, v) \supseteq (F, v)$ is a finite separable extension of Henselian fields with $\overline{L} = \overline{F}$. Recall that for $D \in \mathcal{D}(L)$, ${}^cD \in \mathcal{D}(F)$ denotes the underlying division algebra of $\operatorname{cor}_{L/F}(D)$.

For any valued field (F, v), let $\operatorname{Br}(V_F)$ denote the Brauer group (of equivalence classes of Azumaya algebras) of the valuation ring V_F. There are canonical group homomorphisms $\alpha \colon \operatorname{Br}(V_F) \to \operatorname{Br}(F)$ given by $[A] \mapsto [A \otimes_{V_F} F]$, and $\beta \colon \operatorname{Br}(V_F) \to \operatorname{Br}(\overline{F})$ given by $[A] \mapsto [A / M_F A]$, where $[A]$ is the class of A, an Azumaya algebra over V_F. Then, by [**JW**, Prop. 2.5], α is injective.

Now assume that (F, v) is Henselian. Then define

$$I\operatorname{Br}(F) = \{[D] \in \operatorname{Br}(F) \mid D \in \mathcal{D}_i(F), \text{ i.e., } D \text{ is inertial over } F\}.$$

By [**JW**, Prop. 2.5 and Ex. 2.4 (ii)], $I\operatorname{Br}(F) = \operatorname{im}(\alpha)$, so $I\operatorname{Br}(F)$ is a subgroup of $\operatorname{Br}(F)$. Azumaya proved in [Az, Th. 31] that β is an isomorphism. The composite map $\beta \circ \alpha^{-1} \colon I\operatorname{Br}(F) \to \operatorname{Br}(\overline{F})$ is thus an isomorphism, and it maps $[D]$ to $[\overline{D}]$ for any $D \in \mathcal{D}_i(F)$.

Lemma 4. *If* $D \in \mathcal{D}_i(L)$ *then* ${}^cD \in \mathcal{D}_i(F)$ *and* $\overline{{}^cD} \sim \overline{D}^{\otimes[L:F]}$ *in* $\operatorname{Br}(\overline{F})$. (*Recall that we assume* $\overline{L} = \overline{F}$.)

Proof. Consider the following commutative diagram.

$$\begin{array}{ccccc} I\operatorname{Br}(F) & \xrightarrow[\cong]{\alpha^{-1}} & \operatorname{Br}(V_F) & \xrightarrow[\cong]{\beta} & \operatorname{Br}(\overline{F}) \\ \downarrow \otimes_F L & & \downarrow \otimes_{V_F} V_L & & \downarrow \otimes_{\overline{F}} \overline{L} \\ I\operatorname{Br}(L) & \xrightarrow[\cong]{\alpha^{-1}} & \operatorname{Br}(V_L) & \xrightarrow[\cong]{\beta} & \operatorname{Br}(\overline{L}) \end{array}$$

Since $\overline{L} = \overline{F}$ by assumption, the restriction map $\operatorname{res}_{\overline{L}/\overline{F}} \colon \operatorname{Br}(\overline{F}) \to \operatorname{Br}(\overline{L})$, given by $[\widetilde{D}] \mapsto [\widetilde{D} \otimes_{\overline{F}} \overline{L}]$ for any $\widetilde{D} \in \mathcal{D}(\overline{F})$, is the identity map on $\operatorname{Br}(\overline{F})$.

So the restriction map $\operatorname{res}_{L/F}: I\operatorname{Br}(F) \to I\operatorname{Br}(L)$, given by $[D] \mapsto [D \otimes_F L]$ for any $D \in \mathcal{D}_i(F)$, is an isomorphism. So for any $D \in \mathcal{D}_i(L)$, there is a $D_0 \in \mathcal{D}_i(F)$ such that $[D] = \operatorname{res}_{L/F}([D_0])$, i.e., $D \sim D_0 \otimes_F L$ in $\operatorname{Br}(L)$. Then by the above commutative diagram, $[\overline{D}] = \beta \circ \alpha^{-1}([D]) = \beta \circ \alpha^{-1}(\operatorname{res}_{L/F}([D_0])) = \operatorname{res}_{\overline{L}/\overline{F}}(\beta \circ \alpha^{-1}([D_0])) = [\overline{D_0}]$ in $\operatorname{Br}(\overline{L})(= \operatorname{Br}(\overline{F}))$. Also, by [**Ti**$_2$, Th. 2.5], ${}^cD \sim \operatorname{cor}_{L/F}(D) \sim \operatorname{cor}_{L/F}(D_0 \otimes_F L) \sim D_0^{\otimes[L:F]}$ in $\operatorname{Br}(F)$. Since $[D_0] \in I\operatorname{Br}(F)$ and $I\operatorname{Br}(F)$ is a subgroup of $\operatorname{Br}(F)$, $[{}^cD] = \left[D_0^{\otimes[L:F]}\right]$ is contained in $I\operatorname{Br}(F)$ and $\left[\overline{{}^cD}\right] = \left[\overline{D_0^{\otimes[L:F]}}\right] = \left[\overline{D}^{\otimes[L:F]}\right]$ in $\operatorname{Br}(\overline{F})$, as desired. □

In Theorem 7, we will give relations between D and ${}^cD$ for $D \in \mathcal{D}_{is}(L)$ when $\overline{L} = \overline{F}$. To prove that theorem, we need the following information about the homological corestriction which is of interest in itself.

Let G be a group and A a left G-module. We write A^G for $\{a \in A \mid g(a) = a, \text{ all } g \in G\}$. Let H be a subgroup of G of index $n < \infty$, and N a normal subgroup of G. We have a set of representatives $\mathcal{R} = \{\rho_1, \ldots, \rho_n\}$ of the left cosets of H in HN with $\rho_i \in N$. So, for $n \in N$ and any i there is a j with $n\rho_i = \rho_j h$, and $h \in H \cap N$. Thus, we have a map $\mathcal{N}: A^{H\cap N} \to A^N$ given by $\mathcal{N}(a) = \sum_{i=1}^{n} \rho_i(a)$. Observe that $\mathcal{N}$ is independent of the choice of coset representatives used for $\mathcal{R}$. Then $\mathcal{N}$ and the isomorphism from HN/N to $H/(H\cap N)$ induces the map $\mathcal{N}^*_{HN/H}: H^m(H/(H\cap N), A^{H\cap N}) \to H^m(G/N, A^N)$, $m \geq 0$ given by $(HN/N \overset{\cong}{\to} H/(H\cap N), A^{H\cap N} \overset{\mathcal{N}}{\to} A^N)$.

Theorem 5. *Let G, H, N, A and $\mathcal{N}^*_{HN/H}$ be as above. Suppose $f \in H^m(H/(H\cap N), A^{H\cap N})$, $m \geq 0$. Then*

$$\operatorname{cor}_H^G \circ \left(\inf_{H/(H\cap N)}^H (f)\right) = \inf_{G/N}^G \circ \operatorname{cor}_{HN/N}^{G/N} \circ \mathcal{N}^*_{HN/N}(f).$$

Proof. The theorem follows from the following formula for the special case when $G = HN$ since the corestriction is transitive and commutes with the inflation by [**We**, Prop. 2.4.5].

$$\operatorname{cor}_H^G \left(\inf_{H/(H\cap N)}^H (f)\right) = \inf_{G/N}^G \left(\mathcal{N}^*_{G/H}(f)\right) \tag{6}$$

So, it suffices to prove (6) with assumption that $G = HN$.

For $m = 0$, this is clear. So we may assume $m \geq 1$. For any $\sigma \in G$, there are uniquely determined elements $\rho_\sigma \in \mathcal{R}$ and $h_\sigma \in H$ such that $\sigma = \rho_\sigma h_\sigma$. Also given $\rho_i \in \mathcal{R}$ and $\sigma \in G$, let $\rho_{\sigma_*(i)} \in \mathcal{R}$ and $\delta(\sigma, \rho_i) \in H$ be the elements such that $\sigma\rho_i = \rho_{\sigma_*(i)}\delta(\sigma, \rho_i)$. Since

$$\rho_{\sigma_*(i)}\delta(\sigma, \rho_i) = \sigma\rho_i = \rho_\sigma h_\sigma \rho_i = [\rho_\sigma h_\sigma \rho_i h_\sigma^{-1}]h_\sigma$$

and $\{\rho_{\sigma_*(i)}, \rho_\sigma(h_\sigma \rho_i h_\sigma^{-1})\} \subseteq N$ and $\{\delta(\sigma, \rho_i), h_\sigma\} \subseteq H$, we have $\delta(\sigma, \rho_i) \equiv h_\sigma$ mod $H \cap N$. Also, as $\sigma = \rho_\sigma h_\sigma$ and $\rho_\sigma \in N$, $\sigma \equiv h_\sigma \mod N$. For $h \in H$ (resp. $g \in G$), let $\overline{h}$ (resp. $\tilde{g}$) denote the left coset $h(H \cap N)$ (resp. gN) in $H/(H \cap N)$ (resp. G/N). So, for any $\sigma \in G$ and $\rho_i \in \mathcal{R}$,

$$\text{(7)} \qquad \overline{\delta(\sigma, \rho_i)} = \overline{h_\sigma} \quad \text{and} \quad \tilde{\sigma} = \widetilde{h_\sigma}\,.$$

Let $f \in H^m(H/(H \cap N), A^{H \cap N})$ be represented by an inhomogeneous cocycle, say f again, in $Z^m(H/(H \cap N), A^{H \cap N})$. Then by [**H**$_2$, 1.3], for $\sigma_j \in G$, $1 \le j \le m$,

$$\begin{aligned}
\text{(1)}\quad & \operatorname{cor}_H^G(\inf{}_{H/(H\cap N)}^H(f))(\sigma_1, \dots, \sigma_j, \dots, \sigma_m) \\
&= \sum_{i=1}^n \rho_{(\sigma_1 \cdots \sigma_m)_*(i)} \Big[\inf{}_{H/(H\cap N)}^H(f)(\delta(\sigma_1, \rho_{(\sigma_2 \cdots \sigma_m)_*(i)}), \dots, \\
&\qquad\qquad \delta(\sigma_j, \rho_{(\sigma_{j+1} \cdots \sigma_m)_*(i)}), \dots, \delta(\sigma_m, \rho_i)) \Big] \\
&= \sum_{i=1}^n \rho_{(\sigma_1 \cdots \sigma_m)_*(i)} \left[f\left(\overline{h_{\sigma_1}}, \dots, \overline{h_{\sigma_j}}, \dots, \overline{h_{\sigma_m}}\right)\right] \quad \text{by (7)} \\
&= \sum_{i=1}^n \rho_i \left[f\left(\overline{h_{\sigma_1}}, \dots, \overline{h_{\sigma_j}}, \dots, \overline{h_{\sigma_m}}\right)\right]
\end{aligned}$$

as $(\sigma_1 \cdots \sigma_m)_* \in S_n$, the symmetric group, so we are just rearranging the order of summation. Then, as $\widetilde{\sigma_j} \in G/N$ maps to $\overline{h_{\sigma_j}} \in H/(H \cap N)$ in $G/N \cong H/(H \cap N)$,

$$\begin{aligned}
& \operatorname{cor}_H^G\left(\inf{}_{H/H\cap N}^H(f)\right)(\sigma_1, \dots, \sigma_j, \dots, \sigma_m) \\
&= \mathcal{N}_{G/H}^*(f)(\widetilde{\sigma_1}, \dots, \widetilde{\sigma_j}, \dots, \widetilde{\sigma_m}) \\
&= \inf{}_{G/N}^G(\mathcal{N}_{G/H}^*(f))(\sigma_1, \dots, \sigma_j, \dots, \sigma_m)\,.
\end{aligned}$$

□

Note that Th. 5 is valid for $f \in H_c^m(H/(H \cap N), A^{H \cap N})$, the m-th continuous cohomology group, if G is a profinite group, and H and N are also assumed closed in G, and A is a discrete G-module.

Recall we assume that $(L, v) \supseteq (F, v)$ is a separable extension of degree n of Henselian fields with $\overline{L} = \overline{F}$. Let L_{sep} (resp. F_{sep}) be the separable closure of L (resp. F). So $L_{sep} = F_{sep}$. Let $G = \operatorname{Gal}(F_{sep} / F)$ and $H = \operatorname{Gal}(L_{sep} / L)$. So H is a closed subgroup of G of index $n = [L \colon F]$. Let L_{nr} (resp. F_{nr}) be the maximal inertial extension of L (resp. F) in F_{sep}. Since F_{nr} / F is Galois and $L \cap F_{nr} = F$, L and F_{nr} are linearly disjoint over F and $L \otimes_F F_{nr}$ is the field $L \cdot F_{nr}$. Also by [**JW**, 1.9], $L \cdot F_{nr} = L_{nr}$.

Let $N = \mathrm{Gal}\,(F_{sep} \,/\, F_{nr})$. Then since $F_{nr} \,/\, F$ is Galois and $L \cap F_{nr} = F$, N is normal in G and $G = HN$. Also, $\mathrm{Gal}\,(F_{nr}/F) \cong \mathrm{Gal}\,(\overline{F}_{sep} \,/\, \overline{F}) \cong G/N$, and $\mathrm{Gal}\,(L_{nr}/L) \cong \mathrm{Gal}\,(\overline{L}_{sep}/\overline{L}) \cong H/(H \cap N)$ as $H \cap N = \mathrm{Gal}\,(L_{sep} \,/\, L_{nr})$ with $L_{nr} = L \cdot F_{nr}$. But since $\overline{L} = \overline{F}$, $\mathrm{Gal}\,(\overline{F}_{sep} \,/\, \overline{F}) = \mathrm{Gal}\,(\overline{L}_{sep} \,/\, \overline{L})$. So by identifying $\mathrm{Gal}\,(F_{nr}/F)$ and $\mathrm{Gal}\,(L_{nr}/L)$ with $\mathrm{Gal}\,(\overline{F}_{sep}/\overline{F}) = \mathrm{Gal}\,(\overline{L}_{sep}/\overline{L})$ via canonical isomorphisms, we can identify G/N with $H/(H \cap N)$.

Via the crossed product construction, we have the isomorphisms $\mathrm{Br}\,(L) \cong H^2_c(H, L^*_{sep})$, $\mathrm{Br}\,(F) \cong H^2_c(G, F^*_{sep})$, $\mathrm{Br}\,(L_{nr}/L) \cong H^2_c(H/H \cap N,\, L^*_{nr})$, and $\mathrm{Br}\,(F_{nr}/F) \cong H^2_c(G/N, F^*_{nr})$.

Theorem 8. *Let $(L, v) \supseteq (F, v)$ be a separable extension of degree n of Henselian fields with $\overline{L} = \overline{F}$. Suppose $D \in \mathcal{D}_{is}(L)$, and θ_D is the map of (1). Then, ${}^cD \in \mathcal{D}_{is}(F)$, $\Gamma_{{}^cD} = n\Gamma_D + \Gamma_F$, and $Z(\overline{{}^cD}) = \mathcal{F}(\theta_D(\widetilde{\Gamma}))$, where $\widetilde{\Gamma} = \{\alpha + \Gamma_L \in \Gamma_D/\Gamma_L \,|\, n\alpha \in \Gamma_F\}$. So $Z(\overline{D^n}) \subseteq Z(\overline{{}^cD}) \subseteq Z(\overline{D})$, where D^n is the underlying division algebra of $D^{\otimes n}$, the n-fold product $D \otimes_L \cdots \otimes_L D$. (So if $D \in \mathcal{D}_t(L)$, then ${}^cD \in \mathcal{D}_t(F)$.)*

Proof. Since $L \otimes_F F_{nr}$ is the field L_{nr}, by [**D**$_1$, p. 56, Ex. 1] ${}^cD \otimes_F F_{nr} \sim \mathrm{cor}_{L_{nr}/F_{nr}}(D \otimes_L L_{nr}) \sim \mathrm{cor}_{L_{nr}/F_{nr}}(L_{nr}) \sim F_{nr}$ in $\mathrm{Br}\,(F_{nr})$. So ${}^cD \in \mathcal{D}_{is}(F)$.

Since $[D] \in \mathrm{Br}\,(L_{nr}/L) \subseteq \mathrm{Br}\,(L)$, in $\mathrm{Br}\,(L)$ $[D]$ is represented by $\mathrm{inf}^H_{H/(H\cap N)}(f)$ for some $f \in H^2_c(H/(H \cap N),\, L^*_{nr})$. Since the algebraic corestriction corresponds to the homological corestriction, in $\mathrm{Br}\,(F)$, $[{}^cD]$ is represented by $\mathrm{cor}^G_H\left(\mathrm{inf}^H_{H/(H\cap N)}(f)\right)$. But, by Th. 5 above

$$\mathrm{cor}^G_H\left(\mathrm{inf}^H_{H/(H\cap N)}(f)\right) = \mathrm{inf}^G_{G/N}\left(\mathcal{N}^*_{G/H}(f)\right),$$

where $\mathcal{N}^*_{G/H} : H^2_c(H/(H \cap N),\, L^*_{nr}) \to H^2_c(G/N,\, F^*_{nr})$ is induced by the norm map from L^*_{nr} to F^*_{nr}. Since $[{}^cD] \in \mathrm{Br}\,(F_{nr}/F) \cong H^2_c(G/N,\, F^*_{nr})$, $[{}^cD]$ is represented by $\mathcal{N}^*_{G/H}(f)$.

Let $H' = H/(H \cap N)$ and $G' = G/N$. Since $H' = \mathrm{Gal}\,(L_{nr}/L)$ and $G' = \mathrm{Gal}\,(F_{nr}/F)$, we have homorphisms

$$\gamma\colon H^2_c(H', L^*_{nr}) \to \mathrm{Hom}_c\,(H', \Delta/\Gamma_L)$$

and

$$\gamma\colon H^2_c(G', F^*_{nr}) \to \mathrm{Hom}_c\,(G', \Delta/\Gamma_F),$$

which is helpful when we work with inertially split division algebras. (Δ is the divisible hull of Γ_F.)

Let $(\cdot n)\colon H^2_c(H', \Gamma_L) \to H^2_c(G', \Gamma_F)$ be the map induced by multiplication map $\cdot n$ from Γ_L to Γ_F given by $\alpha \mapsto n\alpha$. (Note that $n\Gamma_L \subseteq \Gamma_F$ as $|\Gamma_L : \Gamma_F|$

divides n.) Let (v) be the maps from $H_c^2(H', L_{nr}^*)$ (resp. $H_c^2(G', F_{nr}^*)$) to $H_c^2(H', \Gamma_L)$ (resp. $H_c^2(G', \Gamma_F)$) induced by valuation v. Since v is Henselian, $v|_{F_{nr}}$ has a unique extension to F_{alg}, the algebraic closure of F. So by the argument in the proof of the theorem in [**W**$_1$], $v(N_{L_{nr}/F_{nr}}(a)) = nv(a)$ for any $a \in L_{nr}^*$ where $n = [L : F] = [L_{nr} : F_{nr}]$. So we have the following commutative diagram:

$$\begin{array}{ccccccc}
H_c^2(H', L_{nr}^*) & & & \xrightarrow{\quad (v) \quad} & & & H_c^2(H', \Gamma_L) \\
 & \searrow^{N_{L_{nr}/F_{nr}}^*} & & & & \swarrow_{(\cdot n)} & \\
\mathcal{N}_{G/H}^* \big\downarrow & & H_c^2(H', F_{nr}^*) & \xrightarrow{(v)} & H_c^2(H', \Gamma_F) & & \big\downarrow (\cdot n) \\
 & \swarrow^{\cong} & & & & \searrow_{\cong} & \\
H_2^c(G', F_{nr}^*) & & & \xrightarrow{\quad (v) \quad} & & & H_c^2(G', \Gamma_F)
\end{array}$$

Since Δ is uniquely k-divisible for each integer $k \geq 1$, the connecting homomorphism $\delta : H_c^1(H', \Delta/\Gamma_L) \to H_c^2(H', \Gamma_L)$ is an isomorphism. So, from the diagram above we have the following commutative diagram:

$$\begin{array}{ccc}
H_c^2(H', L_{nr}^*) & \xrightarrow{\gamma} & \mathrm{Hom}_c\,(H', \Delta/\Gamma_L) \\
\mathcal{N}_{G/H}^* \downarrow & & \downarrow (\cdot n) \\
H_c^2(G', F_{nr}^*) & \xrightarrow{\gamma} & \mathrm{Hom}_c\,(G', \Delta/\Gamma_F)
\end{array} \tag{9}$$

We now identify $H' = \mathrm{Gal}\,(L_{nr}/L)$ and $G' = \mathrm{Gal}\,(F_{nr}/F)$ with

$$\mathrm{Gal}\,(\overline{L}_{sep}/\overline{L}) \quad (= \mathrm{Gal}\,(\overline{F}_{sep}/\overline{F})).$$

Also, we identify $H_c^2(H', L_{nr}^*)$ and $H_c^2(G', F_{nr}^*)$ with $\mathrm{Br}\,(L_{nr}/L)$ and $\mathrm{Br}\,(F_{nr}/F)$, respectively. Let $h_D = \gamma([D]) = \gamma(f)$, and $h_{^cD} = \gamma([{}^cD]) = \gamma(\mathcal{N}_{G/H}^*(f))$. Then by [**JW**, Th. 5.6], the fixed field $\mathcal{F}(\ker(h_D))$ of $\ker(h_D)$ is $Z(\overline{D})$. Let $\widetilde{h}_D \colon \mathrm{Gal}\,(Z(\overline{D}) \,/\, \overline{L}) \to \Gamma_D/\Gamma_L$ be the isomorphism induced by h_D (after identifying $H' \,/\, \ker(h_D)$ with $\mathrm{Gal}\,(Z(\overline{D}) \,/\, \overline{L})$). Then by [**JW**, Th. 5.6] again,

$$\Gamma_D/\Gamma_L = \mathrm{im}\,(h_D), \quad \ker(h_D) = \mathrm{Gal}\,(\overline{L}_{sep}/Z(\overline{D})),$$
$$\Gamma_{^cD}/\Gamma_F = \mathrm{im}\,(h_{^cD}), \quad \ker h_{^cD} = \mathrm{Gal}\,(\overline{F}_{sep}/Z(\overline{{}^cD})),$$

and $\widetilde{h}_D = \theta_D^{-1}$.

Now by the commutative diagram (9), $h_{^cD} = \gamma \circ \mathcal{N}_{G/H}^*(f) = (n\cdot) \circ \gamma(f) = (n\cdot)(h_D)$, so we have $\Gamma_{^cD}/\Gamma_F = \mathrm{im}\,(h_{^cD}) = (n\cdot)(\mathrm{im}\,(h_D)) = (n\cdot)(\Gamma_D/\Gamma_L) =$

$(n\Gamma_D + \Gamma_F)/\Gamma_F$. Hence $\Gamma_{{}^cD} = n\Gamma_D + \Gamma_F$, where $n = [L : F]$. Also, we have $h_{{}^cD}(\sigma) = n(h_D(\sigma)) + \Gamma_F$ for any $\sigma \in G' = H' = \mathrm{Gal}(\overline{F}_{sep}/\overline{F})$, and $\ker(h_D) \subseteq \ker(h_{{}^cD})$. So $h_{{}^cD}$ induces $\widetilde{h_{{}^cD}} : \mathrm{Gal}(Z(\overline{D})/\overline{F}) \to \Delta/\Gamma_F$ as $\mathrm{Gal}(Z(\overline{D})/\overline{F}) = G'/\ker(h_D)$. Also, $\widetilde{h_{{}^cD}}(\tau) = n\widetilde{h}_D(\tau) + \Gamma_F = n\theta_D^{-1}(\tau) + \Gamma_F$ for any $\tau \in \mathrm{Gal}(Z(\overline{D})/\overline{L}) = \mathrm{Gal}(Z(\overline{D})/\overline{F})$. So as $\theta_D : \Gamma_D/\Gamma_L \to \mathrm{Gal}(Z(\overline{D})/\overline{L})$ is an isomorphism by [**JW**, Lemma 5.1], $\ker(\widetilde{h_{{}^cD}}) = \theta_D(\widetilde{\Gamma})$, where $\widetilde{\Gamma} = \{\alpha + \Gamma_L \in \Gamma_D/\Gamma_L \mid n\alpha \in \Gamma_F\}$. Hence $Z(\overline{{}^cD}) = \mathcal{F}(\ker h_{{}^cD}) = \mathcal{F}(\ker(\widetilde{h_{{}^cD}})) = \mathcal{F}(\theta_D(\widetilde{\Gamma}))$.

Note that $\widetilde{\Gamma} \subseteq \widetilde{\Gamma}_1$, where $\widetilde{\Gamma}_1$ is the n-torsion subgroup of Γ_D/Γ_L. But, by [**JW**, Prop. 6.9], $Z(\overline{D^n}) = \mathcal{F}(\theta_D(\widetilde{\Gamma}_1))$ where D^n is the underlying division algebra of $D^{\otimes n}$. So we have $Z(\overline{D^n}) \subseteq Z(\overline{{}^cD})$. Also, as shown above, $\ker h_{{}^cD} \supseteq \ker h_D$, so $Z(\overline{{}^cD}) \subseteq Z(\overline{D})$. Therefore, $Z(\overline{D^n}) \subseteq Z(\overline{{}^cD}) \subseteq Z(\overline{D})$.

The last assertion of the theorem follows from the definition of tame division algebra and the fact that ${}^cD \in \mathcal{D}_{is}(F)$ for $D \in \mathcal{D}_{is}(L)$. □

3. The case when L/F is TRRT.

We begin this section by recalling the features of generalized crossed product algebras which will be needed. For further information on generalized crossed products (and proofs of the properties stated here), see [**Ti**$_1$, **J**], or [**KY**].

Let A be a central simple algebra over a field K, and suppose K is Galois over a subfield F; let $G = \mathrm{Gal}(K/F)$. A *generalized cocycle* of A with respect to G is a pair of functions (α, f) where $\alpha : G \to \mathrm{Aut}_F(A)$ and $f : G \times G \to A^*$, such that for all $\sigma, \tau, \rho \in G$,

(i) $\alpha(\sigma)|_K = \sigma$;

(ii) $\alpha(\sigma) \circ \alpha(\tau) = \mathrm{inn}(f(\sigma,\tau)) \circ \alpha(\sigma\tau)$, where $\mathrm{inn}(f(\sigma,\tau))$ denotes conjugation by $f(\sigma,\tau)$;

(iii) $f(\sigma,\tau)f(\sigma\tau,\rho) = [\alpha(\sigma)(f(\tau,\rho))]f(\sigma,\tau\rho)$.

(α, f) is said to be *normalized* if $\alpha(\mathrm{id}_K) = \mathrm{id}_A$ and $f(\mathrm{id}_K, \sigma) = f(\sigma, \mathrm{id}_K) = 1$ for all $\sigma \in G$. Given a normalized generalized cocycle (α, f) one forms the *generalized crossed product* $(A, G, (\alpha, f))$ as the free left A-module with base $\{x_\sigma \mid \sigma \in G\}$, which is made into a ring by the multiplication rule

$$(cx_\sigma)(dx_\tau) = [c\alpha(\sigma)(d)f(\sigma,\tau)]x_{\sigma\tau} \quad \text{for all} \quad c, d \in A, \ \sigma, \tau \in G.$$

$(A, G, (\alpha, f))$ is a central simple F-algebra. Observe that if S is any central simple F-algebra containing K, then one sees using the Skolem-Noether theorem that there is a normalized generalized crossed product $(A, G, (\alpha, f))$ isomorphic to S, where $A = C_S(K)$. We will need the product theorem for generalized crossed products (cf. [**Ti**$_1$, Th. 4.6] or [**J**, (1.15)] or [**KY**, Th. 3]). This says if $(A, G, (\alpha, f))$ and $(B, G, (\beta, g))$ are generalized crossed products of K over F, then with respect to the obvious induced generalized cocycle $(\alpha \otimes \beta, f \otimes g)$ of $A \otimes_K B$, we have in $\mathrm{Br}(F)$,

$$(10) \qquad (A, G, (\alpha, f)) \otimes_F (B, G, (\beta, g)) \sim (A \otimes_K B, G, (\alpha \otimes \beta, f \otimes g))\,.$$

Proposition 11. *For $i = 1, 2$, let $(T_i, v_i) \in \mathcal{D}_{ttr}(F)$ with $v_1|_F = v_2|_F$. ($v_1|_F$ doesn't need be Heselian.) Suppose there is an extension field K of F of degree n such that $K \subseteq T_i$, $\Gamma_{T_1} \cap \Gamma_{T_2} = \Gamma_K$, and K is Galois over F with Galois group G. So there are normalized generalized crossed products $(C_i, G, (\alpha_i, f_i))$ isomorphic to T_i. Then, $C_1 \otimes_K C_2$ is a division ring with a unique valuation v such that $v|_{C_1} = v_1|_{C_1}, v|_{C_2} = v_2|_{C_2}$, $\Gamma_{C_1 \otimes_K C_2} = \Gamma_{C_1} + \Gamma_{C_2}$ and $(C_1 \otimes_K C_2,\, v) \in \mathcal{D}_{ttr}(K)$. Also let $T = (C_1 \otimes_K C_2,\, G,\, (\alpha_1 \otimes \alpha_2, f_1 \otimes f_2))$. Then $T_1 \otimes_F T_2 \cong M_n(T)$ and T is a division ring with a valuation w such that $w|_{C_1 \otimes_K C_2} = v$ and $(T, w) \in \mathcal{D}_{ttr}(F)$. So $K \subseteq T$ and $\Gamma_{C_1} + \Gamma_{C_2} \subseteq \Gamma_T$.*

Proof. Since $T_i \in \mathcal{D}_{ttr}(F)$, the fundamental inequality (**S**, p. 21) gives $|\Gamma_{C_i} : \Gamma_K| = [C_i : K]$ and $\overline{C_1} = \overline{C_2} = \overline{K}$. Also $\Gamma_K \subseteq \Gamma_{C_1} \cap \Gamma_{C_2} \subseteq \Gamma_{T_1} \cap \Gamma_{T_2} = \Gamma_K$, so $\Gamma_{C_1} \cap \Gamma_{C_2} = \Gamma_K$. So by Prop. 4, $C_1 \otimes_K C_2$ is a division ring with a unique valuation v such that $v|_{C_1} = v_1|_{C_1}$ and $v|_{C_2} = v_2|_{C_2}$. Furthermore, $\Gamma_{C_1 \otimes_K C_2} = \Gamma_{C_1} + \Gamma_{C_2}$. So $|\Gamma_{C_1 \otimes_K C_2} : \Gamma_K| = [C_1 \otimes_K C_2 : K]$. Hence $(C_1 \otimes_K C_2, v) \in \mathcal{D}_{ttr}(K)$ as $\mathrm{char}\,(\overline{K}) \nmid [C_1 \otimes_K C_2 : K]$.

Let $T = (C_1 \otimes_K C_2, G, (\alpha_1 \otimes \alpha_2, f_1 \otimes f_2))$. Then by (10) in $\mathrm{Br}\,(F)$, $T_1 \otimes_F T_2 \sim T$. As $[T : F] = [T_1 \otimes_F T_2 : F]/n^2$ where $n = [K : F]$, $T_1 \otimes_F T_2 \cong M_n(T)$. Our next goal is to define a valuation on T.

We have $T_1 \cong (C_1, G, (\alpha_1, f_1)) = \bigoplus_{\sigma \in G} C_1 x_\sigma$, i.e. the free left C_1-module with base $\{x_\sigma \mid \sigma \in G\}$, which is made into a ring by the multiplication rule

$$(cx_\sigma)(dx_\tau) = [c\alpha_1(\sigma)(d) f_1(\sigma, \tau)] x_{\sigma\tau} \quad \text{for all} \quad c, d \in C_1, \sigma, \tau \in G\,,$$

where (α_1, f_1) is a normalized generalized cocycle of C_1 with respect to G. Likewise, $T_2 \cong (C_2, G, (\alpha_2, f_2)) = \bigoplus_{\sigma \in G} C_2 y_\sigma$, with multiplication rule

$$(cy_\sigma)(\alpha y_\tau) = [c\alpha_2(\sigma)(d) f_2(\sigma, \tau)] y_{\sigma\tau} \quad \text{for all} \quad c, d \in C_2,,\ \sigma, \tau \in G\,,$$

where (α_2, f_2) is a normalized generalized cocycle of C_2 with respect to G.

We now claim that $v_1(x_\sigma) \in \Gamma_{C_1}$ if and only if $\sigma = \mathrm{id}$. For, suppose $v_1(x_\sigma) = v_1(c)$ for some $c \in C_1$. Then by replacing x_σ by $c^{-1}x_\sigma$ we may assume that $v_1(x_\sigma) = 0$. Since $(T_1, v_1) \in \mathcal{D}_{ttr}(F)$, there is a canonical (bilinear) pairing $C_{T_1} : (\Gamma_{T_1}/\Gamma_F) \times (\Gamma_{T_1}/\Gamma_F) \to \mu_\ell(\overline{F})$ given by $(v_1(d) + \Gamma_F, v_1(e) + \Gamma_F) \mapsto \overline{ded^{-1}e^{-1}}$ where $\ell = \exp(\Gamma_{T_1}/\Gamma_F)$ and $\mu_\ell(\overline{F})$ is the set of ℓ distinct ℓ-th roots of unity in $\overline{F}$. (cf. [**TW**, Sec. 3]). As $v_1(x_\sigma) = 0 \in \Gamma_F$ and $x_\sigma k x_\sigma^{-1} = \sigma(k)$ for all $k \in K$, $\bar{1} = C_{T_1}(v_1(k) + \Gamma_F,\ v_1(x_\sigma) + \Gamma_F) =$

$\overline{kx_\sigma k^{-1}x_\sigma^{-1}} = \overline{k/\sigma(k)}$, so that $\overline{k/\sigma(k)} = \bar{1}$ for all $k \in K^*$. But since with respect to $v_1|_K$, K is tame and totally ramified over F and K is Galois over F with Galois group G, by [**TW**, Prop. 1.4] or [**E**, (20.11), pp. 161-162], there is a well-defined nondegenerate bilinear pairing

$$\gamma\colon (\Gamma_K/\Gamma_F) \times G \to \mu(\overline{F}) \quad \text{given by} \quad \gamma(v(k) + \Gamma_F, \tau) = \overline{k/\tau(k)} \in \overline{F}$$

for all $k \in K^*$, $\tau \in G$, where $\mu(\overline{F})$ is the set of roots of unity in $\overline{F}$. Thus $\sigma = \mathrm{id}$ as claimed. Likewise, $v_2(y_\sigma) \in \Gamma_{C_2}$ if and only if $\sigma = \mathrm{id}$.

It follows from the claim above that $v_1(x_\sigma)$ (resp. $v_2(y_\sigma)$), $\sigma \in G$, are distinct modulo Γ_{C_1} (resp. Γ_{C_2}). For, if $v_1(x_\sigma) - v_1(x_\tau) \in \Gamma_{C_1}$, then as $x_{\sigma\tau^{-1}}x_\tau = f(\sigma\tau^{-1}, \tau)x_\sigma$, we have $v(x_{\sigma\tau^{-1}}) = v(f(\sigma\tau^{-1}, \tau)) + [v(x_\sigma) - v(x_\tau)] \in \Gamma_{C_1}$. So, $\sigma\tau^{-1} = \mathrm{id}$ by the claim above, i.e., $\sigma = \tau$.

We will now define a valuation w on $T = (C_1 \otimes_K C_2, G, (\alpha_1 \otimes \alpha_2, f_1 \otimes f_2)) = \bigoplus_{\sigma\in G}(C_1 \otimes_K C_2)z_\sigma$, such that $w|_{C_1\otimes_K C_2} = v$ and $(T, w) \in \mathcal{D}_{ttr}(F)$. Define $w(z_\sigma) = v_1(x_\sigma) + v_2(y_\sigma)$ and

$$w\left(\sum_{\sigma\in G} c_\sigma z_\sigma\right) = \min_{\sigma\in G}\ (v(c_\sigma) + w(z_\sigma))$$

where $c_\sigma \in C_1 \otimes_K C_2$ for all $\sigma \in G$. If $w(z_\sigma) - w(z_\tau) \in \Gamma_{C_1\otimes_K C_2} = \Gamma_{C_1} + \Gamma_{C_2}$, say $w(z_\sigma) - w(z_\tau) = \gamma_1 + \gamma_2$ for some $\gamma_i \in \Gamma_{C_i}$, then $v_1(x_\sigma) - v_1(x_\tau) - \gamma_1 = v_2(y_\tau) - v_2(y_\sigma) + \gamma_2 \in \Gamma_{T_1} \cap \Gamma_{T_2} = \Gamma_K$. So $v_1(x_\sigma) - v_1(x_\tau) \in \Gamma_{C_1}$, showing $\sigma = \tau$ as above. So the $w(z_\sigma)$, $\sigma \in G$, are distinct modulo $\Gamma_{C_1\otimes_K C_2}$. Hence every element $\sum_{\sigma\in G} c_\sigma z_\sigma$ in T, with $c_\sigma \in C_1 \otimes_K C_2$, has a unique summand $c_\tau z_\tau$ with minimum value, which is called the *leading term* of $\sum_{\sigma\in G} c_\sigma z_\sigma$. We will show that w actually defines a valuation on T so that $(T, w) \in \mathcal{D}_{ttr}(F)$.

For $d = \sum_{\sigma\in G} d_\sigma z_\sigma \neq 0$ and $e = \sum_{\sigma\in G} e_\sigma z_\sigma \neq 0$ in T with $d \neq -e$, it is easy to see that $w(d + e) \geq \min(w(d), w(e))$; hence $w(d + e) = \min(w(d), w(e))$ if $w(d) \neq w(e)$. For the behavior of w under products, consider first $(d_\sigma z_\sigma) \cdot (e_\tau z_\tau)$ with d_σ, $e_\tau \neq 0 \in C_1 \otimes_K C_2$. Note that v is the unique extension of $v_1|_F = v_2|_F$ to $C_1 \otimes_K C_2$; so $v(z_\sigma e_\tau z_\sigma^{-1}) = v(e_\tau)$. Thus,

$$\begin{aligned}
w((d_\sigma z_\sigma) \cdot (e_\tau z_\tau)) &= w([d_\sigma(\alpha_1 \otimes \alpha_2)(\sigma)(e_\tau)(f_1 \otimes f_2)(\sigma, \tau)]z_{\sigma\tau}) \\
&= v(d_\sigma \cdot (z_\sigma e_\tau z_\sigma^{-1}) \cdot f_1(\sigma, \tau) \cdot f_2(\sigma, \tau)) + w(z_{\sigma\tau}) \\
&= v(d_\sigma) + v(e_\tau) + v_1(f_1(\sigma, \tau)) + v_2(f_2(\sigma, \tau)) \\
&\qquad + v_1(x_{\sigma\tau}) + v_2(y_{\sigma\tau}) \\
&= v(d_\sigma) + v(e_\tau) + v_1(x_\sigma x_\tau) + v_2(y_\sigma y_\tau) \\
&= [v(d_\sigma) + v_1(x_\sigma) + v_2(y_\sigma)] + [v(e_\tau) + v_1(x_\tau) + v_2(y_\tau)] \\
&= w(d_\sigma z_\sigma) + w(e_\tau z_\tau).
\end{aligned}$$

Hence, if $de \neq 0$,

$$\begin{aligned} w(de) &= w\left(\left(\sum_{\sigma\in G} d_\sigma z_\sigma\right)\cdot\left(\sum_{\tau\in G} e_\tau z_\tau\right)\right) \\ &\geq \min\{w((d_\sigma z_\sigma)(e_\tau z_\tau)) \mid d_\sigma \neq 0\,,\ e_\tau \neq 0\} \\ &= \min\{w(d_\sigma z_\sigma) + w(e_\tau z_\tau) \mid d_\sigma \neq 0\,,\ e_\tau \neq 0\} = w(d) + w(e)\,, \end{aligned}$$

i.e. $w(de) \geq w(d)+w(e)$. Now, let $d_{\sigma_0} z_{\sigma_0}$ be the leading term of d. Then $d = d_{\sigma_0} z_{\sigma_0} + d'$ where $d' = 0$ or $w(d') > w(d_{\sigma_0} z_{\sigma_0})$. Likewise, write $e = e_{\tau_0} z_{\tau_0} + e'$ where $e_{\tau_0} z_{\tau_0}$ is the leading term of e. Then $de = (d_{\sigma_0} z_{\sigma_0})(e_{\tau_0} z_{\tau_0}) + \rho$ where $\rho = d'(e_{\tau_0} z_{\tau_0}) + (d_{\sigma_0} z_{\sigma_0})e' + d'e'$. Now, if $\rho \neq 0$, $w(\rho) > w((d_{\sigma_0} z_{\sigma_0})(e_{\tau_0} z_{\tau_0}))$ by what has been already proved. Hence, $\rho \neq -(d_{\sigma_0} z_{\sigma_0})(e_{\tau_0} z_{\tau_0})$. Therefore, $de \neq 0$ and $w(de) = w((d_{\sigma_0} z_{\sigma_0})(e_{\tau_0} z_{\tau_0})) = w(d) + w(e)$. This shows that T has no zero divisors so T must be a division ring, since $\dim_F T < \infty$. The formula also shows w is a valuation on T. Clearly, $\Gamma_T = \{w(z_\sigma) \mid \sigma \in G\} + \Gamma_{C_1 \otimes_K C_2}$. As the $w(z_\sigma)$, $\sigma \in G$, are distinct modulo $\Gamma_{C_1 \otimes_K C_2}$ and $|G| = [K : F] = n$, we have $|\Gamma_T : \Gamma_F| = n \cdot |\Gamma_{C_1 \otimes_K C_2} : \Gamma_F| = n \cdot [C_1 \otimes_K C_2 : F] = [T : F]$. So $(T, w) \in \mathcal{D}_{ttr}(F)$. □

Next, when $(L, v) \supseteq (F, v)$ is a finite separable, TRRT extension of Henselian fields, we will give relations between $T \in \mathcal{D}_{ttr}(L)$ (resp. $D \in \mathcal{D}_t(L)$) and ${}^cT$ (resp. ${}^cD$) in Theorems 12 and 13. To prove these theorems, we need the following proposition.

Proposition 12. *Let (F, v) be a Henselian field and let p be a prime. Let $L = F(\gamma)$ with $\gamma^p \in F^*$ and $v(\gamma) \notin p\Gamma_F$. Let α, $\beta \in L^*$.*

(1) *Every symbol algebra $(\alpha, \beta; L)_n$ with $\operatorname{char}(\overline{L}) \nmid n$ is isomorphic to a symbol algebra of the form $(a, b\gamma^j; L)_n$ for some $a, b \in F^*$ and $0 \leq j \leq p-1$.*

(2) *If T is a symbol algebra in $\mathcal{D}_{ttr}(L)$, then ${}^cT \in \mathcal{D}_{ttr}(F)$, $p\Gamma_T \subseteq \Gamma_{{}^cT} \subseteq \Gamma_T$, and $\exp(\Gamma_{{}^cT} / \Gamma_F) \mid \exp(\Gamma_T/\Gamma_L)$.*

Proof. Since $\gamma^p \in F^*$ and $v(\gamma) \notin p\Gamma_F$, $v(\gamma) + \Gamma_F$ has order p in $\frac{1}{p}\Gamma_F/\Gamma_F$, so L/F is totally ramified of degree p with $\Gamma_L = \langle v(\gamma)\rangle + \Gamma_F$. Hence L is a TRRT extension of F.

(1) Since $\{1, \gamma, \gamma^2, \ldots, \gamma^{p-1}\}$ is an F-basis of L, $\alpha = a_0 + a_1\gamma + \cdots + a_{p-1}\gamma^{p-1}$, and $\beta = b_0 + b_1\gamma + \cdots + b_{p-1}\gamma^{p-1}$ with all a_k, $b_k \in F$. Since $v(\gamma^k)$, $0 \leq k \leq p-1$, are distinct modulo Γ_F, $\alpha = \sum_{k=0}^{p-1} a_k\gamma^k$ $\left(\text{resp. } \beta = \sum_{k=0}^{p-1} b_k\gamma^k\right)$ has a unique summand, say $a_i\gamma^i$ (resp. $b_j\gamma^j$) with minimal value. Then $\alpha = a_i\gamma^i(1 + \alpha')$ and $\beta = b_j\gamma^j(1 + \beta')$ with $v(\alpha') > 0$ and $v(\beta') > 0$. Since (L, v) is Henselian and $\operatorname{char}(\overline{L}) \nmid n$, $1 + \alpha' = \alpha_0^n$ and $1 + \beta' = \beta_0^n$ for some α_0, $\beta_0 \in L^*$. So $(\alpha, \beta; L)_n \cong (a_i\gamma^i, b_j\gamma^j; L)_n$ for some $a_i, b_j \in F^*$ and i, j, $0 \leq i, j \leq p-1$.

Now recall that $(\alpha_1,\beta_1;L)_n \cong (\beta_1^{-1},\alpha_1;L)_n$ by [$\mathbf{D}_1$, p. 80, Lemma 5]. So if $j=0$, $(\alpha,\beta;L)_n \cong (a_i\gamma^i,b_0;L)_n \cong (b_0^{-1},a_i\gamma^i;L)_n$, as desired. Hence we may assume that $i>0$ and $j>0$. We argue by induction on $i+j$ that $(a_i\gamma^i,b_j\gamma^j;L)_n \cong (a,b\gamma^{j'};L)_n$ for some a,b,j'. First, let's assume $i\le j$. Then since $(a_i\gamma^i,-a_i\gamma^i,L)_n \sim L$ by [$\mathbf{D}_1$, p. 82, Cor. 5], $(a_i\gamma^i,b_j\gamma^j;L)_n \cong (a_i\gamma^i,-a_i^{-1}b_j\gamma^{j-i};L)_n$. As $i+(j-i)<i+j$, we have done by induction. We argue similarly when $i>j$. Note that this proof of (1) does not need the assumption that $[L\colon F]$ is prime.

(2) If T is a symbol division algebra of degree n in $\mathcal{D}_{ttr}(L)$, by (1) above $T\cong(a,b\gamma^j;L)_n$ for some $a,b\in F^*$ and j, $0\le j\le p-1$. (So $\mu_n\subseteq L$.) Since T is tame and totally ramified over L, by [$\mathbf{H}_2$, Prop.3.1] in $\Gamma_L\,/\,n\Gamma_L$, $|\langle v(a)+n\Gamma_L, v(b\gamma^j)+n\Gamma_L\rangle| = n^2$ and $\Gamma_T=\langle \frac{1}{n}v(a), \frac{1}{n}v(b\gamma^j)\rangle+\Gamma_L = \langle \frac{1}{n}v(a), \frac{1}{n}v(b\gamma^j)\rangle+\langle v(\gamma)\rangle+\Gamma_F$. Let $K=L(\sqrt[n]{a})$. Then K is tame and totally ramified over L with $\Gamma_K/\Gamma_L=\langle\frac{1}{n}v(a)+\Gamma_L\rangle$ and $\exp(\Gamma_K/\Gamma_L)=n$ as $|(\langle\frac{1}{n}v(a)\rangle+\Gamma_L)\colon\Gamma_L|=n$. Also, since $\mu_n\subseteq L$, by [$\mathbf{TW}$, Prop. 1.4 (iii)] or [$\mathbf{S}$, p. 64, Th. 3], $\mu_n\subseteq\overline{L}=\overline{F}$, hence $\mu_n\subseteq F$ as (F,v) is Henselian.

(i) If $j=0$, then $T\cong(a,b;L)_n$ for some $a,b\in F^*$ and

$$\Gamma_T=\left\langle \frac{1}{n}v(a),\ \frac{1}{n}v(b)\right\rangle+\langle v(\gamma)\rangle+\Gamma_F.$$

Since $|\Gamma_T\colon\Gamma_F|=n^2p$, we must have $\Gamma_T/\Gamma_F=\langle\frac{1}{n}v(a)+\Gamma_F\rangle\times\langle\frac{1}{n}v(b)+\Gamma_F\rangle\times\langle v(\gamma)+\Gamma_F\rangle\cong\mathbb{Z}_n\times\mathbb{Z}_n\times\mathbb{Z}_p$. Since $\mu_n\subseteq F$, by [$\mathbf{Ti}_2$, Th. 3.1] (Projection formula), as $N_{L/F}(b)=b^p$, ${}^cT\sim(a,b^p;F)_n$ in $\mathrm{Br}\,(F)$.

If $p\nmid n$, then in $\frac{1}{n}\Gamma_F\,/\,\Gamma_F$, $|\langle\frac{1}{n}v(a)+\Gamma_F,\ \frac{1}{n}v(b^p)+\Gamma_F\rangle| = |\langle\frac{1}{n}v(a)+\Gamma_F\ ,\ \frac{1}{n}v(b)+\Gamma_F\rangle|=n^2$. So by [$\mathbf{H}_2$, Prop. 3.1], ${}^cT\in\mathcal{D}_{ttr}(F)$ with

$$\Gamma_{{}^cT}=\left\langle\frac{1}{n}v(a),\ \frac{1}{n}v(b^p)\right\rangle+\Gamma_F,$$

whence $p\Gamma_T\subseteq\Gamma_{{}^cT}\subseteq\Gamma_T$. If $p|n$, then ${}^cT\sim(a,b;F)_{n/p}$ and in $\frac{1}{(n/p)}\Gamma_F/\Gamma_F$,

$$\left|\left\langle\frac{1}{(n/p)},v(a)+\Gamma_F,\frac{1}{(n/p)}v(b)+\Gamma_F\right\rangle\right|=(n/p)^2.$$

So by [$\mathbf{H}_2$, Prop. 3.1], ${}^cT\cong(a,b;F)_{n/p}\in\mathcal{D}_{ttr}(F)$ with $\Gamma_{{}^cT}=\langle\frac{p}{n}v(a),\frac{p}{n}v(b)\rangle+\Gamma_F$ whence $p\Gamma_T\subseteq\Gamma_{{}^cT}\subseteq\Gamma_T$. In either case,

$$\exp(\Gamma_{{}^cT}/\Gamma_F)\,\big|\,\exp(\Gamma_T/\Gamma_L).$$

(ii) If $0<j\le p-1$, then $T\cong(a,\beta;)_n$ where $a\in F^*$ and $\beta=b\gamma^j$ with $b\in F^*$ and $1\le j\le p-1$. Since $v(\beta)=v(b)+jv(\gamma)\in\langle\frac{1}{n}v(\beta)\rangle+\Gamma_F$ and $jk\equiv 1 \mod p$ for some integer k, and $\gamma^p\in F^*$, $v(\gamma)\equiv jkv(\gamma)\equiv kv(\beta)$

mod Γ_F so that $v(\gamma) \in \langle \frac{1}{n}v(\beta)\rangle + \Gamma_F$. Hence $\Gamma_T = \langle \frac{1}{n}v(a),\ \frac{1}{n}v(\beta)\rangle + \langle v(\gamma)\rangle + \Gamma_F = \langle \frac{1}{n}v(a),\ \frac{1}{n}v(\beta)\rangle + \Gamma_F$.

Since $\mu_n \subseteq F$, by [**Ti**$_2$, Th. 3.1] (Projection formula),

$$^cT \sim (a, N_{L/F}(\beta); F)_n$$

in Br (F). Note that $v(N_{L/F}(\beta)) = [L\colon F]v(\beta) = pv(\beta)$ by the argument in the proof of the theorem in [**W**$_1$]. So

$$\left|\Gamma_T\colon \left(\left\langle \frac{1}{n}v(a),\ \frac{1}{n}v(N_{L/F}(\beta))\right\rangle + \Gamma_F\right)\right| \leq p.$$

As $|\Gamma_T/\Gamma_F| = |\langle \frac{1}{n}v(a) + \Gamma_F,\ \frac{1}{n}v(\beta) + \Gamma_F\rangle| = n^2p$,

$$\left|\left\langle \frac{1}{n}v(a) + \Gamma_F,\ \frac{1}{n}v(N_{L/F}(\beta)) + \Gamma_F\right\rangle\right| = n^2.$$

So by [**H**$_2$, Prop. 3.1],

$$^cT \cong (a, N_{L/F}(\beta); F)_n \in \mathcal{D}_{ttr}(F)$$

with $\Gamma_{^cT} = \langle \frac{1}{n}v(a),\ \frac{p}{n}v(\beta)\rangle + \Gamma_F$. Hence $p\Gamma_T \subseteq \Gamma_{^cT} \subseteq \Gamma_T$, and

$$\exp\left(\Gamma_{^cT} \,/\, \Gamma_F\right) \mid \exp\left(\Gamma_T/\Gamma_L\right).$$

□

Theorem 13. *Let* $(L, v) \supseteq (F, v)$ *be a finite separable TRRT extension of Henselian fields. If* $T \in \mathcal{D}_{ttr}(L)$*, then* $^cT \in \mathcal{D}_{ttr}(F)$*,* $[L\colon F]\cdot\Gamma_T \subseteq \Gamma_{^cT} \subseteq \Gamma_T$*, and* $\exp(\Gamma_{^cT} \,/\, \Gamma_F) \mid \exp(\Gamma_T/\Gamma_L)$.

Proof. If $[L\colon F]$ is not prime, then by [**JW**, Remark 4.2] there is a field F_1 such that $F \subsetneqq F_1 \subsetneqq L$ with $[F_1\ F] = p$, prime, and L/F_1, F_1/F TRRT. Assume that we can prove the theorem for F_1/F of prime degree. Since $[L : F_1] < [L : F]$, we can also assume that the theorem holds for L/F_1 by induction on $[L : F]$. Then the theorem is proved for L/F: Let T_1 be the underlying division algebra of $\mathrm{cor}_{L/F_1}(T)$. Then by the assumption for L/F_1, $T_1 \in \mathcal{D}_{ttr}(F_1)$, $[L : F_1]\cdot\Gamma_T \subseteq \Gamma_{T_1} \subseteq \Gamma_T$, and exp $(\Gamma_{T_1} \,/\, \Gamma_{F_1}) \mid \exp(\Gamma_T/\Gamma_L)$. Also since $^cT \sim \mathrm{cor}_{L/F}(T) \sim \mathrm{cor}_{F_1/F}(T_1)$, by the assumption for F_1/F, $^cT \in \mathcal{D}_{ttr}(F)$ and $[F_1 : F]\cdot\Gamma_{T_1} \subseteq \Gamma_{^cT} \subseteq \Gamma_{T_1}$, and $\exp(\Gamma_{^cT} \,/\, \Gamma_F) \mid$ exp $(\Gamma_{T_1} \,/\, \Gamma_{F_1})$. So $[L\colon F]\cdot\Gamma_T = [F_1 : F][L\colon F_1]\Gamma_T \subseteq [F_1 : F]\Gamma_{T_1} \subseteq \Gamma_{^cT} \subseteq \Gamma_{T_1} \subseteq \Gamma_T$, hence $[L\colon F]\Gamma_T \subseteq \Gamma_{^cT} \subseteq \Gamma_T$. Also exp $(\Gamma_{^cT} \,/\, \Gamma_F) \mid \exp(\Gamma_T/\Gamma_L)$.

So we may assume that $[L\colon F] = p$, prime, so that $L = F(\gamma)$ with $\gamma^p \in F^*$ and $v(\gamma) \notin p\Gamma_F$. Also by the primary decomposition we may assume that

$[T\colon F] = q^r$, a power of a prime number q. By [$\mathbf{D}_2$, Th. 1] (Draxl's decomposition theorem for tame and totally ramified division algebras) $T \cong \bigotimes_{i=1}^{k} T_i$ with each T_i a symbol division algebra in $\mathcal{D}_{ttr}(L)$, and $\Gamma_T/\Gamma_L = \bigoplus_{i=1}^{k} (\Gamma_{T_i} / \Gamma_L)$. (So $(\Gamma_{T_1} + \cdots + \Gamma_{T_{j-1}}) \cap \Gamma_{T_j} = \Gamma_L$ for $2 \le j \le k$.) By Prop. 12 $T_i \cong (a_i, b_i\gamma^{e_i}; L)_{n_i}$ for some a_i, $b_i \in F^*$, $0 \le e_i \le p-1$, and n_i, a power of q, and $p\Gamma_{T_i} \subseteq \Gamma_{{}^cT_i} \subseteq \Gamma_{T_i}$. So for $2 \le j \le k$, $(\Gamma_{{}^cT_1} + \cdots + \Gamma_{{}^cT_{j-1}}) \cap \Gamma_{{}^cT_j} \subseteq (\Gamma_{T_1} + \cdots + \Gamma_{T_{j-1}}) \cap \Gamma_{T_j} = \Gamma_L$. As $|\Gamma_L : \Gamma_F| = p$, $\left|\left(\sum_{i=1}^{j-1} \Gamma_{{}^cT_i}\right) \cap \Gamma_{{}^cT_j} : \Gamma_F\right|$ divides p. Also since the (Schur) index of ${}^cT_j$ (i.e. $\sqrt{[{}^cT_j\colon F]}$) is a power of q, $|\Gamma_{{}^cT_j} : \Gamma_F|$ is a power of q, so $\left|\left(\sum_{i=1}^{j-1} \Gamma_{{}^cT_i}\right) \cap \Gamma_{{}^cT_j} : \Gamma_F\right|$ is a power of q.

If $q \neq p$, then for $2 \le j \le k$, $\left(\sum_{i=1}^{j-1} \Gamma_{{}^cT_i}\right) \cap \Gamma_{{}^cT_j} = \Gamma_F$ as the index divides both p and a power of q. Also by Prop. 12 ${}^cT_i \in \mathcal{D}_{ttr}(F)$ for $1 \le i \le k$. So by applying Prop. 2 repeatedly, we have ${}^cT \cong \bigotimes_{i=1}^{k} {}^cT_i \in \mathcal{D}_{ttr}(F)$ and $\Gamma_{{}^cT} = \sum_{i=1}^{k} \Gamma_{{}^cT_i}$. So $p\Gamma_T = p\left(\sum_{i=1}^{k} \Gamma_{T_i}\right) \subseteq \sum_{i=1}^{k} \Gamma_{{}^cT_i} = \Gamma_{{}^cT} = \sum_{i=1}^{k} \Gamma_{{}^cT_i} \subseteq \sum_{i=1}^{k} \Gamma_{T_i} = \Gamma_T$, as desired. So we may assume $q = p$, so that $p \,|\, n_i = \operatorname{ind}(T_i)$ for $1 \le i \le k$.

Now we will prove the theorem by induction on k. If $k = 1$, the assertion is Prop. 12 (2).

If one of T_i, say T_{i_0}, is isomorphic to $(a, b; L)_n$ for some $a, b \in F^*$, then by reindexing we may assume $T_k \cong (a, b; L)_n$. Then $\Gamma_{T_k} / \Gamma_F = \langle \frac{1}{n}v(a) + \Gamma_F\rangle \oplus \langle \frac{1}{n}v(b) + \Gamma_F\rangle \oplus \langle v(\gamma) + \Gamma_F\rangle \cong \mathbb{Z}_n \oplus \mathbb{Z}_n \oplus \mathbb{Z}_p$. As ${}^cT_k = (a, b; F)_{n/p}$ with $p \,|\, n$, and $\Gamma_L = \langle v(\gamma)\rangle + \Gamma_F$, we have $\Gamma_{{}^cT_k} \cap \Gamma_L = \Gamma_F$. This means that $(\Gamma_{{}^cT_1} + \cdots + \Gamma_{{}^cT_{k-1}}) \cap \Gamma_{{}^cT_k} = \Gamma_F$ since $(\Gamma_{{}^cT_1} + \cdots + \Gamma_{{}^cT_{k-1}}) \cap \Gamma_{{}^cT_k} \subseteq \Gamma_L$. Let $T_0 = \bigotimes_{i=1}^{k-1} T_i$. Then we have ${}^cT_0 \in \mathcal{D}_{ttr}(F)$ and $[L : F]\Gamma_{T_0} \subseteq \Gamma_{{}^cT_0} \subseteq \Gamma_{T_0}$ by the inductive assumption on k. As ${}^cT_i \in \mathcal{D}_t(F)$, $\Gamma_{{}^cT_0} \subseteq \Gamma_{{}^cT_1} + \cdots + \Gamma_{{}^cT_{k-1}}$ by [**JW**, Cor. 6.7], so $\Gamma_{{}^cT_0} \cap \Gamma_{{}^cT_k} = \Gamma_F$. Hence by Prop. 2, ${}^cT \cong {}^cT_0 \otimes_F {}^cT_k \in \mathcal{D}(F)$ and $\Gamma_{{}^cT} = \Gamma_{{}^cT_0} + \Gamma_{{}^cT_k}$. So ${}^cT \in \mathcal{D}_{ttr}(F)$ and $p\Gamma_T \subseteq \Gamma_{{}^cT} \subseteq \Gamma_T$, as claimed. Therefore we may assume $T = \bigotimes_{i=1}^{k} T_i$ where $T_i \cong (a_i, b_i\gamma^{e_i}; L)_{n_i}$, $0 < e_i \le p-1$, and a_i, $b_i \in F^*$. As $T_i \in \mathcal{D}_{ttr}(L)$, $\Gamma_{T_i} = \left\langle \frac{1}{n_i}v(a_i), \frac{1}{n_i}v(b_i) + \frac{e_i}{n_i}v(\gamma)\right\rangle + \langle v(\gamma)\rangle + \Gamma_F$. Then ${}^cT \sim \bigotimes_{i=1}^{k} {}^cT_i$ in $\operatorname{Br}(F)$. Also, ${}^cT_i \cong (a_i, b_i^p N_{L/F}(\gamma)^{e_i}; F)_{n_i} \in \mathcal{D}_{ttr}(F)$ and $p\Gamma_{T_i} \subseteq \Gamma_{{}^cT_i} \subseteq \Gamma_{T_i}$ as shown in the last paragraph of the proof of Prop. 12 (2).

Let $K = F\left(\sqrt[p]{N_{L/F}(\gamma)^{e_i}}\right) = F\left(\sqrt[p]{N_{L/F}(\gamma)}\right)$. Note in passing that $K = L$

unless $p = 2$ and $\mu_4 \not\subseteq F$. Since each ${}^cT_i$ contains

$$F\left(\left(b_i^p N_{L/F}(\gamma)^{e_i}\right)^{1/n_i}\right)$$

as a maximal subfield, and n_i is a power of p, each ${}^cT_i$ contains K. Also $\Gamma_K = \Gamma_L$. Since $\mu_{n_i} \subseteq F$ as shown in the first paragraph of proof of Prop. 12 (2), K is Galois over F. Let $G = \operatorname{Gal}(K/F)$, and let C_i be the centralizer of K in ${}^cT_i$. Then ${}^cT_i \cong (C_i, G, (\alpha_i, f_i))$, a generalized crossed product of C_i with respect to G.

We show by induction on k that ${}^cT \in \mathcal{D}_{ttr}(F)$ and ${}^cT \cong (C_1 \otimes_K \cdots \otimes_K C_k,$ $G,$ $(\alpha_1 \otimes \cdots \otimes \alpha_k,$ $f_1 \otimes \cdots \otimes f_k))$, a generalized crossed product. This holds for $k = 1$ by Prop. 12 (2). Assume that it is true for $k-1$, so that if $T_0 = \bigotimes_{i=1}^{k-1} T_i$, then ${}^cT_0 \in \mathcal{D}_{ttr}(F)$ and ${}^cT_0 \cong (C_1 \otimes_K \cdots \otimes_K C_{k-1}, G, (\alpha_1 \otimes \cdots \otimes \alpha_{k-1}, f_1 \otimes \cdots \otimes f_{k-1}))$. Then ${}^cT$ is the underlying division algebra of ${}^cT_0 \otimes_F {}^cT_k$. As ${}^cT_i$ is tame over F for each i, $1 \le i \le k-1$, and ${}^cT_0 \sim \bigotimes_{i=1}^{k-1} {}^cT_i$, by [**JW**, Cor. 6.7] and Prop. 12 (2) we have $\Gamma_{{}^cT_0} \subseteq \sum_{i=1}^{k-1} \Gamma_{{}^cT_i} \subseteq \sum_{i=1}^{k-1} \Gamma_{T_i} = \Gamma_{T_0}$. So $\Gamma_{{}^cT_0} \cap \Gamma_{{}^cT_k} \subseteq \Gamma_{T_0} \cap \Gamma_{T_k} = \Gamma_L = \Gamma_K$, whence $\Gamma_{{}^cT_0} \cap \Gamma_{{}^cT_k} = \Gamma_K$. Since ${}^cT_0$, ${}^cT_k \in \mathcal{D}_{ttr}(F)$, $K \subseteq {}^cT_0$, $K \subseteq {}^cT_k$, and $\Gamma_{{}^cT_0} \cap \Gamma_{{}^cT_k} = \Gamma_K$, by Prop. 11 ${}^cT \in \mathcal{D}_{ttr}(F)$ and ${}^cT \cong (C_1 \otimes_K \cdots \otimes_K C_K,$ $G,$ $(\alpha_1 \otimes \cdots \otimes \alpha_k,$ $f_1 \otimes \cdots \otimes f_k))$, as desired.

Since each C_i is the centralizer of K in ${}^cT_i \in \mathcal{D}_{ttr}(F)$, by [**TW**, Th. 3.8], $\Gamma_{C_i} / \Gamma_F = (\Gamma_K/\Gamma_F)^\perp$ relative to the canonical pairing $\underline{C_{{}^cT_i}} : (\Gamma_{{}^cT_i} / \Gamma_F) \times (\Gamma_{{}^cT_i} / \Gamma_F) \to \mu(\overline{F})$ given by $(v(d)+\Gamma_F, v(e)+\Gamma_F) \mapsto \overline{ded^{-1}e^{-1}}$. Because the pairing is nondegenerate, $|\Gamma_{T_i} : \Gamma_{C_i}| = |\Gamma_{T_i} / \Gamma_F : (\Gamma_K/\Gamma_F)^\perp| = |\Gamma_K/\Gamma_F| = p$. So $p\Gamma_{T_i} \subseteq \Gamma_{C_i}$. Hence $\Gamma_{{}^cT} \supseteq \Gamma_{C_1 \otimes_K \cdots \otimes_K C_k} = \sum_{i=1}^{k} \Gamma_{C_i} \supseteq p\left(\sum_{i=1}^{k} \Gamma_{T_i}\right) = p\Gamma_T$ as $\Gamma_{C_1 \otimes_K \cdots \otimes_K C_k} = \sum_{i=1}^{k} \Gamma_{C_i}$ by Prop. 11. Also since ${}^cT_i \in \mathcal{D}_t(F)$ and $\Gamma_{{}^cT_i} \subseteq \Gamma_{T_i}$, by [**JW**, Cor. 6.7] $\Gamma_{{}^cT} \subseteq \sum_{i=1}^{k} \Gamma_{{}^cT_i} \subseteq \sum_{i=1}^{k} \Gamma_{T_i} = \Gamma_T$. Hence $p\Gamma_T \subseteq \Gamma_{{}^cT} \subseteq \Gamma_T$ as desired.

Since $\Gamma_{{}^cT} \subseteq \sum_{i=1}^{k} \Gamma_{{}^cT_i}$, $\exp(\Gamma_{{}^cT} / \Gamma_F) \mid \exp\left(\left(\sum_{i=1}^k \Gamma_{{}^cT_i}\right) / \Gamma_F\right)$. But each $\exp(\Gamma_{{}^cT_i} / \Gamma_F) \mid \exp(\Gamma_{T_i} / \Gamma_L)$ by Prop. 12. So $\exp\left(\left(\sum_{i=1}^k \Gamma_{{}^cT_i}\right) / \Gamma_F\right) \mid \exp\left(\left(\sum_{i=1}^k \Gamma_{T_i}\right) / \Gamma_L\right) = \exp(\Gamma_T/\Gamma_L)$. Hence $\exp(\Gamma_{{}^cT} / \Gamma_F) \mid \exp(\Gamma_T/\Gamma_L)$. □

While $[L : F] \cdot \Gamma_T \subseteq \Gamma_{{}^cT}$ holds for $T \in \mathcal{D}_{ttr}(L)$ when L/F is TRRT, $\exp(\Gamma_L/\Gamma_F) \cdot \Gamma_T \subseteq \Gamma_{{}^cT}$ need not hold, as the following example illustrates:

Example 14.Let p be a prime, and $m \geq k \geq 1$, integers. Let $n = p^m$. Let F_0 be any field with $\mu_{np^2} \subseteq F_0$. Let $F = F_0((x_1))\cdots((x_{k+2}))$ be the iterated Laurent power series field, with the usual Henselian valuation $v\colon F^* \to \mathbb{Z}^{k+2}$. That is,

$$v\left(\sum_{i_1}\cdots\sum_{i_{k+2}} c_{i_1\ldots i_{k+2}} x_1^{i_1}\cdots x_{k+2}^{i_{k+2}}\right) = \inf\{(i_1,\ldots,i_{k+2}) \mid c_{i_1\cdots i_{k+2}} \neq 0\},$$

where $\mathbb{Z}^{k+2}$ has the right-to-left lexicographical ordering, i.e.,

$$(i_1,\ldots,i_{k+2}) < (j_1,\ldots,j_{k+2})$$

if and only if there is a q with $i_q < j_q$ and $i_r = j_r$ for $q < r \leq k+2$ (cf. [**Rb**, p. 77, Prop. 4 and p. 198, Th. 4]). Let $L = F(\sqrt[p]{x_1},\ldots,\sqrt[p]{x_{k+2}})$. Then L is a TRRT extension of F with $[L\colon F] = p^{k+2}$.

Now, let $T = (\sqrt[p]{x_1}, \sqrt[p]{x_{k+2}}; L)_n$. Then by [$\mathbf{H}_2$, Prop. 3.1], $T \in \mathcal{D}_{ttr}(L)$ and $\Gamma_T = \left\langle \frac{1}{np}v(x_1), \frac{1}{np}v(x_{k+2})\right\rangle + \Gamma_L$. Since $T \sim (x_1, x_{k+2}; L)_{np^2}$, by [$\mathbf{Ti}_2$, Th. 3.1] (Projection formula)

$$\operatorname{cor}_{L/F}(T) \sim (x_1^{p^{k+2}}, x_2; F)_{np^2} \sim (x_1, x_2; F)_{n/p^k}$$

as $N_{L/F}(x_1) = x_1^{p^{k+2}}$. Then by [$\mathbf{H}_2$, Prop. 3.1],

$${}^cT \cong (x_1, x_{k+2}; F)_{n/p^k} \in \mathcal{D}_{ttr}(F)$$

and

$$\Gamma_{{}^cT} = \left\langle \frac{p^k}{n}v(x_1), \frac{p^k}{n}v(x_{k+2})\right\rangle + \Gamma_F.$$

So $p^{k+1}\Gamma_T \subseteq \Gamma_{{}^cT} \subseteq \Gamma_T$. (Hence $[L\colon F]\Gamma_T = p^{k+2}\Gamma_T \subseteq \Gamma_{{}^cT}$.) But

$$\exp(\Gamma_L/\Gamma_F)\Gamma_T = p\Gamma_T \not\subseteq \Gamma_{{}^cT}.$$

Note that $\Gamma_T/\Gamma_F \cong (\mathbb{Z}_{np})^2 \times (\mathbb{Z}_p)^k$, and $\Gamma_{{}^cT}\,/\,\Gamma_F = p^{k+1}(\Gamma_T/\Gamma_F)$.

We end this section by giving relations between $D \in \mathcal{D}_t(L)$ and ${}^cD \in \mathcal{D}_t(F)$ when $(L,v) \supseteq (F,v)$ is a finite separable TRRT extension of Henselian fields.

Theorem 15. *Let $(L,v) \supseteq (F,v)$ be as above. If $D \in \mathcal{D}_t(L)$, then $[L : F]\Gamma_D \subseteq \Gamma_{{}^cD} \subseteq \Gamma_D$ and $Z(\overline{{}^cD}) \subseteq Z(\overline{D})$.*

Proof. (1) $[L\colon F]\Gamma_D \subseteq \Gamma_{{}^cD} \subseteq \Gamma_D$: By Prop. 3, in $\operatorname{Br}(L)$, $D \sim S \otimes_L T$ for some $S \in \mathcal{D}_{is}(L)$ and $T \in \mathcal{D}_{ttr}(L)$, and $\Gamma_D = \Gamma_S + \Gamma_T$. Then ${}^cD \sim {}^cS \otimes_F {}^cT$

in $\mathrm{Br}(F)$ where ${}^cS \in \mathcal{D}_{is}(F)$, with $\Gamma_{{}^cS} = [L:F]\Gamma_S + \Gamma_F$ by Th. 8, and ${}^cT \in \mathcal{D}_{ttr}(F)$, with $[L:F]\Gamma_T \subseteq \Gamma_{{}^cT} \subseteq \Gamma_T$ by Th. 13 above. So by Prop. 3, $\Gamma_{{}^cD} = \Gamma_{{}^cS} + \Gamma_{{}^cT}$. Therefore, $[L:F]\Gamma_D = [L:F]\Gamma_S + [L:F]\Gamma_T \subseteq \Gamma_{{}^cS} + \Gamma_{{}^cT} = \Gamma_{{}^cD} \subseteq \Gamma_S + \Gamma_T = \Gamma_D$.

(2) $Z(\overline{{}^cD}) \subseteq Z(\overline{D})$: Since $D \in \mathcal{D}_t(L)$, $Z(\overline{D})$ is separable (so abelian Galois) over $\overline{L} = \overline{F}$ by [**JW**, Lemma 6.1]). Let Z be the inertial lift of $Z(\overline{D})$ over F. Then Z is Galois over F and $L \cap Z = F$ as L/F is TRRT and Z/F is inertial. So $L \otimes_F Z$ is the field $L \cdot Z$. Then by [$\mathbf{D}_1$, p. 56, Ex.1] ${}^cD \otimes_F Z \sim \mathrm{cor}_{LZ/Z}(D \otimes_L LZ)$ in $\mathrm{Br}(Z)$. Let D_{LZ} and $({}^cD)_Z$ be the underlying division algebras of $D \otimes_L LZ$ and ${}^cD \otimes_F Z$, respectively. Since LZ/L is inertial, by [**JW**, Th. 3.1] $Z(\overline{D_{LZ}}) = Z(\overline{D}) \cdot \overline{LZ} = \overline{Z} \cdot \overline{LZ} = \overline{LZ}$. Then by [**JW**, Cor. 2.11], in $\mathrm{Br}(LZ)$, $D_{LZ} \sim I \otimes_{LZ} T$ for some $I \in \mathcal{D}_i(LZ)$ and $T \in \mathcal{D}_{ttr}(LZ)$. So $({}^cD)_Z \sim {}^cD \otimes_F Z \sim \mathrm{cor}_{LZ/Z}(D_{LZ}) \sim \mathrm{cor}_{LZ/Z}(I) \otimes_Z \mathrm{cor}_{LZ/Z}(T)$.

Let I' and Z' be the underlying division algebras of $\mathrm{cor}_{LZ/Z}(I)$ and $\mathrm{cor}_{LZ/Z}(T)$, respectively. Since LZ/Z is TRRT, by Lemma 6 $I' \in \mathcal{D}_i(Z)$, and by Th. 13 $T' \in \mathcal{D}_{ttr}(Z)$. So $Z(\overline{({}^cD)_Z}) = \overline{Z} = Z(\overline{D})$. But since Z/F is inertial, by Th. [**JW**, Th. 3.1] again $Z(\overline{({}^cD)_Z}) = Z(\overline{{}^cD}) \cdot \overline{Z}$ so we have $Z(\overline{{}^cD}) \cdot Z(\overline{D}) = Z(\overline{D})$, hence $Z(\overline{{}^cD}) \subseteq Z(\overline{D})$. □

4. The case when L/F is tame.

Suppose $(L, v) \supseteq (F, v)$ is a finite separable extension of Henselian fields such that $\overline{L}/\overline{F}$ is separable and L/K is TRRT where K is the inertial lift of $\overline{L}$ over F in L (i.e. the inertial extension of F with $\overline{K} = \overline{L}$). Then we can combine the previous results with ones of [$\mathbf{H}_2$] to obtain relations between $D \in \mathcal{D}_t(L)$ and ${}^cD \in \mathcal{D}(F)$ since L/K is TRRT and K/F is inertial and ${}^cD \sim \mathrm{cor}_{K/F}(\mathrm{cor}_{L/K}(D))$ in $\mathrm{Br}(F)$. Notably, if L is tame over F (i.e. $\mathrm{char}(\overline{F}) \nmid [L:F]$), then L/F is such an extension: Note that $[L:F] = |\Gamma_L : \Gamma_F| \cdot [\overline{L}:\overline{F}] \cdot q^b$ for some nonnegative integer b, where $q = \mathrm{char}(\overline{F})$ if $\mathrm{char}(\overline{F}) \neq 0$, or $q = 1$ otherwise. (This is proved in [**E**, 20.21] when L is normal over F. By passing to the normal closure as done in the proof of [**M**, Cor. 3], this can be proved in general.) Since $\mathrm{char}(\overline{F}) \nmid [L:F]$, necessarily $q^b = 1$, so $[L:F] = |\Gamma_L:\Gamma_F| \cdot [\overline{L}:\overline{F}]$. As $\mathrm{char}(\overline{F}) \nmid [\overline{L}:\overline{F}]$, $\overline{L}/\overline{F}$ is separable. If K is the inertial lift of $\overline{L}$ over F in L, then $[L:K] = [L:F]\,/\,[\overline{L}:\overline{F}] = |\Gamma_L:\Gamma_F| = |\Gamma_L:\Gamma_K|$, and $\mathrm{char}(\overline{K}) = \mathrm{char}(\overline{F}) \nmid [L:K]$. So L/K is tame and totally ramified. Since (K, v) is Henselian, L/K is TRRT by [**S**, p. 64, Th. 3].

Throughout this section, we assume that $(L, v) \supseteq (F, v)$ is a finite separable tame extension of Henselian fields and K is the inertial lift of $\overline{L}$ over F in L. (So L/K is TRRT as we just showed and K/F is inertial.) Also, for any $D \in \mathcal{D}_t(L)$, let ${}^cD \in \mathcal{D}(F)$ denote the underlying division algebra of $\operatorname{cor}_{L/F}(D)$ as before.

Theorem 16. *Let $T \in \mathcal{D}_{ttr}(L)$ and let T_1 be the underlying division algebra of $\operatorname{cor}_{L/K}(T)$. Let $t = \exp(\Gamma_T/\Gamma_L)$ and $t_1 = \exp(\Gamma_{T_1}/\Gamma_K)$. (So $t_1 \mid t$ by Th. 13.) Let $e = \gcd([\overline{L} : \overline{F}], t)$ and $e_1 = \gcd([\overline{L} : \overline{F}], t_1)$. If $\mu_{t_1} \subseteq F$, then $Z(\overline{{}^cT}) \subseteq \overline{F}\left((N_{\overline{L}/\overline{F}}(\overline{L}))^{1/e_1}\right) \subseteq \overline{F}\left((N_{\overline{L}/\overline{F}}(\overline{L}))^{1/e}\right)$, where $N_{\overline{L}/\overline{F}}$ is the norm map from $\overline{L}$ to $\overline{F}$.*

Proof. Since L/K is TRRT, $T_1 \in \mathcal{D}_{ttr}(K)$ by Th. 13. Also, since ${}^cT$ is the underlying division algebra of $\operatorname{cor}_{K/F}(T_1)$ and K/F is inertial and $\mu_{t_1} \subseteq F$, by [**H**$_2$, Th. 3.9], $Z(\overline{{}^cT}) \subseteq \overline{F}\left((N_{\overline{K}/\overline{F}}(\overline{K}))^{1/e_1}\right) = \overline{F}\left((N_{\overline{L}/\overline{F}}(\overline{L}))^{1/e_1}\right)$ as $[K: F] = [\overline{L}: \overline{F}]$ and $\overline{K} = \overline{L}$.

The second inclusion is clear as $e_1 \mid e$. □

Recall that $\mathcal{N}(M/F)$ denotes the normal closure of M over F where M/F is an algebraic extension of fields.

Theorem 17. *Let L/F be tame.*

(a) If $I \in \mathcal{D}_i(L)$, then ${}^cI \in \mathcal{D}_i(F)$ and $\overline{{}^cI} \sim \operatorname{cor}_{\overline{L}/\overline{F}}(\overline{I}^{\otimes|\Gamma_L:\Gamma_F|})$ in $\operatorname{Br}(\overline{F})$.

(b) If $D \in \mathcal{D}_{is}(L)$ then ${}^cD \in \mathcal{D}_{is}(F)$, $\Gamma_{{}^cD} \subseteq |\Gamma_L : \Gamma_F| \cdot \Gamma_D + \Gamma_F$, and $Z(\overline{{}^cD}) \subseteq \mathcal{N}\left(\mathcal{F}(\theta_D(\widetilde{\Gamma}))/\overline{F}\right)$, where $\widetilde{\Gamma} = \{\alpha + \Gamma_L \in \Gamma_D/\Gamma_L \mid |\Gamma_L : \Gamma_F| \cdot \alpha \in \Gamma_F\}$. (So $\Gamma_{{}^cD} \subseteq \Gamma_D$ and $Z(\overline{{}^cD}) \subseteq \mathcal{N}(Z(\overline{D})/\overline{F})$.)

(c) If $D \in \mathcal{D}_t(L)$, then ${}^cD \in \mathcal{D}_t(F)$ and $\Gamma_{{}^cD} \subseteq \Gamma_D$.

Proof. (a) Let I_1 be the underlying division algebra of $\operatorname{cor}_{L/K}(I)$. Since $\overline{L} = \overline{K}$, by Lemma 4 $I_1 \in \mathcal{D}_i(K)$ and $\overline{I_1} \sim \overline{I}^{\otimes|\Gamma_L:\Gamma_F|}$ in $\operatorname{Br}(\overline{K})$ as $[L : K] = |\Gamma_L : \Gamma_F|$. Since ${}^cI \sim \operatorname{cor}_{K/F}(I_1)$ in $\operatorname{Br}(F)$, and K/F is inertial, by [**H**$_2$, Th. 2.4] ${}^cI \in \mathcal{D}_i(F)$ and $\overline{{}^cI} \sim \operatorname{cor}_{\overline{K}/\overline{F}}(\overline{I_1}) \sim \operatorname{cor}_{\overline{L}/\overline{F}}(\overline{I}^{\otimes|\Gamma_L:\Gamma_F|})$ in $\operatorname{Br}(\overline{F})$.

(b) Let D_1 be the underlying division algebra of $\operatorname{cor}_{L/K}(D)$. Since $\overline{L} = \overline{K}$, by Th. 8 $D_1 \in \mathcal{D}_{is}(K)$, $\Gamma_{D_1} = [L: K] \cdot \Gamma_D + \Gamma_K = |\Gamma_L: \Gamma_F| \cdot \Gamma_D + \Gamma_F$, and $Z(\overline{D_1}) = \mathcal{F}(\theta_D(\widetilde{\Gamma}))$ where $\widetilde{\Gamma} = \{\alpha + \Gamma_L \in \Gamma_D/\Gamma_L \mid [L : K] \cdot \alpha \in \Gamma_K\} = \{\alpha + \Gamma_L \in \Gamma_D/\Gamma_L \mid |\Gamma_L : \Gamma_F| \cdot \alpha \in \Gamma_F\}$. Then since K/F is inertial and ${}^cD \sim \operatorname{cor}_{K/F}(D_1)$ in $\operatorname{Br}(F)$, by [**H**$_2$, Th. 2.4] ${}^cD \in \mathcal{D}_{is}(F)$, $\Gamma_{{}^cD} \subseteq \Gamma_{D_1}$, and $Z(\overline{{}^cD}) \subseteq \mathcal{N}\left(Z(\overline{D_1})/\overline{F}\right)$. Therefore, ${}^cD \in \mathcal{D}_{is}(F)$, $\Gamma_{{}^cD} \subseteq |\Gamma_L : \Gamma_F| \cdot \Gamma_D + \Gamma_F$, and $Z(\overline{{}^cD}) \subseteq \mathcal{N}(\mathcal{F}(\theta_D(\widetilde{\Gamma}))/\overline{F})$. So since $\mathcal{F}(\theta_D(\widetilde{\Gamma})) \subseteq Z(\overline{D})$, we have $Z(\overline{{}^cD}) \subseteq \mathcal{N}(Z(\overline{D})/\overline{F})$.

(c) If $D \in \mathcal{D}_t(L)$, then ${}^cD \in \mathcal{D}_t(F)$ by definition of tameness and (b) above. Let D_1 be the underlying division algebra of $\operatorname{cor}_{L/K}(D)$. Since L/K

is TRRT, $\Gamma_{D_1} \subseteq \Gamma_D$ by Th. 15. Since K/F is inertial and ${}^cD \sim \operatorname{cor}_{K/F}(D_1)$ in $\operatorname{Br}(F)$, $\Gamma_{{}^cD} \subseteq \Gamma_{D_1}$ by [$\mathbf{H}_2$, Th.4.5]. So $\Gamma_{{}^cD} \subseteq \Gamma_D$. □

Theorem 18. *Let L/F be tame and let K be the inertial lift of $\overline{L}$ over F in L. Let $D \in \mathcal{D}_t(L)$ and let D_1 be the underlying division algebra of $\operatorname{cor}_{L/K}(D)$. Let $t_1 = \exp(\ker \theta_{D_1}) = 2^{e_0}p_1^{e_1}\cdots p_r^{e_r}$ where p_i are odd primes, and $e_0 \geq 0$, $e_i > 0$ are integers, $1 \leq i \leq r$. Suppose $\mu_{p_i} \subseteq F$ for $1 \leq i \leq r_0$ and $\mu_{p_i} \not\subseteq F$ for $r_0+1 \leq i \leq r$. Let $s = 2^{e_0}p_1^{e_1}\cdots p_{r_0}^{e_{r_0}}$ and $s' = s/2^{e_0}$. Then*

(a) *if $\mu_4 \subseteq F$ or $4 \nmid t_1$, then*

$$Z(\overline{{}^cD}) \subseteq \mathcal{N}(Z(\overline{D_1})\,/\,\overline{F})^{1/n} \subseteq \mathcal{N}(Z(\overline{D})\,/\,\overline{F})^{1/n},$$

where $n = (t_1/s)\cdot\gcd([\overline{L}:\overline{F}], s)$;

(b) *if $4 \mid t_1$ and $\mu_4 \not\subseteq F$, then*

$$Z(\overline{{}^cD}) \subseteq \mathcal{N}(Z(\overline{D_1})\,/\,\overline{F})^{1/n'} \subseteq \mathcal{N}(Z(\overline{D})\,/\,\overline{F})^{1/n'},$$

where $n' = (t_1/s')\cdot\gcd([\overline{L}:\overline{F}], s')$. So in either case,

$$Z(\overline{{}^cD}) \subseteq \mathcal{N}(Z(\overline{D_1})\,/\,\overline{F})^{1/t_1} \subseteq \mathcal{N}(Z(\overline{D})\,/\,\overline{F})^{1/t}$$

where $t = \exp(\ker \theta_D)$.

Proof. Since L/K is TRRT and $D_1 \sim \operatorname{cor}_{L/K}(D)$, $D_1 \in \mathcal{D}_t(K)$ and $Z(\overline{D_1}) \subseteq Z(\overline{D})$ by Th. 8 and Th. 15. Also, since K/F is inertial and ${}^cD \sim \operatorname{cor}_{K/F}(D_1)$, by [$\mathbf{H}_2$, Th. 4.6] we have (a) and (b).

Since $n \mid t_1$ and $n' \mid t_1$, $Z(\overline{{}^cD}) \subseteq \mathcal{N}(Z(\overline{D_1})\,/\,\overline{F})^{1/t_1}$. By Prop. 3, we have $D \sim S \otimes_L T$ for some $S \in \mathcal{D}_{is}(L)$ and $T \in \mathcal{D}_{ttr}(L)$, and $\Gamma_T/\Gamma_L = \ker(\theta_D)$. So $t = \exp(\Gamma_T/\Gamma_L)$. Let S_1 and T_1 be the underlying division algebras of $\operatorname{cor}_{L/K}(S)$ and $\operatorname{cor}_{L/K}(T)$, respectively. Since L/K is TRRT, $D_1 \sim S_1 \otimes_K T_1$ where $S_1 \in \mathcal{D}_{is}(K)$ and $T_1 \in \mathcal{D}_{ttr}(K)$ and $\exp(\Gamma_{T_1}/\Gamma_K) \mid \exp(\Gamma_T/\Gamma_L)$ by Th. 8 and Th. 13. So by Prop. 3, $\ker(\theta_{D_1}) = \Gamma_{T_1}/\Gamma_K$, hence $t_1 = \exp(\Gamma_{T_1}/\Gamma_K) \mid \exp(\Gamma_T/\Gamma_L) = t$. So we have $\mathcal{N}\left(Z(\overline{D_1})\,/\,\overline{F}\right)^{1/t_1} \subseteq \mathcal{N}\left(Z(\overline{D})\,/\,\overline{F}\right)^{1/t_1} \subseteq \mathcal{N}\left(Z(\overline{D})\,/\,\overline{F}\right)^{1/t}$ as $Z(\overline{D_1}) \subseteq Z(\overline{D})$. □

Remark. (The corestriction of central simple algebras with Dubrovin valuation rings.) There are generalizations of Theorems 17 and 18 above for central simple algebras S over a valued field (L, v) where v is not Henselian. We describe the generalizations here in Th. 20, but omit proofs, which can be found in [$\mathbf{H}_1$, Chap. 5]. When v is not Henselian, it may not extend to a valuation on S, but there is always a unique (up to isomorphism) Dubrovin valuation ring B of S extending the valuation ring V of v on F, and B has a value group Γ_B and a residue central simple algebra $\overline{B} = B/J(B)$, where $J(B)$ is the Jacobson radical of B (cf. [**W**$_2$]). The following proposition is used in proving the generalizations of Th. 17 and 18.

Proposition 19 [**H**$_1$, Th. 5.4]. *Let (F, v) be a Henselian field. If $D_i \in \mathcal{D}_t(F)$ for $1 \le i \le n$, and D is the underlying division algebra of $D_1 \otimes_F \cdots \otimes_F D_n$, then $Z(\overline{D}) \subseteq Z(\overline{D_1})^{1/t_1} \cdots Z(\overline{D_n})^{1/t_n}$ where $t_i = \exp(\ker(\theta_{D_i}))$.*

To state Th. 20, we introduce the following notation:

Let L be a finite separable extension of a field with valuation ring (F, V) and let $W_1, \ldots, W_k$ be all the valuation rings of L extending V. Let $L_i = (L, W_i)$, (F^h, V^h) = the Henselization of (F, V) and (L_i^h, W_i^h) = the Henselization of (L, W_i) for $1 \le i \le k$. Let S be a central simple L-algebra and let ${}^{\text{cor}}S = \text{cor}_{L/F}(S)$, the corestriction of S. Let A be a Dubrovin valuation ring of ${}^{\text{cor}}S$ with $A \cap F = V$ and let B_i be a Dubrovin valuation ring of S with $B_i \cap L = W_i$. Set $S_i = (S, B_i)$. Let S_i^h be the underlying division algebra of $S_i \otimes_L L_i^h$ and let $({}^{\text{cor}}S)^h$ be the underlying division algebra of ${}^{\text{cor}}S \otimes_F F^h$. Since $\Gamma_{B_i} = \Gamma_{S_i^h}$, $\Gamma_A = \Gamma_{({}^{\text{cor}}S)^h}$, $Z(\overline{B_i}) = Z(\overline{S_i^h})$ and $Z(\overline{A}) = Z\left(\overline{({}^{\text{cor}}S)^h}\right)$ by [**W**$_2$, Th. B], we can obtain information about A by applying Th. 17 and 18 to the Henselizations. Thereby, we obtain the following theorem.

Theorem 20 [**H**$_1$, Th. 5.15]. *Assume all L_i^h/F^h are tame for $1 \le i \le k$.*

(1) *If S_i^h is inertially split over L_i^h for each i, $1 \le i \le k$, then $({}^{\text{cor}}S)^h$ is inertially split over F^h, $\Gamma_A \subseteq \sum_{i=1}^{k} \Gamma_{B_i}$ and*

$$Z(\overline{A}) \subseteq \mathcal{N}\left(\prod_{i=1}^{k} Z(\overline{B_i}) \,/\, \overline{F}\right).$$

(2) *If S_i^h is tame over L_i^h for each i, $1 \le i \le k$, then $({}^{\text{cor}}S)^h$ is tame over F^h and $\Gamma_A \subseteq \sum_{i=1}^{k} \Gamma_{B_i}$. Further, if $t_i = \exp(\ker(\theta_{S_i^h}))$ for $1 \le i \le k$, then $Z(\overline{A}) \subseteq \prod_{i=1}^{k} \mathcal{N}\left(Z(\overline{B_i}) \,/\, \overline{F}\right)^{1/m_i}$ where $m_i = t_i$ if $4 \nmid t_i$ or $\mu_4 \subseteq F$, or $m_i = 2t_i$ if $4 \mid t_i$ and $\mu_4 \not\subseteq F$. (The condition that S_i^h is inertially split (or tame) over L_i^h can be expressed in terms of B_i itself. See* [**H**$_1$, Chap. 5, Sec. 2] *for details.)*

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DEPARTMENT OF MATHEMATICS
KOREA UNIVERSITY
ANAM-DONG 5-1, SUNGBUK-KU
SEOUL 136-701, KOREA
E-MAIL ADDRESS: YHWANG@SEMI.KOREA.AC.KR