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# SMOOTH DECOMPOSITION OF FINITE MULTIPLICITY MONOMIAL REPRESENTATIONS FOR A CLASS OF COMPLETELY SOLVABLE HOMOGENEOUS SPACES.

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**Given a finite multiplicity monomial representation $\tau$ of a completely solvable Lie group $G$, a smooth decomposition of $\tau$ is a concrete direct integral decomposition into irreducibles parametrized by a manifold $\Sigma$, with the property that compactly-supported elements of $\mathcal{H}_\tau^\infty$ are mapped to smooth sections on $\Sigma$ by the intertwining operator. A natural way of constructing such a decomposition is by means of the distribution-theoretic Plancherel formula for $\tau$ and a cross-section $\Sigma$ for coadjoint orbits. However, for irreducible representations $\pi_l$, $l \in \Sigma$, the determination of appropriate distributions $\beta_l \in \mathcal{H}_l^{-\infty}$ is problematic. For the case where $\tau$ is induced from a "Levi" component, we overcome these problems and give an explicit and natural construction for a smooth decomposition. In the process we show that in this situation the nilradical of $G$ must be two-step.**

## 0. Introduction.

We are interested in the decomposition of the representation $\tau$ of a solvable Type I group $G$ induced from a unitary character of closed, connected subgroup $H$. In one sense, to decompose $\tau$ means to describe the spectrum of $\tau$, the multiplicities, and the equivalence class of the Plancherel measure, in terms of the coadjoint orbit picture. But there is also a stronger sense of what it means to decompose $\tau$: one would like to give a construction for a direct integral, a unitary intertwining map, and a distribution-theoretic Plancherel formula. The goal here is that the construction be as explicit as possible, but at the same time natural: all objects should be naturally and uniquely determined up to the choice of a certain Jordan-Holder basis for the Lie algebra. The base space for the direct integral should be a smooth manifold which naturally parametrizes an explicitly determined set of coadjoint orbit data, and compactly supported smooth vectors for $\tau$ should be mapped under the intertwining map to smooth functions on this manifold. Under these circumstances we shall say that we have a *smooth* decomposition of $\tau$. The present paper carries out this program for a particular class of completely solvable homogeneous spaces.

Let $G$ and $H$ be as above but also completely solvable. Then $G$ is exponential, hence $G$ is of Type I and one has the canonical bijection between the unitary dual $\hat{G}$ and the space of coadjoint orbits [**3**]. For each $\lambda \in \hat{G}$, let $O_\lambda$ denote the corresponding coadjoint orbit. Lie algebras of designated Lie groups will be denoted by corresponding German gothic letters. Given a unitary representation $\pi$ of $G$ acting in a Hilbert space $\mathcal{H}$, denote by $\mathcal{H}^\infty$ the Frechet space of smooth vectors for $\pi$, and by $\mathcal{H}^{-\infty}$ the space of continuous anti-linear functionals on $\mathcal{H}^\infty$; recall that $\mathcal{H}^\infty \subset \mathcal{H} \subset \mathcal{H}^{-\infty}$. Elements of $\mathcal{H}^{-\infty}$ will be called generalized vectors for $\pi$.

Let $f \in \mathfrak{g}^*$ have the property that $\mathfrak{h}$ is subordinate to $f$ (that is, $[\mathfrak{h}, \mathfrak{h}] \subset \ker(f)$), let $\chi$ be the corresponding unitary character of $H$, and let $\tau = \tau_f$ be the representation of $G$ induced from $\chi$. Denote by $\mathfrak{h}^\perp$ the space of linear functionals that vanish on $\mathfrak{h}$. It was proved first by Corwin, Greenleaf, and Grelaud for $G$ nilpotent [**6**], and by Lipsman for $G$ completely solvable [**14, 15**], that the spectrum of $\tau$ consists of those $\lambda \in \hat{G}$ for which $O_\lambda \cap (f + \mathfrak{h}^\perp) \neq \emptyset$, and that the multiplicity $m_\lambda$ of $\lambda$ is given by the number of $H$-orbits contained in $O_\lambda \cap (f + \mathfrak{h}^\perp)$. Either $m_\lambda = \infty$ for a.e. $\lambda$, or there is $M > 0$ such that for a.e. $\lambda$, $m_\lambda < M$. One has the finite (indeed bounded) multiplicity case if and only if $\dim(Gl) = 2\dim(Hl)$ for a.e. $l \in f + \mathfrak{h}^\perp$, whence each $H$-orbit in $O_\lambda$ is a connected component of $O_\lambda \cap (f + \mathfrak{h}^\perp)$. Of course the preceding references contain more information than is conveyed here. We also remark that the spectral decomposition of $\tau$ for the more general class of exponential $G$ was obtained by Fujiwara [**10**] and is similar to the above, though some questions surrounding the finite multiplicity case are still unresolved.

When $\dim(Gl) = 2\dim(Hl)$ holds for generic $l \in f + \mathfrak{h}^\perp$, there is a one to one correspondence between $H$-orbits in $O_\lambda$ and "appearances" of $\lambda$ in decomposition of $\tau$ (for generic $\lambda$). The spectral decomposition formula of [**14**] is

$$[\tau] = \int_{(f+\mathfrak{h}^\perp)/H} \lambda_\theta \, d[\nu](\theta) \tag{0.1}$$

Here $[\tau]$ stands for the equivalence class of $\tau$, and $\theta$ is an $H$-orbit contained in $O_{\lambda_\theta} \cap (f + \mathfrak{h}^\perp)$. The equivalence class $[\nu]$ is that of pushforwards of finite measures on $f + \mathfrak{h}^\perp$ equivalent to Lebesgue measure. Note that in contrast with the usual spectral decomposition with base space $\hat{G}$, the multiplicities are "spread out" within the $H$-orbit picture. Besides being elegant, the formula (0.1) calls our attention to the possibility of a smooth decomposition of $\tau$ over an *$H$-orbit* cross-section, wherein different realizations for the multiple "appearances" of each $\lambda$ are allowed. The derivation (and application) of this kind of decomposition of $\tau$ first appears in [**5**], where $G$ is nilpotent, but the approach and methods there differ greatly from those of the present work.

Here the smooth decomposition is just a consequence of an explicit Plancherel formula. The distribution-theoretic Plancherel formula (the Penney-Fujiwara Plancherel formula) which is analogous to (0.1) is

$$\langle \tau(\omega)\, \alpha_\tau, \alpha_\tau \rangle = \int_{(f+\mathfrak{h}^\perp)/H} \langle \pi_\theta(\omega)\, \beta_\theta, \beta_\theta \rangle \, d\nu(\theta). \tag{0.2}$$

where $\alpha_\theta$ is the canonical cyclic generalized vector for $\tau$, $\pi_\theta$ is a realization of $\lambda_\theta$, and $\beta_\theta$ is an (appropriately $H$-covariant) generalized vector for $\pi_\theta$. The choice of $\nu$ depends on choices of various Haar measures. In the case that $G$ is nilpotent, (0.2) was obtained by Fujiwara (in a different form) [**9**], and derives from the fundamental work of Penney [**19**]. Groundbreaking work on extending results of [**9**] to other classes of homogeneous spaces has been done by Fujiwara and Yamagami [**11**] and Lipsman [**16, 17, 18**]. However, beyond the nilpotent case, the technical difficulties involved in (0.2) are considerable. One constructs the model $\pi_\theta$ and the generalized vector $\beta_\theta$ for generic $\theta$ by first choosing $l \in \theta$ and a polarization $\mathfrak{b} = \mathfrak{b}(l)$ at $l$ (satisfying the Pukanzsky condition), then $\beta_\theta$ is obtained by integrating $f \in (\mathcal{H}_{\pi_\theta})^\infty$ over $H \cap B\backslash H$ with respect to a certain appropriate measure. At issue is convergence of the integral, as well as the fact that $\beta_\theta$ must be appropriately $H$-covariant. One must make "good" choices for $l$ and $\mathfrak{b}(l)$: examples show that not all choices will produce $\beta_\theta$ with the required properties. What exactly are the generic $\theta$, and whether good choices for $\mathfrak{b}(l)$ actually exist are questions which are not settled in general. In any case, one would like to have a natural procedure for determining generic $\theta$ and making these choices, in the process describing $(f + \mathfrak{h}^\perp)/H$ by a smooth orbital cross-section $\Sigma$, and the measure $\nu$ as an explicit measure on $\Sigma$. In this paper we consider a special case in which the difficulties surrounding the construction of the $\beta_\theta$ are nevertheless very much present. Our main task is to overcome these difficulties and obtain an explicit version of (0.2) by the procedure outlined above.

The class of homogeneous spaces $H\backslash G$ with which we are concerned is that for which $G$ is the semi-direct product $G = NH$, where $N$ is nilpotent and normal in $G$, and $H$ is abelian and acts semi-simply on $N$ with real eigenvalues. In the context of algebraic groups $H$ is sometimes called a Levi component [**13**] (but we need not assume algebraic here). The orbital spectrum formula for the quasi-regular representation $\tau_0$ on this class of homogeneous spaces was known before the more general results of [**10, 14, 15**]: in [**13**] the spectrum of $\tau_0$ is computed using the Mackey machine, and it is shown that $\tau_0$ has uniform multiplicity (either a power of 2, or $+\infty$), which is in turn the number of $H$-orbits in a generic orbital intersection $O_\lambda \cap \mathfrak{h}^\perp$.

Now since $H$ is co-normal, $H$-orbits in $f+\mathfrak{h}^{\perp}$ are just translates of $H$-orbits in $\mathfrak{h}^{\perp}$, so that if $\tau_f$ has finite multiplicity for some $f$, then it does for all $f$. If this holds we simply say that $H\backslash G$ is finite multiplicity, and we assume that this is the case for the present paper. Our first main result - one which was at first surprising to the present author - is that if $H\backslash G$ is finite multiplicity, then $N$ must be two-step (in fact, a particular type of two-step group which includes the Heisenberg groups). It was already well-known that if $H\backslash G$ as above is symmetric, then the quasi-regular representation $\tau_0$ is multiplicity free and $N$ must be abelian. In our context we deduce that each $\tau_f$ has uniform multiplicity $2^u$, where $u = \dim(\mathrm{cent}(N) \sim N)/2$. The reduction to two-step nilpotent groups here parallels a similar reduction that occurs in the situation where $N$ is as above, but $H$ is *compact* (whence $G$ is no longer necessarily solvable). The result there is that if $(H, N)$ is a Gelfand pair, then $N$ is two-step [**1**].

A key to our method is a precise definition of what it means to be a generic element of $f+\mathfrak{h}^{\perp}$ by means of "jump sets" of indices, and this is the first instance known to this author where such techniques have been used to derive canonical structural information about the group itself. For generic $l$ in $f+\mathfrak{h}^{\perp}$ the so-called Vergne polarizations $\mathfrak{b}(l)$ vary rationally with $l$ and have central intersection with $\mathfrak{h}$. On the other hand, we show that the set of generic $H$-orbits admits a natural, smooth, algebraic cross-section $\Sigma$. Choosing $l \in \Sigma$, and Vergne polarizations $\mathfrak{b}(l)$, we obtain our models $\pi_\theta$. The main result of Section 2 is that an appropriate integral formula for $\beta_\theta$ converges absolutely for every $\pi_\theta$-smooth vector; thus the natural, smoothly varying $\mathfrak{b}(l)$ are in fact "good" polarization choices. In Section 3 we derive the Plancherel formula in terms of $\Sigma$. For this class of homogeneous spaces, the choices for Haar measures are natural and the resulting Plancherel measure on $\Sigma$ is seen to be rational.

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## 1. Algebraic structure of $\mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{n}+\mathfrak{h}$ where $\mathfrak{n}$ is nilpotent, $\mathfrak{n} \supset [\mathfrak{g},\mathfrak{g}]$, and where $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ such that $\mathrm{ad}(\mathfrak{h})$ consists of semisimple endomorphisms with real eigenvalues. For each $A \in \mathfrak{h}$, $a \in \mathbb{R}$, let $R(A,a) = \{X \in \mathfrak{n} : [A,X] = aX\}$. For any real numbers $a$ and $b$, and for $A \in \mathfrak{h}$, we have the usual inclusion $[R(A,a), R(A,b)] \subset R(A,a+b)$. We fix once and for all a basis $\{Z_1, Z_2, \dots, Z_n\}$ for $\mathfrak{n}$ with the properties that

(i) $\operatorname{span}\{Z_1, Z_2, \ldots, Z_i\} = \mathfrak{n}_i$ is an ideal in $\mathfrak{g}$, and
(ii) for each $A \in \mathfrak{h}$, $Z_i$ is an eigenvector for $\operatorname{ad}(A)$, $1 \leq i \leq n$.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the linear functional on $\mathfrak{h}$ such that $[A, Z_i] = \lambda_i(A) Z_i$, $A \in \mathfrak{h}$, $1 \leq i \leq n$, and for each $i$ let $\Lambda_i$ be the corresponding positive character of $G$: $\Lambda_i(\exp Z) = e^{\lambda_i(Z)}$. We select a subset $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_n}$ as follows: $i_1 = \min\{1 \leq i \leq n : \lambda_i \neq 0\}$, $i_2 = \min\{1 \leq i \leq n : \lambda_i$ is not a multiple of $\lambda_{i_1}\}$, $i_3 = \min\{1 \leq i \leq n : \lambda_i \notin \operatorname{span}\{\lambda_{i_1}, \lambda_{i_2}\}\}$, and so on. We thus obtain a minimal spanning set $\{\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_d}\}$ for the root system $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. The minimality of our selection with respect to the ordering of the root system (as well as to cardinality) is crucial. Set $\Psi = \{i_1 < i_2 < \cdots < i_d\}$. For each $k, 1 \leq k \leq u$, let $A_k \in \mathfrak{h}$ be chosen such that $\lambda_{i_r}(A_s) = \delta_{rs}, 1 \leq r, s \leq d$.

**Lemma 1.1.** *For each $k, 1 \leq k \leq d$, we have*

$$[A_k, \mathfrak{n}_{i_k - 1}] = [Z_{i_k}, \mathfrak{n}_{i_k - 1}] = (0).$$

*Proof.* By minimality of the selection of $\Psi$, we have $[A_k, \mathfrak{n}_{i_k-1}] = (0)$. But since $\lambda_{i_k}(A_k) = 1, [Z_{i_k}, \mathfrak{n}_{i_k-1}] \subset R(A_k, 1) \cap \mathfrak{n}_{i_k-1} = (0)$. □

We have $\mathfrak{h} \cap \operatorname{cent}(\mathfrak{g}) = \bigcap\{\ker \lambda_{i_k} : 1 \leq k \leq d\}$; choose any basis $A_{d+1}, \ldots, A_u$ for $\mathfrak{h} \cap \operatorname{cent}(\mathfrak{g})$. This determines a Jordan-Holder sequence $\mathfrak{g} = \mathfrak{g}_m \supset \mathfrak{g}_{m-1} \supset \cdots$ for $\mathfrak{g}$: namely

$$\mathfrak{g}_j = \mathfrak{n} + \operatorname{span}\{A_u, A_{u-1}, \ldots, A_{m-j+1}\}, n < j \leq m,$$

and $\mathfrak{g}_j = \mathfrak{n}_j, 1 \leq j \leq n$. The corresponding basis elements are $\{Z_1, Z_2, \ldots, Z_m\}$, where $Z_{n+1} = A_u, Z_{n+2} = A_{u-1}, \ldots, Z_m = A_1$.

Now that a (conveniently chosen) Jordan-Holder basis is in place, we can employ the "layering" construction of [**7**]. As is well-known, each $l \in \mathfrak{g}^*$ determines a degenerate alternating bilinear form

$$B_l(Z, W) = l([Z, W]), \ Z, W \in \mathfrak{g}.$$

For any subset $\mathfrak{s}$ of $\mathfrak{g}$, let $\mathfrak{s}^l$ denote the orthogonal complement of $\mathfrak{s}$, and set $\mathfrak{r}(l, \mathfrak{s}) = \mathfrak{s}^l \cap \mathfrak{s} =$ the "radical" of the restriction of $B_l$ to $\mathfrak{s}$. We also denote $\mathfrak{r}(l, \mathfrak{g})$ by $\mathfrak{g}(l)$. It is well-known that for each l,

$$\mathfrak{b}(l) = \sum_j \mathfrak{r}(l, \mathfrak{g}_i)$$

is a subalgebra of $\mathfrak{g}$ which is totally isotropic for $B_l$. Note that for any $f \in \mathfrak{n}^\perp$, $\mathfrak{b}(l + f) = \mathfrak{b}(l)$. Also associated to $l$ and to the above Jordan-Holder sequence for $\mathfrak{g}$, we have the index pair $\alpha(l) = (i(l), j(l))$ (cf. [**7**] or [**8**]). Here

$$i(l) \cup j(l) = e(l) = \{1 \leq j \leq m : \mathfrak{g}_j + \mathfrak{g}(l) \neq \mathfrak{g}_{j-1} + \mathfrak{g}(l)\},$$

and

$$j(l) = \{1 \le j \le m : \mathfrak{g}_j + \mathfrak{b}(l) \ne \mathfrak{g}_{j-1} + \mathfrak{b}(l)\}.$$

Let $p : \mathfrak{g}^* \to \mathfrak{n}^*$ denote the restriction mapping. Obviously $p|_{\mathfrak{h}^\perp}$ is one-to-one, onto, and $G$-equivariant. In this way we can identify $\mathfrak{h}^\perp$ and $\mathfrak{n}^*$. In a similar way we identify $\mathfrak{n}^\perp$ and $\mathfrak{h}^*$ (though in this case the identification is not $G$-equivariant).

For each index pair $\alpha$, the corresponding "layer" in $\mathfrak{g}^*$ is the set $\Omega_\alpha = \{l : \alpha(l) = \alpha\}$. Each layer is a real algebraic subset of $\mathfrak{g}^*$, determined by the polynomials which depend only on $p(l)$. We totally order the set of all non-empty layers as in [**8**, Prop. 1.2] and let $\Omega_0$ denote the minimal layer, with $\alpha^0 = (i^0, j^0)$ its sequence pair. $\Omega_0$ is Zariski open in $\mathfrak{g}^*$ and consists of $G$-orbits having maximal dimension. Since the condition $l \in \Omega_0$ depends only on $p(l)$, we have that for $f \in \mathfrak{h}^*$, $\Omega_0 \cap (f + \mathfrak{h}^\perp) \ne \emptyset$ and hence $\Omega_0 \cap (f + \mathfrak{h}^\perp)$ is an $H$-invariant Zariski open subset of $F + \mathfrak{h}^\perp$. It is necessary that our notion of "generic" $H$-orbits require that they be contained in $\Omega_0 \cap (f + \mathfrak{h}^\perp)$, but this is not sufficient.

For any $l$, set $\mathfrak{h}(l) = \mathfrak{g}(l) \cap \mathfrak{h}$, let

$$\Omega_1 = \{l \in \mathfrak{g}^* : l(Z_{i_k}) \ne 0, 1 \le k \le d\}.$$

It is clear that for each $l \in \Omega_1$, $\mathfrak{h}(l) = \mathfrak{h} \cap \mathrm{cent}(\mathfrak{g})$, that $\Omega_1$ is $H$-invariant and consists of $H$-orbit of maximal dimension, and that $\Omega_1 \cap (f + \mathfrak{h}^\perp)$ is Zariski open in $f + \mathfrak{h}^\perp$. Note however that $\Omega_1$ is not necessarily $G$-invariant.

**Lemma 1.2.** *For every $l \in \Omega_1$, $i(l) \supset \Psi$ and $\mathfrak{b}(l) \cap \mathfrak{h} = \mathrm{cent}(\mathfrak{g}) \cap \mathfrak{h}$.*

*Proof.* For $l \in \Omega_1$, and for $i_k \in \Psi$, we have $A_k \in \mathfrak{g}^l_{i_k - 1} \sim \mathfrak{g}^l_{i_k}$, from which it follows that $i_k \in e(l)$. But by Lemma 1.1, $Z_{i_k} \in \mathfrak{r}(l, \mathfrak{g}_{i_k}) \subset \mathfrak{b}(l)$. Thus $i_k \in e(l) \sim j(l) = i(l)$.

Now let $A \in \mathfrak{h} \sim \mathrm{cent}(\mathfrak{g})$. For some $i \in \Psi$, $\lambda_i(A) \ne 0$, hence $l([A, Z_i]) \ne 0$. But $Z_i \in \mathfrak{b}(l)$, so $A \notin \mathfrak{b}(l)$. □

Set $\Omega = \Omega_0 \cap \Omega_1$; the functionals in $\Omega$ will be the "generic" ones: given any $f \in \mathfrak{h}^*$, the irreducible representations which correspond to $G$-orbits $Gl$, $l \in \Omega \cap (f + \mathfrak{h}^\perp)$, are sufficient to decompose $\tau_f$. Containment in $\Omega_0$ will insure that the subalgebras $\mathfrak{b}(l)$ vary smoothly with $l$, while containment in $\Omega_1$ insures a nice cross-section for the $H$-orbits in $\Omega$. The abelian group $\mathrm{cent}(G) \cap H$ is a direct factor of $G$ contained in $H$, and thus will have no effect on the analysis of $\tau_f$. The point of the preceding lemma is therefore that for the remainder of this paper, we can assume that $\mathfrak{b}(l) \cap \mathfrak{h} = (0)$ for every $l \in \Omega$.

Now we introduce the finite multplicity assumption, and derive from it additional algebraic information about $\mathfrak{g}$. In our context here we have

$H(f+l) = f + Hl, f \in \mathfrak{h}^*, l \in \mathfrak{n}^*$, so that $H\backslash G$ is a finite multiplicity homogeneous space simply means that for each $l \in \Omega$, $\dim(Gl) = 2\dim(Hl)$. In the case of finite multiplicity, the above results mean the following.

**Corollary 1.3.** *Assume that $H\backslash G$ is a finite multplicity homogeneous space. Then $i^0 = \Psi$, and for every $l \in \Omega$, $\mathfrak{b}(l) = \mathfrak{g}(l) \oplus \mathfrak{k} = \mathfrak{k}^l$, where $\mathfrak{k} = \operatorname{span}\{Z_{i_1}, Z_{i_2}, \dots, Z_{i_d}\}$ is an abelian subalgebra of $\mathfrak{g}$.*

*Proof.* We have $\#(\Psi) = \dim(Hl) = \dim(Gl)/2 = \#(i^0)$, so $i^0 = \Psi$. In the proof of Lemma 1.2, we saw that $\mathfrak{k} \subset \mathfrak{b}(l)$ and so by definition of $i^0$, $\mathfrak{b}(l) = \mathfrak{g}(l) \oplus \mathfrak{k}$. □

To derive more algebraic information about $\mathfrak{g}$, we use the pairing between elements of $i^0$ and $j^0$, and between their corresponding basis elements, established in [7]. There the set $j^0$ is written as a (not necessarily increasing) sequence $\{j_1, j_2, \dots, j_d\}$, and subalgebras $\mathfrak{b}_k(l), 1 \le k \le d$, are defined, according to the following inductive scheme: for $l \in \Omega$, set $\mathfrak{b}_0(l) = \mathfrak{g}$, define $\mathfrak{b}_k(l) = \mathfrak{b}_{k-1}(l) \cap (\mathfrak{g}_{i_k} \cap \mathfrak{b}_{k-1}(l))^l$, and

$$j_k = \min\{1 \le j \le m : \mathfrak{g}_j \cap \mathfrak{b}_{k-1}(l) \not\subset \mathfrak{b}_k(l)\},$$

$k = 1, 2, \dots, d$. (The sequence $i^0 = \Psi = \{i_1 < i_2 < \dots < i_d\}$ can be obtained within this scheme also by setting

$$i_k = \min\{1 \le j \le m : \mathfrak{g}_j \cap \mathfrak{b}_{k-1}(l) \not\subset \mathfrak{r}(l, \mathfrak{b}_{k-1}(l))\},$$

but in this context that is not necessary.) Thus

$$\mathfrak{b}_1(l) = \mathfrak{g}_{i_1}^l, \mathfrak{b}_2(l) = (\mathfrak{g}_{i_2} \cap \mathfrak{g}_{i_1}^l)^l,$$

and so on. One has $\mathfrak{g} = \mathfrak{b}_0(l) \supset \mathfrak{b}_1(l) \supset \mathfrak{b}_2(l) \supset \dots \supset \mathfrak{b}_d(l) = \mathfrak{b}(l)$. In our case here it is easily seen from what we have done that for each $k$,

$$\mathfrak{b}_k(l) = (\operatorname{span}\{Z_{i_1}, Z_{i_2}, \dots, Z_{i_k}\})^l.$$

This gives the pairing of $j_k$ with $i_k$, $1 \le k \le d$, and hence a pairing of the basis elements $Z_{i_k}$ and $Z_{j_k}$, $1 \le k \le d$. For simplicity of notation, set $V_k = Z_{i_k}, W_k = Z_{j_k}, 1 \le k \le d$. Note that by Corollary 1.3, the $V_k$ lie in $\mathfrak{n}$, but the $W_k$ may or may not lie in $\mathfrak{n}$. Of course if $W_k \notin \mathfrak{n}$, then $W_k$ is one of the basis elements of $\mathfrak{h}$. Let $\mathcal{R} = \{1 \le k \le d : W_k \in \mathfrak{n}\}$.

**Lemma 1.4.** *For each $k, 1 \le k \le d$, if $k \notin \mathcal{R}$, then $W_k = A_k$.*

*Proof.* Suppose that $W_k \notin \mathfrak{n}$, that is, $j_k > n$. Since $[A_r, \mathfrak{g}_{i_k}] = (0)$ for $r > k$, then $A_r \in \mathfrak{b}_k(l), k \le r \le d$. On the other hand, $[A_k, \mathfrak{g}_{i_k - 1}] = (0)$

and $A_k \notin \mathfrak{g}^l_{i_k}$. By definition of the algebras $\mathfrak{b}_r(l), 1 \le r \le d$, this implies $A_k \in \mathfrak{b}_{k-1}(l) \sim \mathfrak{b}_k(l)$, hence $W_k = A_k$. ☐

**Lemma 1.5.** *For each $k, 1 \le k \le d, [W_k, V_k] \neq 0, [\mathfrak{g}_{j_k-1}, V_k] = 0, [W_k, \mathfrak{g}_{i_k-1}] = 0$, and $[W_k, V_r] = 0$ for $r \neq k$. Also if $k \in \mathcal{R}$, then $\lambda_{j_k}(A_k) = -1$ and $\lambda_{j_k}(A_r) = 0$ for $r > k$.*

*Proof.* We proceed by induction on $k, 1 \le k \le d$. Suppose $k = 1$. Then by definition of $i_1$ and $j_1$, $[W_1, V_1] \neq 0$, and for every $l \in \Omega, j < j_1, l([Z_j, V_1]) = 0$. Since $\Omega$ is dense in $\mathfrak{n}^*$, this means $[Z_j, V_1] = 0, 1 \le j < j_1$. Suppose that $1 \in \mathcal{R}$. For $j_1 \le j \le n$, we have $[Z_j, V_1] \in R(A_1, \lambda_j(A_1) + \lambda_{i_1}(A_1)) \cap \mathfrak{g}_{i_1-1}$, so that if $[Z_j, V_1] \neq 0$, then $\lambda_j(A_1) = -1$. In particular $\lambda_{j_1}(A_1) = -1$. For $r > 1, 0 \neq [W_1, V_1] \in R(A_r, \lambda_{j_1}(A_r)) \cap \mathfrak{g}_{i_1-1}$ yields $\lambda_{j_1}(A_r) = 0$. But now

$$[W_1, V_r] \in R(A_r, \lambda_{j_1}(A_r) + \lambda_{i_r}(A_r)) \cap \mathfrak{g}_{i_r-1} = R(A_r, 1) \cap \mathfrak{g}_{i_r-1} = (0).$$

Similarly $[W_1, \mathfrak{g}_{i_1-1}] \subset r(A_1, -1) \cap \mathfrak{g}_{i_1-1} = (0)$. On the other hand, if $1 \notin \mathcal{R}$, then $W_1 = A_1$ and $[W_1, V_r] = 0$ if and only if $r > 1$, and $[W_1, \mathfrak{g}_{i_1-1}] = (0)$, by definition of $A_1$. This proves the lemma for the case $k = 1$.

Suppose that $k > 1$, and that the lemma is true for each $h, 1 \le h < k$. Now by definition of $j_k$, for each $l \in \Omega$ and for each $j, 1 \le j \le j_k$, we have real numbers $c_{h,j}(l), 1 \le h < k$, such that

$$Z_j(l) = Z_j + \sum_{1 \le h < k} c_{h,j}(l) W_h \in \mathfrak{g}_{j_k} \cap \mathfrak{b}_{k-1}(l),$$

and again by definition of $j_k$, $l([Z_j(l), V_k]) \neq 0$. But by induction, $[W_h, V_k] = 0, 1 \le h < k$, hence we have $[W_k, V_k] \neq 0$, and for $j < j_k$, $l([Z_j, V_k]) = 0$ for all $l \in \Omega$ so that $[Z_j, V_k] = 0$. If $k \in \mathcal{R}$, then arguing as in the case $k = 1$ we find that $\lambda_{j_k}(A_k) = -1, \lambda_{j_k}(A_r) = 0, r > k$, and $[W_k, \mathfrak{g}_{i_k-1}] \in R(A_k, -1) \cap \mathfrak{g}_{i_k-1} = (0)$. In particular, $[W_k, V_r] = 0$ for $r < k$, since $i_r < i_k$, and for $r > k$, $[W_k, V_r] \in R(A_r, 1) \cap \mathfrak{g}_{i_r-1} = (0)$. If $k \notin \mathcal{R}$, then $W_k = A_k$, and in this case, $[W_k, \mathfrak{n}_{i_k-1}] = 0$ and $[W_k, V_r] = 0$ if and only if $r \neq k$ just because of the definition of $A_k$. This proves the lemma. ☐

**Lemma 1.6.** *If both $k$ and $r$ belong to $\mathcal{R}$, then $[W_k, W_r] = 0$.*

*Proof.* Assume $k > r$, and set $Z = [W_k, W_r]$. It is easily seen from Lemma 1.5 that $Z \in \mathfrak{b}(l)$ (for every $l \in \Omega$), and that $[A_k, Z] = -Z$. Let $i_0 = \min\{1 \le i \le n : Z \in \mathfrak{n}_i\}$, and set

$$a_k(l) = l([A_k, V_k]) / l([W_k, V_k]) = l(V_k) / l([W_k, V_k]), \; l \in \Omega.$$

The function $a_k(l)$ depends only on $l|_{\mathfrak{n}_{i_k}}$, and by Corollary 1.3 and Lemma 1.5, $A_k - a_k(l) W_k$ belong to $\mathfrak{b}(l)$. Set

$$P(l) = l([A_k - a_k(l) W_k, Z_k]) = -l(Z) - a_k(l) l([W_k, Z]).$$

Then $P$ is identically zero on $\Omega$, hence on all of $\mathfrak{n}^*$. Now if $i_0 < i_k$, then $P(l) = -l(Z)$ so $Z = 0$, while if $i_0 > i_k$, then we have $P(l) = -l(Z) + P_0(l)$ where $P_0(l)$ depends only on $l|_{\mathfrak{n}_{i_0-1}}$, so again $Z = 0$. But $i_0 = i_k$ is impossible, since if $i_0 = i_k$, say $Z = cV_k + Z_0$ with $c \neq 0$ and $Z_0 \in \mathfrak{n}_{i_0-1}$, then we find that

$$P(l) = -2cl(V_k) - l(Z_0) - l([W_k, Z_0]) / l([W_k, V_k]) = -2cl(V_k) + P_0(l)$$

where $P_0(l)$ depends only on $l|_{\mathfrak{n}_{i_0-1}}$. If this were the case then $P(l)$ could not be identically zero. Hence $Z = 0$. □

**Proposition 1.7.** *If $k \in \mathcal{R}$, then $[W_k, V_k]$ belongs to the center of $\mathfrak{n}$, $[W_k, Z_j] = 0$ for all $j \neq i_k, j \leq n$, and $\lambda_j(A_k) = 0$ for all $j \neq i_k$ or $j_k$. Moreover, for each $k, 1 \leq k \leq d$, we have $[Z_j, V_k] = 0$ for all $j \neq j_k$, and for any $i \notin e, j \notin e, 1 \leq i, j \leq n$, we have $[Z_j, Z_i] = 0$. In particular, $\mathfrak{n}$ is two-step.*

*Proof.* Let $k \in \mathcal{R}$ and set $Z = [W_k, V_k]$. For any $j \leq n$, $[Z_j, W_k] \in \mathfrak{g}_{j_k-1}, [Z_j, V_k] \in \mathfrak{g}_{i_k-1}$, and so by Lemma 1.5, $[Z_j, Z] = [[Z_j, W_k], V_k] + [W_k, [Z_j, V_k]] = 0$.

Secondly we show that if $k \in \mathcal{R}$, and $j \neq i_k$ or $j_k$, then $\lambda_j(A_k) = 0$. Note that if $j \in i^0$, or if $j < i_k$, then we have $\lambda_j(A_k) = 0$ just by definition of $A_k$. Suppose that $j \in j^0$, say $j = j_r, Z_j = W_r$. We may assume that $r \in \mathcal{R}$, and by Lemma 1.4, we may assume that $r > k$. Now set $Z = [W_r, V_r]$. We have $[A_k, Z] = \lambda_j(A_k) Z$. For $l \in \Omega$, we have $A_k - a_k(l) W_k \in \mathfrak{b}(l)$ as in Lemma 1.6, and $Z \in \operatorname{cent}(\mathfrak{n}) \subset \mathfrak{b}(l)$, so

$$P(l) = l([A_k - a_k(l) W_k, Z])$$

is therefore identically zero. But $P(l) = \lambda_j(A_k) l(Z)$, and since $Z \neq 0$ we must have $\lambda_j(A_k) = 0$. Next suppose that $j \notin e, j > j_k$, and for $1 \leq r \leq d$ set

$$c_r(l) = l([Z_j, V_r]) / l([W_r, V_r]) .$$

Then $Z_j - \sum c_r(l) W$ belongd to $\mathfrak{b}(l)$ Now by Lemma 1.5, $c_r(l) = 0$ unless $j_r < j$, hence $l \to c_r(l)$ depends only on $l|_{\mathfrak{n}_{j-1}}$. Thus

$$P(l) = l\left(\left[A_k - a_k(l) W_k, Z_j - \sum c_r(l) W_r\right]\right)$$

is identically zero, but $P(l) = \lambda_j(A_k) l(Z_j) + P_0(l)$ with $P_0(l)$ depending only on $l|_{\mathfrak{n}_{j-1}}$, which gives $\lambda_j(A_k) = 0$.

Now fix $k, 1 \leq k \leq d$, and $j \neq j_k$. To show that $Z = [Z_j, V_k] = 0$, from above results we may assume $j \notin e$ and $n \geq j > j_k$. Thus $k \in \mathcal{R}$ and so by the above $\lambda_j(A_k) = 0$. But then $Z \in R(A_k, 1) \cap \mathfrak{n}_{i_k-1} = (0)$.

Next we show that $[Z_j, W_k] = 0$ for $k \in \mathcal{R}$ and $j \neq i_k$, $j \leq n$, and here we can assume that $j \notin e$ and $j > j_k$. Note that from the above we have $[Z_j, V_k] = 0$, $1 \leq k \leq d$, hence $Z_j \in \mathfrak{b}(l)$, for every $l \in \Omega$. We claim that $Z = [Z_j, W_k] \in \mathfrak{b}(l)$. Let $1 \leq r \leq d$; if $r \neq k$, then $[V_r, W_k] = [V_r, Z_j] = 0$ by above results and so $[V_r, Z] = 0$. On the other hand $[V_k, Z] = [Z_j, [V_k, W_k]]$, and since $[V_k, W_k] \in \operatorname{cent}(\mathfrak{n})$ we have $l([V_k, Z]) = 0$. This proves the claim. Now $P(l) = l([A_k - a_k(l) W_k, Z])$ is identically zero. But $P(l) = -l(Z) - a_k(l)\, l([W_k, Z])$, and now an argument exactly like that of Lemma 1.6 gives $Z = 0$.

Finally, for $i \notin e, j \notin e, 1 \leq i, j \leq n$, the above shows that both $Z_i$ and $Z_j$ belong to $\mathfrak{b}(l)$ for every $l \in \Omega$. So $l([Z_i, Z_j]) = 0$ for every $l \in \Omega$ and this implies $[Z_i, Z_j] = 0$. This completes the proof. □

Write $\mathcal{R} = \{r_1 < r_2 < \cdots < r_u\}$ and

$$\{1, 2, \ldots, d\} \sim \mathcal{R} = \{s_1 < s_2 < \cdots < s_\nu\}$$

(here $\sim$ denotes "set minus"). For the remainder of the paper we change notation for the basis elements of $\mathfrak{h}$: set $A_h = A_{r_h}, 1 \leq h \leq u$, and write $B_k = A_{s_k}, 1 \leq k \leq \nu$. We will use the coordinates

$$(t, s) = \exp(t_1 A_1) \ldots \exp(t_u A_u) \exp(s_1 B_1) \ldots \exp(s_\nu B_\nu)$$

for $H$ freely, e.g., $q_{B,G}(t, s), \Lambda_j(t, s)$, etc.

Let us summarize what we know about the structure of $G$. Set $X_h = W_{r_h}, Y_h = V_{r_h}, 1 \leq h \leq u$.

**Theorem 1.8.** *Let $G$ be the semi-direct product $NH$, with $N$ nilpotent and normal in $G$, and with $H$ abelian, $\operatorname{Ad} H$ consisting of semi-simple transformations. If $H \backslash G$ is finite multiplicity, then $N$ is two-step. Moreover, there are elements $X_1, X_2, \ldots, X_u, Y_1, Y_2, \ldots, Y_u$ in $\mathfrak{n}$ such that*
(i) $[X_k, Y_r] = 0$ *if and only if* $r \neq k$, *and* $[X_k, Y_k]$ *is central in* $\mathfrak{n}, 1 \leq k \leq u$,
(ii) *for every* $r, k, [X_k, X_r] = [Y_k, Y_r] = 0$,
(iii) $\mathfrak{n} = \operatorname{cent}(\mathfrak{n}) + \operatorname{span}\{X_1, X_2, \ldots, X_u, Y_1, Y_2, \ldots, Y_u\}$, *and*
(iv) *each* $X_k$ *and* $Y_k$ *is an eigenvector for* $\operatorname{Ad}(h), h \in H$.

It is clear that for each $l \in \Omega, \mathfrak{n}/\ker(l) \cap \operatorname{cent}(\mathfrak{n})$ is Heisenberg, and

$$\begin{aligned} \mathfrak{p} = \mathfrak{b}(l) \cap \mathfrak{n} &= \operatorname{span}\{Z_i : 1 \leq i \leq n, i \notin j^0\} = \\ &= \operatorname{cent}(\mathfrak{n}) + \operatorname{span}\{Y_1, Y_2, \ldots, Y|_u\} \end{aligned}$$

is an abelian ideal in $\mathfrak{g}$ (and of course a polarization at $p(l) = l|_{\mathfrak{n}}$).

It follows that

$$\mathfrak{b}(l) = \mathfrak{p} + \operatorname{span}\{A_1 + a_1(l) X_1, A_2 + a_2(l) X_2, \ldots, A_u + a_u(l) X_u\},$$

where the $a_h(l)$ are defined as in Lemma 1.6. Each $A_k$ commutes with $\mathrm{cent}(\mathfrak{n})$ and satisfies $[A_k, Y_k] = Y_k$, $[A_k, X_k] = -X_k$, and $[A_k, X_r] = [A_k, Y_k] = 0$, $r \neq k$. In particular $B(l)$ is a semi-direct product of $P = \exp(\mathfrak{p})$ with a vector group $W(l)$ of dimension $u$.

It is immediate from the above that there is a *single* subset $\Gamma$ which is a cross-section for all of the coset spaces $B(l)\backslash G$.

**Corollary 1.9.** *For* $(x_1, x_2, \ldots, x_u) \in \mathbb{R}^u$ *and* $(s_1, s_2, \ldots, s_\nu) \in \mathbb{R}^\nu$, *set* $\gamma(x,s) = \exp(x_1X_1)\ldots\exp(x_uX_u)\exp(s_1B_1)\ldots\exp(s_\nu B_\nu)$. *For any* $l \in \Omega$, *the set* $\Gamma = \{\gamma(x,s) : (x,s) \in \mathbb{R}^u \times \mathbb{R}^\nu\}$ *is a cross-section for* $B(l)\backslash G$.

**Remark 1.10.** As mentioned in the introduction, our class of homogeneous spaces $H\backslash G$ has also been studied in [**13**]. There the irreducibles are constructed by means of the Mackey machine, and the spectrum of the quasi-regular representation $\tau$ is described by "Mackey parameters". Let $G$ be as in the hypothesis of Theorem 1.8, but without the assumption that $H\backslash G$ is finite multiplicity. Let $\sigma \in \hat{N}$, and let $H_\sigma$ be the stabilizer in $H$ of $\sigma$. One of the main ideas of [**13**] is to choose $f \in \mathfrak{n}^*$, belonging to the coadjoint orbit corresponding to $\sigma$, such that $H_\sigma$ coincides with the stabilizer $H(f)$ of $f$ in $H$ [**13**, Theorem 3.2]; such a linear functional is said to be *aligned.* Then the natural map $\alpha : H_\sigma \to \mathrm{Sp}(\mathfrak{n}/\mathfrak{n}(f))$ is considered. A result of this (though not explicitly stated there) is that $H\backslash G$ has uniform multiplicity $2^u$ if

$$\dim(\alpha(H_\sigma)) = \dim(\mathfrak{n}/\mathfrak{n}(f))/2 \tag{1.1}$$

holds for generic $\sigma \in \hat{N}$, where $u = \dim(\alpha(H_\sigma))$.

On the other hand, suppose that $H\backslash G$ is finite multiplicity, and set $\mathfrak{h}_0 = \mathrm{span}\{A_k : k \in \mathcal{R}\}$, $H_0 = \exp(\mathfrak{h}_0)$. Let $l \in \Omega_0 \subset \mathfrak{g}^*$, and let $\sigma \in \hat{N}$ be the irreducible representation (equivalence class) corresponding to the $N$-orbit of $f = p(l)$. It is easily seen that $\mathfrak{n}+\mathfrak{b}(l) = \mathfrak{n}+\mathfrak{n}^l = \mathfrak{n}+\mathfrak{b}_0$, and hence $H_0 = H_\sigma$. As we said above, a natural choice for $l \in \Omega_0$ when using the Mackey machine is one for which $f$ is aligned, and in this setting that means $H_0 = H(f)$, hence $f(Y_k) = f(X_k) = 0, 1 \leq k \leq u$. Thus the aligned linear functionals which are used in [**13**] are *not* in $\Omega$. In the present work we shall construct the irreducibles as monomial representations by means of polarizations, and we shall use generic $H$-orbit parameters for the concrete Plancherel formula. For linear functionals in generic $H$-orbits we have $f(Y_k) \neq 0$. Thus for $H$-orbit parameters we use linear functionals in $\Omega$, while for Mackey parameters one uses linear functionals that are not in $\Omega$.

There is a strong parallel between the present work and the theory of Gelfand pairs $(H,N)$ where $N$ is nilpotent and $H$ acts on $N$ by automorphisms, but now $H$ is a compact Lie group [**1**, **2**]. To begin with, in [**1**] it is shown that if $(H, N)$ is a Gelfand pair, then $N$ is two-step. Secondly, we can relate our situation to a result of Carcano [**4**] concerning Gelfand pairs. Let $\sigma \in \hat{N}$; realizing $\sigma$ in a Hilbert space $\mathcal{H}_\sigma$, the Weil representation associated with $\sigma$ is a representation $\omega_\sigma$ of $H_\sigma$ acting in $\mathcal{H}_\sigma$ that "extends" $\sigma$. It is well-known that in the setting of the present paper, $\omega_\sigma$ is quasi-equivalent to the regular representation of $H_\sigma$ and has uniform multiplicity, and that (1.1) holds if and only if $\omega_\sigma$ has finite multiplicity, in which case that multiplicity is $2^u$ [**13**, Prop. 3.4]. Thus from the work of [**13**] and the present work, we can say the following, which parallels the above-mentioned result of Carcano. If $\omega_\sigma$ has finite multiplicity for almost every $\sigma \in \hat{N}$ (with respect to Plancherel measure), then $H\backslash G$ is finite multiplicity (and in this case both multiplicities are $2^u$). Conversely, if $H\backslash G$ is finite multiplicity, then from our structural results on $N$ it is easily seen that for *every* $\sigma \in \hat{N}$, the multiplicity of $\omega_\sigma$ is $2^{u'}$, where $u' \leq u$.

We conclude this section with the observation that Theorem 1.8 provides a coordinate-free description of all nilpotent groups that can arise in the class of homogeneous spaces we are considering. For $X \in \mathfrak{n}$ let $\mathfrak{c}(X)$ be the centralizer of $X$ in $\mathfrak{n}$. If $\mathfrak{n}$ is as in the Theorem 1.8, then $\dim(\mathfrak{n}/\mathfrak{c}(X)) \leq 1$, for every $X \in \mathfrak{n}$. On the other hand if $\mathfrak{n}$ is a nilpotent Lie algebra such that $\dim(\mathfrak{n}/\mathfrak{c}(X)) \leq 1$, for every $X \in \mathfrak{n}$, then $\mathfrak{n}$ is two-step (or abelian) and there are elements $X_1, X_2, \ldots, X_u, Y_1, Y_2, \ldots, Y_u$ in $\mathfrak{n}$ that satisfy conditions (i), (ii), (iii) of the theorem. If $\mathfrak{n}$ does have this form, it is clear that there is $H$ as in the theorem such that $H\backslash NH$ is finite multiplicity. Hence we have the following.

**Corollary 1.10.** *Let $N$ be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$. Then the following are equivalent.*
(i) *There is a vector subgroup $H$ of* $\mathrm{Aut}(N)$ *whose derived group in* $\mathrm{Aut}(\mathfrak{n})$ *consists of semi-simple transformations and so that if $G$ is the resulting semi-direct product, then $H\backslash G$ is finite multiplicity.*
(ii) *For every $X \in \mathfrak{n}, \dim(\mathfrak{n}/\mathfrak{c}(X)) \leq 1$.*

We remark that the above class of two-step nilpotent groups is very different from the class of nilpotent groups known as Heisenberg-type (or H-type) groups [**12**] (that occur naturally in the study of Gelfands pairs.) A two-step nilpotent Lie algebra is H-type if $\dim(\mathfrak{n}/\mathfrak{c}(X)) = \dim(\mathrm{cent}(\mathfrak{n}))$ holds for every $X \notin \mathrm{cent}(\mathfrak{n})$. Hence if $\mathfrak{n}$ is H-type and satisfies (ii) above then $\mathfrak{n}$ is a Heisenberg Lie algebra. There seems to be no simple description of the class of two-step $N$ that can arise in a Gelfand pair $(H, N)$.

## 2. Smooth vectors and generalized vectors.

Given a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$, let $dk$ be a right Haar measure on $K = \exp(\mathfrak{k})$. Let $\Delta_K$ be the modular function of $K$ (the derivative of right Haar measure with respect to a left Haar measure). In particular, one can take $\Delta_G(g) = \prod_{1 \leq i \leq n} \Lambda_i(g)$. For exponential solvable groups $G$, it is well know that there is a positive character $q$ on $G$ such that $q(k) = \Delta_K(k)/\Delta_G(k), k \in K$, and that the space $K \backslash G$ carries a relatively invariant measure $d\gamma$ with modulus $q^{-1}$, that is, a measure $d\gamma$ which satisfies

$$\int_{K\backslash G} f(\gamma g)\, d\gamma = \int_{K\backslash G} f(\gamma)\, q(g)^{-1}\, d\gamma$$

for compactly-supported $f$ on $K\backslash G$. We want to make natural choices of $dk, d\gamma$ for $K = B(l)$, but before addressing that issue, we make some more general comments. Let $\chi$ be a unitary character of $K$. Let $C^\infty(G, K, \chi)$ denote the space of smooth functions $f$ on $G$ which satisfy $f(kg) = \chi(k) f(g)$, and let $C_c^\infty(G, K, \chi)$ be the subspace of $C^\infty(G, K, \chi)$ consisting of those $f$ which are compactly supported mod $K$. The Hilbert space $L^2(G, K, \chi)$ is the completion of $C_c^\infty(G, K, \chi)$ under the norm $\|f\|_2 = \left[\int_{K\backslash G} |f(\gamma)|^2\, d\gamma\right]^{1/2}$. Let $\pi_\chi$ be the irreducible representation induced from the character $\chi$ of $K$, so that $\pi_\chi$ acts in the space $\mathcal{H}_\chi = L^2(G, K, \chi)$ by the formula

$$\pi_\chi(s) f(g) = f(gs)\, q(s)^{1/2}.$$

Let $\mathcal{H}_\chi^\infty$ be the Frechet space of smooth vectors for $\pi_\chi$ in $\mathcal{H}_\chi$, and let $\mathcal{H}_\chi^{-\infty}$ denote its antidual. It is well-known that $\mathcal{H}_\chi^\infty \subset C^\infty(G, K, \chi)$ [**20**].

Fix $l \in \Omega$, and let $B = B(l) = \exp(\mathfrak{b}(l))$. We have seen that $B = PW$, where $P = \exp(\mathfrak{p})$ is the polarization in $\mathfrak{n}$ at $p(l)$, and $W = \exp(\mathfrak{w}(l))$ is an abelian group of dimension $u$. A basis for $\mathfrak{p}$ is $\{Z_j : 1 \leq j \leq n, j \notin j^0\}$ and for $\mathfrak{w}(l)$ is $\{A_k - a_k(l) X_k : 1 \leq k \leq u\}$. Letting $dp$ and $dw$ be the Lebesgue measures on $P$ and $W$ resp. obtained from these coordinates, a natural choice for right Haar measure on $B$ is just $db = dp\,dw$. Recall that we have the index set $\mathcal{R} = \{1 \leq k \leq d : j_k \leq n\}$. Define a positive character $q_{B,G}$ on $G$ by

$$q_{B,G} = \prod_{k \in \mathcal{R}} \Lambda_{j_k}^{-1}(g).$$

Then for $Y \in \mathfrak{b}$,

$$q_{B,G}(\exp Y) = e^{-\operatorname{tr} \operatorname{ad}_{\mathfrak{g}/\mathfrak{b}} Y} = \Delta_B(\exp Y)/\Delta_G(\exp Y).$$

Note that this is not the only choice for $q_{B,G}$ that we could have taken (one can extend $\Delta_B/\Delta_G$ in many ways), and the choice of $q_{B,G}$ affects the relatively invariant measure $d\gamma$ as well as the growth properties of functions in the resulting space $\mathcal{H}_\chi$. The above choice is natural and more importantly, will result in manageable growth properties.

Recall that we have the coordinates $\gamma(x,s)$ on $B\backslash G$ given in the previous section.

**Lemma 2.1.** *Let $d\gamma$ be the measure on $B\backslash G$ defined by*

$$\int_{B\backslash G} f(\gamma)\, d\gamma = \int_{\mathbb{R}^d} f(\gamma(x,s))\, q_{B,G}(\gamma(0,s))\, dxds$$

*where $dxds$ denotes Lebesgue measure on $\mathbb{R}^u \times \mathbb{R}^\nu$. Then $d\gamma$ is relatively invariant with modulus $q_{B,G}^{-1}$.*

*Proof.* Let $g \in G$, and define the diffeomorphism $T_g : \mathbb{R}^u \times \mathbb{R}^\nu \to \mathbb{R}^u \times \mathbb{R}^\nu$ by $B\gamma(x,s)\,g = B\gamma(T_g(x,s))$. If $g \in P$, then normality of $P$ gives that $T_g = \mathrm{Id}$. Let $t$ be any real number; we compute $T_g$ in the cases (a) $g = \exp(tB_h)$ for some $h, 1 \le h \le \nu$, (b) $g = \exp(tX_k)$, for some $k, 1 \le k \le u$, and (c) $g = \exp(t(A_k - a_k(l)X_k))$, for $1 \le k \le u$.

(a) Here we have

$$\gamma(x,s)\,g = \exp(x_1X_1)\dots\exp(x_uX_u) \times \\ \times \exp(s_1B_1)\dots\exp([t+s_h]B_h)\dots\exp(s_\nu B_\nu)$$

hence $T_g(x,s) = (x, s_1, \dots, t+s_h, \dots, s_\nu)$.

(b) In this case

$$\gamma(x,s)\,g = \exp(x_1X_1)\dots\exp([e^a t + x_k]X_k)\dots\exp(x_uX_u) \times \\ \times \exp(s_1B_1)\dots\exp(s_\nu B_\nu)$$

where $a = \sum_r \lambda_{j_k}(B_r)$, so $T_g(x,s) = (x_1, x_2, \dots, e^a t + x_k, \dots, x_u, s)$.

(c) Here $g = x(t)\exp(tA_k)$ and $\exp(tA_k) = gy(t)$ where

$$x(t) = \exp\left(a_k(l)\left(e^{-t}-1\right)X_k\right)$$

and

$$y(t) = \exp\left(a_k(l)\left(e^t - 1\right)X_k\right).$$

We have

$$\begin{aligned}\gamma(x,s)\exp(tA_k) &= \exp(tA_k) \times \\ &\times \gamma\left((x_1,\dots,e^t x_k,\dots,x_u),s\right) = \\ &= gy(t)\gamma\left((x_1,\dots,e^t x_k,\dots,x_u),s\right) = \\ &= g\gamma\left((x_1,\dots,e^t x_k + a_k(l)\left(e^t-1\right),\dots,x_u),s\right)\end{aligned}$$

hence

$$\begin{aligned}\gamma(x,s)\,g &= \gamma(x,s)\exp(t(A_k - a_k(l)X_k)) = \gamma(x,s)\,x(t)\exp(tA_k) = \\ &= \gamma\left(\left(x_1,\dots,x_k + a_k(l)\,e^{a(s)}\left(e^{-t}-1\right),\dots,x_u\right),s\right)\exp(tA_k) = \\ &= \exp(tA_k)\,\gamma\left(\left(x_1,\dots,e^t x_k + a_k(l)\,e^{a(s)}\left(1-e^t\right),\dots,x_u\right),s\right) = \\ &\qquad = g\gamma\left(\left(x_1,\dots,e^t x_k + a_k(l)\left(1-e^t\right)\left(e^{a(s)}-1\right),\dots,x_u\right),s\right).\end{aligned}$$

Thus

$$T_g(x,s) = \left(\left(x_1,\dots,e^t x_k + a_k(l)\left(1-e^t\right)\left(e^{a(s)}-1\right),\dots,x_u\right),s\right).$$

To finish the proof one need only check that in each case,

$$q_{B,G}\left(\gamma\left(T_g(x,s)\right)\right)J_g(x,s) = q_{B,G}(g),$$

where $J_g(x,s)$ is the Jacobian determinant of $T_g(x,s)$. We leave this to the reader. □

Let $\chi = \chi_l$ be the character of $B$ defined by $\chi_l(\exp Y) = e^{il(Y)}, Y \in B$. Set $\pi_l = \pi_\chi$, $\mathcal{H}_l = \mathcal{H}_\chi$, etc. Since $BH$ is an open subset of $G$, $H$ may be regarded as an open subset of $B\backslash G$. Using the coordinates $(t,s)$ for $H$ and the coordinates $(x,s)$ for $B\backslash G$, we compute that the map $\varphi : H \to \mathbb{R}^u \times \mathbb{R}^\nu$ defined by

$$\varphi(t,s) = \left(a_1(l)\left(e^{t_1}-1\right),\dots,a_u(l)\left(e^{t_u}-1\right),s\right)$$

satisfies $B\gamma(\varphi(t,s)) = B(t,s)$.

We want to construct an appropriately covariant generalized vector for $\pi_l$, that is, an element of

$$\left(\mathcal{H}_l^{-\infty}\right)^{q_{H,G}^{-1/2}} = \left\{\beta \in \mathcal{H}_l^{-\infty} : \pi_l(h)\,\beta = q_{H,G}(h)^{-1/2}\beta, \text{for every } h \in H\right\}.$$

Following Fujiwara and Yamagami [**11**], and Lipsman [**16**], we define *formally*

$$\beta_l(f) = \int_H \bar{f}(h)\,q_{B,G}^{1/2}q_{H,G}^{-1/2}\chi_f(h)\,dh, f \in \mathcal{H}_l^\infty. \tag{2.1}$$

It is not at all obvious that (2.1) is convergent for all $f \in \mathcal{H}_l^\infty$. Note for example that if $f \in C_c^\infty(B,G,\chi)$, then $f|_H$ may not be compactly supported (if $\mathcal{R} \neq \emptyset$, then the image of $H$ in $B\backslash G$ is not closed). Hence it is not immediate that (2.1) is finite even for $f \in C_c^\infty(G, B, \chi)$. However we shall prove the following.

**Theorem 2.2.** *The integral* (2.1) *is absolutely convergent for every* $f \in \mathcal{H}_l^\infty$, *and* $\beta_l$ *is continuous on* $\mathcal{H}_l^\infty$.

This result is a generalization of the proof of convergence in [**18**], in which it is assumed the $N$ is abelian. We have seen that $N$ is abelian if and only if $\mathcal{R} = \emptyset$, so in what follows we assume that $\mathcal{R} \neq \emptyset$. To prove the result we need information about the growth properties of $f$ on $H$. For simplicity of notation we shall write $f(t,s)$ for $f|_{H(t,s)}$. We make a couple of observations. First, $f \in \mathcal{H}_l$ implies that $f|_H$ is square integrable on $H$ with respect to the measure $q_{B,G}(t,s)\,dtds$, or in other words, $(f|_H)(q_{B,G})^{1/2} \in L^2(H)$, and $\left\|(f|_H)(q_{B,G})^{1/2}\right\|_{L^2(H)} \leq \|f\|_{\mathcal{H}_l}$. Second, for $f \in \mathcal{H}_l^\infty$, the differential operators $\pi(Z)$, $Z \in \mathbf{U}(\mathfrak{g}_c)$ (=the enveloping algebra of the complexification $\mathfrak{g}_c$ of $\mathfrak{g}$), act on $f|_H$, and $\left\|(q_{B,G})^{1/2}\pi(Z)(f|_H)\right\|_{L^2(H)} \leq \|\pi(Z)f\|_{\mathcal{H}_l}$ for every $Z \in \mathbf{U}(\mathfrak{g}_c)$. We can compute $\pi(Z)$ as an operator on $\varphi(H)$ (in the $G$-coordinates $(x,s)$ ) or as an operator on $H$ (in the $H$-coordinates $(t,s)$). In the latter coordinates the algebra $\pi(\mathbf{U}(\mathfrak{g}_c))$ is more easily described and provides us with more useful information about $\mathcal{H}_l^\infty$.

First we set some notation that will be convenient. For $J \subset \{1,2,\dots,n\}$, set $\Lambda_J(t,s) = \prod_{j\in J}\Lambda_j(t,s)$. Recall we have written $\mathcal{R} =\{r_1 < r_2 < \cdots < r_u\}$; set $q_k(t,s) = \Lambda_{j_{r_k}}(t,s), 1 \leq k \leq u$, and for $K \subset \{1,2,\dots,u\}$, set $q_K(t,s) = \prod_{k\in K} q_k(t,s)$ (so that for $K = \{1,2,\dots,u\}, q_K = q_{B,G}^{-1}$). Denote by $C_0(H)$ the space of continuous functions on $H$ that vanish at infinity. We recall (a weak form of) a standard regularity result that if $f$ and its partial derivatives of all orders belong to $L^2(H)$, then $f \in C_0(H)$. In fact if we choose a fixed constant-coefficient partial differential operator $D$ on $H$ such that the reciprocal of its "symbol" $P = \hat{D}$ belongs to $L^2(H)$, then we have $\|f\|_\infty \leq \|1/P\|_{L^2(H)}\|Df\|_{L^2(H)}$.

**Lemma 2.3.** *Let* $f \in \mathcal{H}_l^\infty$, *let* $J \subset \{1,2,\dots,n\}$, $J' \subset \{1,2,\dots,n\} \sim j^0$, *and let* $K \subset \{1,2,\dots,u\}$. *Set* $D_K = \prod_{k\in K}\partial_{t_K}$. *Then*
(a) *the function*

$$\phi(t,s) = q_{B,G}^{1/2}(t,s)\,\Lambda_J(0,s)\,\Lambda_{J'}(0,s)\,f(t,s) \in C_0(H),$$

*and there is* $V \in \mathbf{U}(\mathfrak{g}_c)$ *and a constant* $M$, *depending only on* $J$ *and* $J'$, *such that* $\|\phi\|_\infty \leq M\|\pi(V)f\|_{\mathcal{H}_l}$ *and*
(b) *the function*

$$\phi(t,s) = q_{B,G}^{1/2}(t,s)\,\Lambda_J(0,s)\,\Lambda_{J'}(t,s)\,q_K(t,s)\,D_K f(t,s) \in C_0(H),$$

*and there is $W \in \mathbf{U}(\mathfrak{g}_c)$ and a constant $M$, depending only on $J, J'$, and $K$, such that $\|\phi\|_\infty \leq M \|\pi(W) f\|_{\mathcal{H}_j}$.*

*Proof.* We begin by computing $\pi(Z)$ as an operator on $H$, for certain $Z \in \mathfrak{g}$. First, consider a basis element $Z_j$ which belongs to the center of $\mathfrak{n}$. We have $Z_j \in \mathfrak{p}$ and by Proposition 1.7, $\lambda_j(A_k) = 0, 1 \leq k \leq u$, so one finds that $\pi(Z_j) = i\Lambda_j(t,s) = i\Lambda_j(0,s)$. Next, let $r = r_k \in \mathcal{R}$. If $j = i_r$, then $\pi(Z_j) = i\Lambda_j(t,s) = e^{t_k}$ and $\Lambda_j(0,s) = \Lambda_j(0,0) = 1$. Suppose that $j = j_r$. Then $[A_h, [X_k, Y_k]] = 0, 1 \leq h \leq u$, and since $[B_h, Y_k] = 0, [B_k, [X_k, Y_k]] = \lambda_{j_r}(B_h)[X_k, Y_k], 1 \leq h \leq \nu$. Hence $\pi([X_k, Y_k]) = i\Lambda_{j_r}(0,s)$. Thus by taking the appropriate element $U$ of $\mathbf{U}(\mathfrak{g}_c)$, we have $\pi(U) = \Lambda_J(0,s)\Lambda_{J'}(t,s)$ as an operator on $H$ and

$$q_{B,G}^{1/2}(t,s)\Lambda_J(0,s)\Lambda_{J'}(t,s) f(t,s) \in L^2(H).$$

Now $\pi(A_k) = \partial_{t_k}, 1 \leq k \leq u$ and $\pi(B_k) = \partial_{s_k}, 1 \leq k \leq \nu$, so if $D_\alpha$ is any mixed partial of order $|\alpha|$, then $D_\alpha \in \pi(\mathbf{U}(\mathfrak{h}))$ and so

$$q_{B,G}^{1/2}(t,s)\Lambda_J(0,s)\Lambda_{J'}(t,s) D_\alpha f(t,s)$$

belongs to $L^2(H)$ also. But since the function $q_{B,G}^{1/2}(t,s)\Lambda_{J'}(t,s)\Lambda_J(0,s)$ involves only exponentials in $t$ and $s$, then $D_\alpha\left(q_{B,G}^{1/2}(t,s)\Lambda_J(0,s) f(t,s)\right)$ can be written as a sum of terms of the form

$$c_\beta q_{B,G}^{1/2}(t,s)\Lambda_J(0,s)\Lambda_{J'}(t,s) D_\beta f(t,s), |\beta| < |\alpha|, c_\beta \in \mathbb{R}.$$

Hence all partials of $\phi(t,s) = q_{B,G}^{1/2}(t,s)\Lambda_J(0,s)\Lambda_{J'}(t,s) f(t,s)$ also belong to $L^2(H)$, and so $\phi \in C_0(H)$. Now choose $Z \in \mathbf{U}(\mathfrak{h}_c)$ for which the reciprocal of the Fourier transform $\hat{\pi(Z)}$ belongs to $L^2(H)$ and we have $\pi(Z)\phi(t,s) = q_{B,G}^{1/2}(t,s)\sum_{|\beta|<|\alpha|}\pi(Z_\beta) f(t,s)$ where in this case $\alpha$ is the order of $\pi(Z)$. Hence

$$\|\phi\|_\infty \leq M \|\pi(Z)\phi\|_{L^2(H)} = M \left\| \sum_{|\beta|<|\alpha|} \pi(Z_\beta) f(t,s) \right\|_{\mathcal{H}_l}.$$

As for the function (b), we compute that for $1 \leq k \leq u, \pi(X_k) = q_k(t,s)\partial_{t_k}$. Now by Lemma 1.5 and Proposition 1.7, $q_k(t,s) = e^{-t_k} q_k(0,s), 1 \leq k \leq u$, so if $X_K = \prod_{k\in K} X_k$, then $\pi(X_K) = q_K(t,s) D_K$, and we have

$$q_{B,G}^{1/2}(t,s)\Lambda_J(0,s)\Lambda_{J'}(t,s) q_K(t,s) D_K f(t,s) \in L^2(H).$$

Now in a similar manner as before we find that

$$q_{B,G}^{1/2}(t,s)\Lambda_J(0,s)\Lambda_{J'}(t,s) q_K(t,s) D f(t,s)$$

belongs to $C_0(H)$ and the indicated estimate is obtained. $\square$

Note that by taking $K = \{k\}$, Lemma 2.3 tells us something about the growth of $\partial_{t_k} f$ as $T_k \to -\infty, 1 \le k \le u$, (in particular, if all other variables are held constant, then $\partial_{t_k} f(t,s) \to 0$ rapidly as $t \to -\infty$). In the next lemma we derive information about the growth of $f$ itself as $t_k \to -\infty$.

**Lemma 2.4.** *Let $f \in \mathcal{H}_l^\infty$, let $P \subset \{1,2,\dots,u\}$, and let*

$$U(P) = \{(t,s) \in H : \log(q_k(t,s)) > 0, \textit{for every} \, k \in P\}.$$

*Let $\tilde{P} = \{1,2,\dots,u\} \sim P$ and let $K$ be any subset of $P, \tilde{K} = P \sim K$. Write $K = \{k_1,k_2,\dots,k_a\}, \tilde{K} = \{h_1,h_2,\dots,h_b\}$, and write $t \in \mathbb{R}^u$ as $t = (t_K, t_{\tilde{K}}, t_{\tilde{P}})$ (with the obvious meaning). Finally write $Q_{\tilde{K}}(s) = (\log(q_{h_1}(0,s)), \log(q_{h_2}(0,s)), \dots, \log(q_{h_b}(0,s)))$.*

*Then for any $J \subset \{1,2,\dots,n\}$, $J' \subset \left\{i_{r_k} : k \in \tilde{P}\right\}$, the function*

$$\phi(t,s) = \Lambda_J(0,s)\, \Lambda_{J'}(t,s)\, q_{\tilde{P}}(t,s)^{-1/2} f((t_K, Q_{\tilde{K}}(s), t_{\tilde{P}}), s)$$

*is bounded on $U(P)$. Moreover, there is a finite set of positive constants $\{M_1, M_2, \dots\}$, and elements $\{W_1, W_2, \dots\}$ in $\mathbf{U}(\mathfrak{g}_c)$, depending only on $J$, $J'$ and $K$, such that*

$$\sup_{U(P)} |\phi(t,s)| \le \sum_\beta M_\beta \left\|\pi(W_\beta) f\right\|_{\mathcal{H}_l}.$$

*Proof.* Note that if $P = \emptyset$, then $t = t_{\tilde{P}}$ and $q_{\tilde{P}}^{-1/2} = q_{B,G}^{1/2}$, so in this case we are done by Lemma 2.3. Assume that $P \neq \emptyset$. We proceed by induction on $a = \#(K)$. If $a = 0$, then

$$\phi(t,s) = \Lambda_J(0,s)\, \Lambda_{J'}(t,s)\, q_{B,G}(t,s)^{1/2} f(t,s)\,|_{t_k = \log(q_k(0,s)), k \in P}$$

so again by Lemma 2.3 we are done. Suppose that $a > 0$, and that the lemma holds for all $K'$ with $\#(K') < a$. Now for each $k = 1,2,\dots,u$, $q_k(t,s) = e^{-t_k} q_k(0,s)$ so $\log(q_k(0,s)) > 0$ means $t_k < \log(q_k(0,s))$. For each $(t,s) \in U(P)$ let $E = E(t,s)$ be the subset of $\mathbb{R}^a$ defined by

$$E(t,s) = \{\tau \in \mathbb{R}^a : t_{k_\alpha} < \tau_\alpha < \log(q_{k_\alpha}(0,s)), 1 \le \alpha \le a\}$$

and set $D_K = \prod_{k \in K} \partial_{t_k}$.

Replacing $t_K = (t_{k_1}, t_{k_2}, \dots, t_{k_a})$ by $\tau = (\tau_1, \tau_2, \dots, \tau_a)$ in $f((t_K, Q_{\tilde{K}}(s), t_{\tilde{P}}), s)$ and integrating $D_K f$ over $E$, repeated application of the fundamental theorem of calculus gives

$$\int_E D_K f((\tau, Q_{\tilde{K}}(s), t_{\tilde{P}}), s)\, d\tau = \sum (-1)^{\#(K \sim K')} f((t_{K'}, Q_{\tilde{K}'}, t_{\tilde{P}}), s)$$

where the sum is taken over all subsets $K'$ of $K$. Now multiply both sides of the above by $\Lambda_J(0,s)\,\Lambda_{J'}(t,s)\,q_{\tilde P}(t,s)^{-1/2}$, and we get

$$
\begin{aligned}
&|\phi(t,s)| = \\
&= \left|\int_E \Lambda_J(0,s)\,\Lambda_{J'}(t,s)\,q_{\tilde P}(t,s)^{-1/2}\,D_K f\left(\left(\tau, Q_{\tilde K}(s), t_{\tilde P}\right), s\right)\,d\tau\right| + \\
&\qquad + \sum_{K'\neq K}\left|\Lambda_J(0,s)\,\Lambda_{J'}(t,s)\,q_{\tilde P}(t,s)^{-1/2}\,f\left(\left(t_{K'}, Q_{\tilde K'}, t_{\tilde P}\right), s\right)\right|.
\end{aligned}
$$

Let $g_{K'}$ be a term in the right hand sum with $K'\neq K$. Then by induction, there are finitely many constants $M_{\beta,K'}$ and elements $W_{\beta,K'}\in\mathbf{U}(\mathfrak{g}_c)$ depending only on $J$, $J'$, and $K'$, such that

$$
\sup_{U(P)}|g_{K'}(t,s)| \le \sum_\beta M_{\beta,K'}\,\|\pi(W_{\beta,K'})\,f\|_{\mathcal{H}_l}.
$$

Therefore it remains to show that the function

$$
I(t,s) = \int_E \Lambda_J(0,s)\,\Lambda_{J'}(t,s)\,q_{\tilde P}(t,s)^{-1/2}\,D_K f\left(\left(\tau, Q_{\tilde K}(s), t_{\tilde P}\right), s\right)\,d\tau
$$

is bounded on $U(P)$ in a similar way. To see this, note that

$$
\begin{aligned}
q_{B,G}\left(\left(\tau, Q_{\tilde K}(s), t_{\tilde P}\right), s\right) &= q_K(\tau,s)^{-1}\,q_{\tilde P}(t,s)^{-1} \text{ and}\\
q_{B,G}\left(\left(\tau, Q_{\tilde K}(s), Q_{\tilde P}(s)\right), s\right) &= q_K(\tau,s)^{-1}
\end{aligned}
$$

where $Q_{\tilde P}(s)$ means we have replaced $t_k$ by $\log(q_k(0,s))$ for all $k\in\tilde P$. Hence

$$
\begin{aligned}
&\Lambda_J(0,s)\,\Lambda_{J'}(t,s)\,q_{\tilde P}^{-1/2} D_K f\left(\left(\tau, Q_{\tilde K}(s), t_{\tilde P}\right), s\right) = \\
&\Big[q_{B,G}^{1/2}\left(\left(\tau, Q_{\tilde K}(s), t_{\tilde P}\right), s\right)\Lambda_J(0,s)\,\Lambda_{J'}(t,s)\,q_K(\tau,s)\,D_K \times \\
&\qquad \times f\left(\left(\tau, Q_{\tilde K}(s), t_{\tilde P}\right)\right)\Big]\,q_{B,G}^{1/2}\left(\left(\tau, Q_{\tilde K}(s), Q_{\tilde P}(s)\right), s\right).
\end{aligned}
$$

But we can apply Lemma 2.4 to the function inside the brackets above, and in so doing obtain $M>0$ and $W\in\mathbf{U}(\mathfrak{g}_c)$ such that

$$
\begin{aligned}
I(t,s) &\le M\,\|\pi(W)\,f\|_{\mathcal{H}_l}\int_E q_{B,G}^{1/2}\left(\left(\tau, Q_{\tilde K}(s), Q_{\tilde P}(s)\right), s\right)\,d\tau = \\
&= M\,\|\pi(W)\,f\|_{\mathcal{H}_l}\int_E q_K(\tau,s)^{-1}\,d\tau.
\end{aligned}
$$

But it is easily seen that $\int_E q_K(\tau,s)^{-1}\,d\tau \le 1$ on $U(P)$, and this proves the lemma. □

*Proof of Theorem* 2.2. Let $S = \{1 \leq j \leq n : \Lambda_j(t,s) \neq 1 \text{ for some } (t,s) \in H\}$. For each $j \in S$, let $\varepsilon_j$ be a choice of sign, $\varepsilon_j = \pm 1$. Set

$$U_\varepsilon = \{(t,s) \in H : \log \Lambda_j (t,s) \varepsilon_j > 0, \text{for every } j \in S\}.$$

(Of course some of the $U_\varepsilon$ may be empty). $H$ is the disjoint union of the sets $U_\varepsilon$, and for each $\varepsilon$, set $P = \left\{1 \leq k \leq u : \varepsilon_{j_{r_k}} = +1\right\}$ so that $U_\varepsilon \subset U(P)$.

Fix $U_\varepsilon \neq \emptyset$; we need to show that $f q_{B,G}^{1/2} q_{H,G}^{-1/2}$ is integrable on $U_\varepsilon$. We begin by noting that from Lemma 1.5 and Proposition 1.7, we have $q_{H,G}^{-1/2}(t,s) = q_{H,G}^{-1/2}(0,s)$ for every $(t,s) \in H$. Recall that for each $k = 1, 2, \dots, u$, $\Lambda_{i_{r_k}}(t,s) = e^{t_k}$ and $\Lambda_{j_{r_k}}(t,s) = q_k(t,s) = e^{-t_k} q_k(0,s)$, and recall also that for all other $j, \Lambda_j(t,s) = \Lambda_j(0,s), (t,s) \in H$. Next we observe that for each $k$ the sign of $\log(q_k(0,s))$ is constant on $U_\varepsilon$, for since $[X_k, Y_k] \in \operatorname{cent}(\mathfrak{n})$ and $\operatorname{Ad}(t,s)([X_k,Y_k]) = q_k(0,s)[X_k,Y_k]$, there is some $j$ (with $Z_j \in \operatorname{cent}(\mathfrak{n})$) such that $\Lambda_j(t,s) = \Lambda_{j_{r_k}}(0,s) = q_k(0,s)$ for all $(t,s) \in H$. Hence the sign of $\log(\Lambda_j(0,s))$ is constant on $U_\varepsilon$ for each $j \in S, j \neq i_{r_k}$. Let $I = S \sim \{i_{r_k} : 1 \leq k \leq u\}$ and let $I^+ = \{j \in I : \log \Lambda_j(0,s) > 0 \text{ on } U_\varepsilon\}, I^- = I \sim I^+$. Then

$$\begin{aligned} q_{H,G}^{-1/2}(t,s) &= q_{H,G}^{-1/2}(0,s) = \prod_{j \in I} \Lambda_j(0,s)^{1/2} = \\ &= \prod_{j \in I^+} \Lambda_j(0,s) \prod_{j \in I^+} \Lambda_j(0,s)^{-1/2} \prod_{j \in I^-} \Lambda_j(0,s)^{1/2}. \end{aligned}$$

On the other hand

$$q_{B,G}^{1/2}(t,s) = q_P(t,s)^{-1/2} q_{\tilde{P}}(t,s)^{-1/2}.$$

We partition $\tilde{P}$: let

$$Q = \left\{k \in \tilde{P} : \log\left(\Lambda_{i_{r_k}}\right) > 0 \text{ on } U_\varepsilon\right\} = \left\{k \in \tilde{P} : t_k > 0 \text{ on } U_\varepsilon\right\}$$

and let $\tilde{Q} = \tilde{P} \sim Q$. Applying Lemma 2.4 with $K = P, J = I^+$, and $J' = \{i_{r_k} : k \in Q\}$, we have that

$$\phi(t,s) = \left(\prod_{j \in I^+} \Lambda_j(0,s)\right)\left(\prod_{k \in Q} e^{t_k}\right) q_{\tilde{P}}(t,s)^{-1/2} f(t,s)$$

is bounded on $U_\varepsilon$. We claim that the function

$$\Psi(t,s) = q_P^{-1/2}(t,s) \prod_{k \in Q} e^{-t_k} \prod_{j \in I^+} \Lambda_j(0,s)^{-1/2} \prod_{j \in I^-} \Lambda_j(0,s)^{1/2}$$

is integrable on $U_\varepsilon$. If this is so then

$$f(t,s)\, q_{B,G}^{1/2}(t,s)\, q_{H,G}^{-1/2}(t,s) = \phi(t,s)\,\Psi(t,s)$$

is integrable on $U_\varepsilon$ and we are done. To prove the claim, we partition $Q$: set $Q^+ = \{k \in Q : \log(q_k(0,s)) > 0\}, Q^- = Q \sim q^+$. Set

$$\begin{aligned}
\Psi_{Q^+}(t,s) &= \prod_{k\in Q^+} e^{-t_k} q_k(0,s)^{-1/2}\\
\Psi_{Q^-}(t,s) &= \prod_{k\in Q^-} e^{-t_k} q_k(0,s)^{1/2}\\
\Psi_{\tilde{Q}}(s) &= \prod_{k\in\tilde{Q}} q_k(0,s)^{1/2}\\
\Psi_+(s) &= \prod_{j\in I^+\sim j^0} \Lambda_j(0,s)^{-1/2}\\
\Psi_-(s) &= \prod_{j\in I^-\sim j^0} \Lambda_j(0,s)^{1/2}.
\end{aligned}$$

Then $\Psi(t,s) = q_P(t,s)^{-1/2}\,\Psi_{Q^+}(t,s)\,\Psi_{Q^-}(t,s)\,\Psi_{\tilde{Q}}(s)\,\Psi_+(t,s)\,\Psi_-(s)$. Note that $I \sim j^0$ consists of indices $j$ for which $Z_j \in \mathrm{cent}(\mathfrak{n})$. Next we describe the set $U_\varepsilon$: define

$$V_\varepsilon = \{s : \log(\Lambda_j(0,s))\,\varepsilon_j > 0 \ \text{ for } \ j \in I \sim j^0\}$$

and for $s \in V_\varepsilon$, set

$$W_\varepsilon(s) = \left\{t : t_k \varepsilon_{i_{r_k}} > 0 \text{ and } \left(-t_k + \log(q_k(0,s))\right)\varepsilon_{j_{r_k}} > 0, 1 \le k \le u\right\},$$

so that $U_\varepsilon = \{(t,s) : s \in V_\varepsilon, t \in W_\varepsilon(s)\}$ and

$$\int_{U_\varepsilon} \Psi(t,s)\, dtds = \int_{V_\varepsilon}\left[\int_{W_\varepsilon(s)} \Psi(t,s)\, dt\right] ds.$$

It is enough to show that the function $s \to \int_{W_\varepsilon(s)} \Psi(t,s)\, dt$ is exponentially decreasing on $V_\varepsilon$. To see this, note first that the function $\Psi_+(s)\,\Psi_-(s)$ is exponentially decreasing on $V_\varepsilon$. Now for each $s \in V_\varepsilon, W_\varepsilon(s)$ is simply a $u$-dimensional cube in $\mathbb{R}^u$, and we consider each interval which makes up $W_\varepsilon(s)$. If $k \in \tilde{Q}$, then $\log(q_k(0,s)) < t_k < 0$, and

$$\int_{(\log(q_k(0,s)),0)} q_k(0,s)\, dt_k = \log(q_k(0,s))\, q_k(0,s).$$

If $k \in Q^+$ then $0 < \log(q_k(0,s)) < t_k$, and

$$\int_{(\log(q_k(0,s)),+\infty)} e^{-t_k} q_k(0,s)^{-1/2}\, dt_k < q_k(0,s)^{-1/2},$$

while if $k \in Q^-$, we have $0 < t_k < +\infty$ and $\int_{(0,+\infty)} e^{-t_k} q_k(0,s)^{1/2}\, dt_k = q_k(0,s)^{1/2}$. Finally if $k \in P$, then $t_k < \log(q_k(0,s))$ and

$$\int_{(-\infty,\log(q_k(0,s)))} e^{-t_k/2} q_k(0,s)^{1/2}\, dt_k = 2.$$

Thus for each $s \in V_\varepsilon$,

$$\begin{aligned}
&\int_{W_\varepsilon(s)} \Psi(t,s)\, dt = \\
&= \int_{W_\varepsilon(s)} q_P(t,s)^{-1/2}\, \Psi_{Q^+}(t,s)\, \Psi_{Q^-}(t,s)\, \Psi_{\tilde{Q}}(s)\, \Psi_+(s)\, \Psi_-(s)\, dt \le \\
&\qquad \le 2^{\#(P)} \left[\prod_{k\in Q^+} q_k(0,s)^{-1/2}\right]\left[\prod_{k\in Q^-} q_k(0,s)^{1/2}\right] \times \\
&\qquad\qquad \times \left[\prod_{k\in\tilde{Q}} \log(q_k(0,s))\, q_k(0,s)^{1/2}\right] \Psi_+(s)\, \Psi_-(s).
\end{aligned}$$

From the definition of the index sets $Q^+$, $Q^-$ and $\tilde{Q}$ we now see that the function $s \to \int_{W_\varepsilon(s)} \Psi(t,s)\, dt$ is exponentially decreasing on $V_\varepsilon$, and the integral (2.1) is convergent.

Now to show that $\beta_l$ is continuous on $\mathcal{H}_l^\infty$, we apply the estimate of Lemma 2.4, and the above analysis, to each set $U_\varepsilon$. We have $\phi_\varepsilon$ and $\Psi_\varepsilon$ as above, and Lemma 2.4 gives $\sup_{U_\varepsilon} |\phi_\varepsilon(t,s)| \le \sum_\beta M_{\beta,\varepsilon} \|\pi(W_{\beta,\varepsilon}) f\|_{\mathcal{H}_l}$, for some constants $M_{\beta,\varepsilon}$ and elements $W_{\beta,\varepsilon} \in \mathbf{U}(\mathfrak{g}_c)$ independent of $f$. Hence

$$|\beta_l(f)| \le \sum_\varepsilon \|\Psi_\varepsilon\|_1 \sup_{U_\varepsilon} |\phi_\varepsilon(t,s)| \le \sum_\varepsilon \sum_\beta \|\Psi_\varepsilon\|_1 M_{\beta,\varepsilon} \|\pi(W_{\beta,\varepsilon}) f\|_{\mathcal{H}_l}.$$

By definition of the topology on $\mathcal{H}_l^\infty$, this finishes the proof. $\square$

## 3. The Plancherel formula.

Now that we have generalized vectors $\beta_l \in \left(\mathcal{H}_l^{-\infty}\right)^{q_{H,G}^{-1/2}}, l \in \Omega$, the results of [**16**] show that we have a Plancherel formula. Here we shall derive it by simple Fourier inversion. To do so we must choose some $l$ in each $H$-orbit, and we want to do this in a smooth natural way. Hence the first task is to compute a nice cross-section for $H$-orbits in $\Omega \cap (f + \mathfrak{h}^\perp)$. In our scenario here, since $H(f+l) = f + Hl$ and $H$ acts only by "dilations", a cross-section is easy to find. Specifically, given $l \in \mathfrak{h}^\perp$, set $l_j = l(Z_j)$, and set $\varepsilon_r(l) = \operatorname{sign}(l_{i_r}), 1 \le i \le d$. We have analytic functions $Q_j(w,l), 1 \le j \le n$, such that

$$Hl = \left\{ \sum_j Q_j(w,l)\, Z_j^* : w = (w_1, w_2, \dots, w_d) \in ((0,+\infty))^d \right\}.$$

For each $j$, if $j = i_r \in \Psi$, then $Q_j(w,l) = \varepsilon_j(l) w_r$, and if $j \notin \Psi$, say $i_r < j < i_{r+1}$, then $Q_j(w,l) = \rho_j(w_1, w_2, \ldots, w_r) l_j$, where $\rho_j$ is an analytic function of the form $\rho_j(w_1, w_2, \ldots, w_r) = w_1^{\alpha_1} \ldots w_r^{\alpha_r}, \alpha_h \in \mathbb{R}$. For each $w$, the function $Q_j(w, \cdot)$ is $H$-invariant on $\Omega \cap \mathfrak{h}^\perp$. A nice cross-section $\Sigma$ for $H$-orbits in $\Omega \cap \mathfrak{h}^\perp$ is given by simply putting $w_r = 1, 1 \le r \le d$, that is $\Sigma = \{l \in \Omega \cap \mathfrak{h}^\perp : |l_{i_r}| = 1, 1 \le r \le d\}$. The cross-section in $f + \mathfrak{h}^\perp$ is $f + \Sigma$. Fixing a choice of signs $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_d), \varepsilon_r = \pm 1$, we have $\Sigma = \bigcup \Sigma_\varepsilon$ where $\Sigma_\varepsilon$ is the flat variety

$$\Sigma_\varepsilon = \{l \in \Omega \cap \mathfrak{h}^\perp : l_{i_r} = \varepsilon_r\}.$$

We choose coordinates for each $\Sigma_\varepsilon$. The center of $\mathfrak{g}$ has as a basis

$$\{Z_j : j \notin e, j \le n\} = \{C_1, C_2, \ldots, C_a\},$$

and we set $l(C_h) = \zeta_h, 1 \le h \le a$, and $l(X_k) = \mu_k, 1 \le k \le u$. Then there is a dense, open subset $D_\varepsilon$ of $\mathbb{R}^a \times \mathbb{R}^u$ such that

$$\Sigma_\varepsilon = \left\{\sum \zeta_h C_h^* + \sum \varepsilon_h V_r^* + \sum \mu_k X_k^* : (\zeta, \mu) \in D_\varepsilon\right\}.$$

Given a function $\Theta$ on $f + \mathfrak{h}^\perp$, we shall write

$$\int_{f+\Sigma} \Theta(l)\, dl = \\ = \sum_{\varepsilon \in \{1,-1\}^d} \int_{\mathbb{R}^a \times \mathbb{R}^u} \Theta\left(f + \sum \zeta_h C_h^* + \sum \varepsilon_r V_r^* + \sum \mu_k X_k^*\right) d\zeta d\mu.$$

Before deriving the Plancherel formula we may as well compute multiplicities, which amounts to just counting the number of $H$-orbits in each $G$-orbit intersection with $\Omega \cap (f + \mathfrak{h}^\perp)$. Fix $l_0 = \sum \zeta_h C_h^* + \sum \varepsilon_r V_r^* + \sum \mu_k X_k^* \in \Sigma_\varepsilon$, and let $l = f + l_0$. Set $f(A_k) = \alpha_k, 1 \le k \le u$. An ordered "co-exponential" basis for $\mathfrak{g} \bmod \mathfrak{g}(l)$ is $\{Z_{e_1}, Z_{e_2}, \ldots, Z_{e_{2d}}\}$ (where we have written $e = \{e_1 < e_2 < \ldots < e_{2d}\}$) and consists of the $V_r$'s, the $X_k$'s, and the $B_h$'s. Using the methods of [**7**, **8**], we find that the $G$-orbit of $l$ is the set of all $l'$ of the form

$$l' = \sum \zeta_h C_h^* + \sum \varepsilon_{s_h} w_{s_h} V_{s_h}^* + \sum y_k Y_k^* + \sum x_k X_k^* + \\ + \sum P_k(w, x_k, y_k, l') A_k^* + \sum z_h B_h^*$$

where $W_{s_h}$ runs through $(0, +\infty), 1 \le h \le \nu$, $x_k$ and $y_k$ run through $\mathbb{R}, 1 \le k \le u$, and $z_h$ runs through $\mathbb{R}, 1 \le h \le \nu$. Recall that $A_k$ commutes with every basis element except $X_k$ and $Y_k$, that $X_k$ commutes with every element except $A_k, Y_k$ and possibly some of the $B_h, 1 \le h \le \nu$, and $Y_k$ commutes with every element except $X_k$ and $A_k$. This is why $P_k$ depends only on $x_k, y_k$, and the $w_{s_h}, 1 \leq h \leq \nu$. In fact the function $P_k(w, x_k, y_k, l')$ can be computed as

$$P_k(w, x_k, y_k, l') = \alpha_k + (\sigma_k(w) x_k y_k - \varepsilon_{r_k} \mu_k) / l'([X_k, Y_k]),$$

where $\sigma_k(w)$ is a positive analytic function in the positive variables $w_{s_1}, w_{s_2}, \ldots, w_{s_\nu}$. (The function $\rho_{j_{r_k}}$ above involves only the variables $w_{s_1}, w_{s_2}, \ldots, w_{s_\nu}$ and $w_{r_k}$, and $\sigma_k(w_{s_1}, w_{s_2}, \ldots, w_{s\nu}) = \rho_{j_{r_k}}(w_{s_1}, w_{s_2}, \ldots, w_{s_\nu}, w_{r_k})^{-1}|_{w_{r_k}=1})$. Thus

$$\begin{aligned} Gl \cap (f + \mathfrak{h}^\perp) &= f + \{l' \in \mathrm{p}(Gl) : P_k(w, x_k, y_k, l') = \alpha_k, 1 \leq k \leq u\} = \\ &= f + \{l' \in \mathrm{p}(Gl) : \sigma_k(w) x_k y_k = \varepsilon_{r_k} \mu_k, 1 \leq k \leq u\}. \end{aligned}$$

Now $\#((Gl \cap (f + \mathfrak{h}^\perp))/H) = \#((Gl \cap (\mathfrak{h}^\perp) \cap (f + \Sigma)))$. If $l'$ belongs to this intersection, we must put each $w_{s_h} = 1, 1 \leq h \leq \nu$, and so every coordinate of $l'$ is fixed except the $X_k$ and $Y_k$ coordinates, where we are allowed $l'(Y_k) = \varepsilon_{r_k}(l') = \pm 1, 1 \leq k \leq u$ while $l'(X_k) = \mu'_k$ is determined by

$$\varepsilon_{r_k}(l') \mu'_k = \varepsilon_{r_k} \mu_k, 1 \leq k \leq u.$$

Hence the intersection $Gl \cap (f + \mathfrak{h}^\perp) \cap (f + \Sigma)$ consists of $2^u$ elements corresponding to the possible choices of signs for $\varepsilon_{r_1}(l'), \varepsilon_{r_2}(l'), \ldots, \varepsilon_{r_u}(l')$, and we have proved the following.

**Proposition 3.1.** *For any $f \in \mathfrak{h}^*$, the representation $\tau_f$ is uniform multiplicity $2^u$, where $u = \dim(\mathrm{cent}(N) \sim N)/2$.*

We turn now to the Plancherel formula and the intertwining operator. Let $f \in \mathfrak{h}^*, \tau = \tau_f$ the representation induces by $\chi_f$, and let $\alpha_\tau$ be the canonical cyclic generalized vector for $\tau$, that is, $\alpha_\tau(\phi) = \tilde{\phi}(e), \phi \in \mathcal{H}_\tau^\infty$. Then for any test function $\omega \in \mathcal{D}(G), \tau(\omega)\alpha_\tau$ belongs to $\mathcal{H}_\tau^\infty$, in fact to $C_c^\infty(G, H, \chi_f)$, and is given by the formula (cf. [**16**])

$$\begin{aligned} \tau(\omega)\alpha_\tau(g) &= \omega_{H,f}(g) = \\ &= \Delta_G^{-1}(g) q_{H,G}^{-1/2}(g) \int_H \omega(g^{-1}h^{-1}) \Delta_G^{-1}(h) q_{H,G}^{-1/2}(h) \chi_f(h)^{-1}\, dh. \end{aligned}$$

One also has

$$\langle \tau(\omega)\alpha_\tau, \alpha_\tau \rangle = \omega_{H,f}(e) = \int_H \omega(h^{-1}) \Delta_G^{-1}(h) q_{H,G}^{-1/2}(h) \chi_f(h)^{-1}\, dh.$$

Let $l \in f + \mathfrak{h}^\perp$, and let $\pi_l$ and $\beta_l$ be as in Section 2. Then $\pi_l(\omega)\beta_l \in \mathcal{H}_l^\infty$ is given by

$$\pi_l(\omega)\beta_l(g) = \int_B \omega_{H,f}(bg) \chi_l(b)^{-1} q_{B,G}^{-1/2}(bg) q_{H,G}^{1/2}(bg) \Delta_B(b)\, db$$

and

$$\begin{aligned}\langle \pi_l(\omega)\beta_l,\beta_l\rangle &= \int_H\int_B \omega_{H,f}(bh)\,\chi_l(b)^{-1}\,q_{B,G}^{-1/2}(b)\times\\ &\quad\times q_{H,G}^{1/2}(b)\,\Delta_B(b)\,\chi_f(h)^{-1}\,dbdh\\ &= \int_H\int_B \omega_{H,f}(h^{-1}bh)\,\chi_l(b)^{-1}\,q_{B,G}^{-1/2}(b)\times\\ &\quad\times q_{H,G}^{1/2}(b)\,\Delta_B(b)\,dbdh\end{aligned}$$

(cf. **[16]** or **[17]** for the computations). For any $\phi\in C_c(G,H,\chi)$ (where $\chi$ is any unitary character of $H$), we set

$$I_l(\phi)=\int_H\int_B \phi(h^{-1}bh)\,\chi_l(b)^{-1}\,q_{B,G}^{-1/2}(b)\,q_{H,G}^{1/2}(b)\,\Delta_B(b)\,dbdh$$

so that $I_l(\omega_{H,f})=\langle\pi_l(\omega)\beta_l,\beta_l\rangle$ when $l\in f+\mathfrak{h}^\perp$.

**Theorem 3.2.** *Let $\chi$ be any unitary character of $H$, and let $\phi\in C_c(G,H,\chi)$. Then the integral $\int_{f+\Sigma} I_l(\phi)\,|R(l)|\,dl$ is independent of the choice of $f\in\mathfrak{h}^*$ and we have*

$$\phi(e)=\int_{f+\Sigma} I_l(\phi)\,|R(l)|\,dl$$

*where $R(l)=\left((2\pi)^n\,l([X_1,Y_1])\,l([X_2,Y_2])\ldots l([X_u,Y_u])\right)^{-1}$. In particular*

$$\langle\tau(\omega)\alpha_\tau,\alpha_\tau\rangle=\int_{f+\Sigma}\langle\pi_l(\omega)\beta_l,\beta_l\rangle\,|R(l)|\,dl.$$

*Proof.* Let $f\in\mathfrak{h}^*$ and $l\in f+\Sigma$, writing $l=f+\sum\zeta_hC_h^*+\sum\varepsilon_rV_r^*+\sum\mu_kX_k^*\in f+\Sigma_\varepsilon$ as above. We use the following coordinates on $B(l)$: an element

$$\begin{aligned}b=\prod\exp(c_hC_h)\prod\exp(z_hV_{s_h})\prod\exp(y_kY_k)\times\\ \times\prod\exp\left(w_k\left(A_k-a_k(l)\right)X_k\right)\end{aligned}$$

is identified with $(c,z,y,w)\in\mathbb{R}^a\times\mathbb{R}^\nu\times\mathbb{R}^u\times\mathbb{R}^u$. Recalling the formula for $a_k(l)$ we have $a_k(l)=\varepsilon_{r_k}/l([X_k,Y_k])$ in this case, and so $|R(l)|=(2\pi)^{-n}\prod|a_k(l)|$. We denote $\prod e^{t_k}$ by $e^t$, $\prod e^{s_h}$ by $e^s$, $(e^{s_1}z_1,e^{s_2}z_2,\ldots,e^{s_\nu}z_\nu)$ by $e^sz$, etc. We compute explicitly that

$$q_{B,G}^{-1/2}(b)\,q_{H,G}^{1/2}(b)\,\Delta_B(b)=e^{w_1/2}e^{w_2/2}\ldots e^{w_u/2}=e^{w/2}.$$

We have

$$I_l(\phi) = \iint_H \iiiint_B \phi\left(c, e^{-s}z, e^{-t}y, \mathrm{Ad}^{-1}(t,s)\,w\right) e^{i\left[\sum w_k(a_k(l)\mu_k - \alpha_k)\right]}$$
$$\times\, e^{-i\left[\sum y_k \varepsilon_{r_k} + \sum z_h \varepsilon_{s_h} + \sum c_k \zeta_k\right]} e^{w/2}\, dcdzdydwdsdt$$
$$= \iint_H \iiiint_B \phi\left(c, z, y, \mathrm{Ad}^{-1}(t,s)\,w\right) e^{i\left[\sum w_k(a_k(l)\mu_k - \alpha_k)\right]}$$
$$\times\, e^{-i\left[\sum y_k e^{t_k} \varepsilon_{r_k} + \sum z_h e^{s_h} \varepsilon_{s_h} + \sum c_k \zeta_k\right]} e^t e^s e^{w/2}\, dcdzdydwdsdt.$$

It is easy to check that for $t, s, c, z$, and $y$ fixed, the function

$$w \to \phi\left(c, z, y, \mathrm{Ad}^{-1}(t,s)\,w\right) e^{w/2}$$

is rapidly decreasing. By the change of variables $\mu_k \to a_k(l)(\mu_k + \alpha_k)$, $1 \leq k \leq u$, and Fourier inversion in the variables $w, \mu$, we get

$$(2\pi)^{n-u} \int_{f+\Sigma} I_l(\phi)\, |R(l)|\, dl$$
$$= \sum_{\varepsilon} \int_{\mathbb{R}^a} \int_{\mathbb{R}^u} \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^u} \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^a} \phi(c, z, y, 0)$$
$$\times\, e^{-i\left[\sum y_k e^{t_k} \varepsilon_{r_k} + \sum z_h e^{s_h} \varepsilon_{s_h} + \sum c_k \zeta_k\right]} e^t e^s\, dcdzdydsdtd\zeta.$$

Note that the above is independent of the choice of $f \in \mathfrak{h}^*$. Set $\nu_k = e^{t_k}$, $1 \leq k \leq u$, $\rho_h = e^{s_h}$, $1 \leq h \leq \nu$, and set $I = (0, +\infty)$. Note that $a + \nu + u = n - u$. A simple computation gives

$$\int_{f+\Sigma} I_l(\phi)\, |R(l)|\, dl = (2\pi)^{u-n} \sum_{\varepsilon} \int_{\mathbb{R}^a} \int_{\mathbb{R}^u} \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^a} \int_{I^\nu} \int_{I^u} \phi(c, z, y, 0)$$
$$\times\, e^{-i\left[\sum y_k \nu_k \varepsilon_{r_k} + \sum z_h \rho_h \varepsilon_{s_h} + \sum c_k \zeta_k\right]}\, d\nu d\rho dcdzdyd\zeta$$
$$= (2\pi)^{u-n} \int_{\mathbb{R}^a} \int_{\mathbb{R}^u} \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^a} \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^u} \phi(c, z, y, 0)$$
$$\times\, e^{-i\left[\sum y_k \gamma_k + \sum z_h \xi_h + \sum c_k \zeta_k\right]}\, d\gamma d\xi dcdzdyd\zeta$$
$$= \phi(c).$$

□

Define $T : C_c(G, H, \chi_f) \to \int_{f+\Sigma} \mathcal{H}_l\, |R(l)|\, dl$ by

$$T(\omega_{H,f}) = \{\pi_l(\omega)\, \beta_l(g)\}_{l \in f+\Sigma}.$$

By [**16**, Prop 3.2], $T$ extends to an intertwining operator

$$L^2(G, H, \chi_f) \to \int_{f+\Sigma} \mathcal{H}_l\, |R(l)|\, dl.$$

Explicitly,

$$\begin{aligned} T\left(\omega_{H,f}\right)_l(g) &= \pi_l(\omega)\,\beta_l(g) \\ &= \iiiint_B \phi\left(\left(c,z,y,w,\right)g\right) e^{i\left[\sum w_k\left(a_k(l)\mu_k-\alpha_k\right)\right]} \\ &\quad \times e^{-i\left[\sum y_k\varepsilon_{r_k}+\sum z_h\varepsilon_{s_h}+\sum c_k\zeta_k\right]} e^{w/2}\, dc\,dz\,dy\,dw \\ &\quad \times q_{B,G}^{-1/2}(g)\, q_{H,G}^{1/2}(g)\,. \end{aligned}$$

Identifying $\mathcal{H}_l$ with $L^2\left(\mathbb{R}^u \times \mathbb{R}^\nu\right)$ via the mapping $\gamma(x,s)\,, l \in f+\Sigma$, it is clear that for any $\Phi \in L^2\left(\mathbb{R}^u \times \mathbb{R}^\nu\right)$, the function

$$l \to \left\langle \pi_l(\omega)\,\beta_l, \Phi\right\rangle$$

is $C^\infty$, so $\left\{\pi_l(\omega)\,\beta_l(g)\right\}_{l\in f+\Sigma}$ is a smooth section of $\left\{\mathcal{H}_l\right\}_{l\in f+\Sigma}$.

## 4. Examples.

We provide two examples. The first will be the basic split oscillator group, wherein most of the essential difficulties of the subject are already exhibited. In the second example we let $\mathbb{R}$ act semi-simply on the split oscillator so as to be non-trivial on the center, in order to show the differences created by a non-trivial action of $H$ on the center of the nilradical. By the results of Section 1, in some sense the general case just amounts to taking higher dimensional analogues of these examples (or of the $ax+b$ group), where the commutator $[\mathfrak{n},\mathfrak{n}]$ is allowed to be arbitrarily large, and where the portion of $H$ acting non-trivially on $\operatorname{cent}(\mathfrak{n})$ can act on $\mathfrak{n}$ in a fairly arbitrary manner. In an attempt to make the techniques of Section 2 more transparent, we have related results of that section to computations in these examples.

1. $\mathfrak{g} = \operatorname{span}\{A,X,Y,Z\}$ with non-vanishing brackets $[A,X]=-X, [A,Y]=Y, [X,Y]=Z$. Here $\mathfrak{h}=\mathbb{R}A, \mathfrak{n}=\operatorname{span}\{X,Y,Z\}$. $G$ is the semi-direct product of the 3-dimensional Heisenberg group with $\mathbb{R}$, and is diffeomorphic with $\mathbb{R}^4$ by identifying $(z,y,x,t)$ with $\exp(zZ)\exp(yY)\exp(xX)\exp(tA)$. The multiplication is

$$(z,y,x,t)\left(z',y',x',t'\right) = \left(z+z'+e^t xy', y+e^t y', x+e^{-t}x', t+t'\right).$$

The Jordan-Holder sequence is given by $\mathfrak{g}_1=\mathbb{R}Z$, $\mathfrak{g}_2=\operatorname{span}\{Z,Y\}$, $\mathfrak{g}_3=\operatorname{span}\{Z,Y,X\}$, $\mathfrak{g}_4=\mathfrak{g}$. For $l\in\mathfrak{g}^*$, write $l=(\lambda,\gamma,\mu,\alpha)$, where $\lambda=l(Z), \gamma=l(Y), \mu=l(X), \alpha=l(A)$. $\Omega_0=\{l\in\mathfrak{g}^*:\lambda\neq 0\}$ and $\Omega_1=\{l\in\mathfrak{g}^*:\gamma\neq 0\}$. Fix $l\in\Omega=\Omega_0\cap\Omega_1$; then $\mathfrak{b}(l)=\operatorname{span}\{A_l,Y,Z\}$ where

$A_l = A - a\,(l)\,X, a\,(l) = \gamma/\lambda$. We have

$$B\,(l) = \{\exp\,(zZ)\exp\,(yY)\exp\,(wA_l) \\ = (z, y, a(l)\,(e^{-w} - 1)\,w) : w \in \mathbb{R}, y \in \mathbb{R}, z \in \mathbb{R}\}.$$

Haar measure on $G$ is $dg = dzdydxdt$, on $B$ is $db = dzdydt$, and on $H, dh = dt$. $G$ is unimodular but $\Delta_B\,(z, y, a\,(l)\,(e^{-w} - 1)\,, w) = e^{-w}$ and $q_{B,G}\,(z, y, x, t) = e^t$. Let $f \in \mathcal{H}_l^\infty$, and write $f\,(t)$ for $f|_H$. Then the generalized vector $\beta_l$ is defined formally by

$$\beta_l\,(f) = \int_H \bar{f}\,(t)\,e^{t/2}\,dt. \tag{4.1}$$

How do we see that this integral converges absolutely and defines a generalized vector? Using $H$-coordinates, we compute that $\pi\,(A)\,f\,(t) = f'\,(t)$, $\pi\,(Y)\,f\,(t) = ie^t f\,(t)$, and $\pi\,(X)\,f\,(t) = e^{-t}f'\,(t)$. Set

$$\phi_1\,(t) = q_{B,G}\,(t)\,\pi\,(Y)\,f\,(t)\,.$$

Since all derivatives of $\phi_1$ are square integrable on $H$, then $\phi_1\,(t) \in C_0\,(H)$ and we have $\|\phi_1\|_\infty \le M_1\,\|\pi\,(V)\,f\|_{\mathcal{H}_l}$ (as in Lemma 2.3, part (a)). In particular $f\,(t)\,e^{t/2} = \phi_1\,(t)\,e^{-t}$ is absolutely integrable over $0 \le t \le +\infty$. Let $\phi_2\,(t) = e^{-t/2}f'\,(t) = q_{B,G}\,(t)\,\pi\,(X)\,f'\,(t)$; then all derivatives of $\phi_2$ are square integrable on $H$ so $\phi_2 \in C_0\,(H)$ and $\|\phi_2\|_\infty \le M_2\,\|\pi\,(W)\,f\|_{\mathcal{H}_l}$ (this is Lemma 2.3, part (b)). Hence as $t \to -\infty, f' \to 0$ faster than $e^{t/2}$, and we apply the fundamental theorem of calculus to see that $f$ is bounded on $-\infty \le t \le 0$, and that $\sup\{|f(t)| : -\infty \le t \le 0\} \le 2M_2\,\|\pi(W)f\|_{\mathcal{H}_l} + |f(0)|$. (This is a special case of Lemma 2.4.) Now as in the proof of Theorem 2.2, it follows that (4.1) converges absolutely and

$$\begin{aligned} |\beta_l(f)| &\le \int_{(0,+\infty)} |f(t)|\,e^{t/2}dt + \int_{(-\infty,0)} |f(t)|\,e^{t/2}dt \\ &\le \sup\{|\phi_1(t)| : 0 \le t \le +\infty\} \int_{(0,+\infty)} e^{-t}dt + \\ &\quad + \sup\{|f(t)| : -\infty \le t \le 0\} \int_{(-\infty,0)} e^{t/2}dt \\ &\le M_1\,\|\pi(V)f\|_{\mathcal{H}_l} + 4M_2\,\|\pi(W)f\|_{\mathcal{H}_l} + 2\,|f(0)|\,. \end{aligned}$$

The cross-section for $H$-orbits in $\mathfrak{h}^\perp \cap \Omega$ is

$$\Sigma = \{(\lambda, \varepsilon, \mu, 0) : \lambda \in \mathbb{R} \sim \{0\}\,, \mu \in \mathbb{R}, \varepsilon = \pm 1\} = \bigcup_{\varepsilon = \pm 1} \Sigma_\varepsilon$$

and Theorem 3.2 says that the Plancherel measure is given on $\Sigma_\varepsilon$ by $(2\pi)^{-3}\lambda^{-1}d\lambda d\mu$. The matrix elements for $\pi_l, l \in \Sigma$, are

$$\begin{aligned}\langle \pi_l(\omega)\beta_l, \beta_l\rangle &= \int_H \iiint_B \phi\left(\exp(zZ)\exp\left(e^{-t}yY\right)\exp\left(\mathrm{Ad}^{-1}(t)w\right)\right) \\ &\quad \times e^{iwa(l)\mu}e^{-i[y\varepsilon+z\lambda]}e^{w/2}\,dzdydwdt = \\ &= \int_H \iiint_B \phi\left(z,y,\mathrm{Ad}^{-1}(t)\,w\right) \\ &\quad \times e^{iw\mu}e^{-i\left[ye^{-t}\varepsilon+z\lambda\right]}e^{t}e^{w/2}\,|a(l)|^{-1}\,dzdydwdt\end{aligned}$$

where $\omega \in \mathcal{D}(G)$ and $\phi = \tau(\omega)\,\alpha_\tau$. Now $R(l) = (2\pi)^{-3}\,a(l)$ here. Thus

$$\begin{aligned}&\int_\Sigma \langle \pi_l(\omega)\,\beta_l, \beta_l\rangle\,|R(l)|\,dl \\ &\qquad = \sum_{\varepsilon=\pm1}\iint_{\Sigma_\varepsilon}\int_H \iiint_B \phi\left(z,y,\mathrm{Ad}^{-1}(t)\,w\right)e^{iw\mu}e^{-i\left[ye^{-t}\varepsilon+z\lambda\right]} \\ &\qquad\quad \times e^{t}e^{w/2}\,dzdydwdt\,(2\pi)^{-3}\,d\lambda d\mu \\ &\qquad = (2\pi)^{-2}\sum_{\varepsilon=\pm1}\iiiint \phi(z,y,0)\,e^{-i\left[ye^{t}\varepsilon+z\lambda\right]}\,dye^{t}dtdzd\lambda \\ &\qquad = (2\pi)^{-2}\iint\left[\int_{(-\infty,0)}+\int_{(0,+\infty)}\right]\int \phi(z,y,0) \\ &\qquad\quad \times e^{-i[y\gamma+z\lambda]}\,dyd\gamma dzd\lambda = \\ &\qquad = \phi(e) = \langle \tau(\omega)\,\alpha_\tau, \alpha_\tau\rangle\,.\end{aligned}$$

2. $\mathfrak{g} = \mathrm{span}\{B,A,X,Y,Z\}$ where $\mathfrak{g}_4 = \mathrm{span}\{A,X,Y,Z\}$ is the split oscillator, and $[B,X] = X, [B,Z] = Z, [B,A] = [B,Y] = 0$. $G_4$ is realized as above, and $G = G_4\exp(\mathbb{R}B)$ so that the multiplication is

$$\begin{aligned}&(z,y,x,t,s)\,(z',y',x',t',s') \\ &\qquad = \left(z+e^{s}z'+e^{t}xy', y+e^{t}y', x+e^{s-t}x', t+t', s+s'\right).\end{aligned}$$

The Jordan-Holder sequence is the obvoius extension of that which was chosen above for $\mathfrak{g}_4$, the set $\Omega$ of generic linear functionals is the same as before, and for $l \in \Omega$, the polarization $\mathfrak{b}(l)$ is the same as before, but now $q_{B,G}(s,t) = e^{t-s}$. $G$ is no longer unimodular: $\Delta_G(z,y,x,t,s) = q_{H,G}(z,y,x,t,s)^{-1} = e^{2s}$. Thus the formula for $\beta_l$ now is

$$\beta_l(f) = \int_H \bar{f}(s,t)\,e^{s}e^{(t-s)/2}\,dsdt \tag{4.2}$$

where we have written $f(s,t)$ for $f|_H$. One computes that, for smooth vectors restricted to $H$, $\pi(A) = \partial_t, \pi(B) = \partial_s, \pi(X) = e^{s-t}\partial_t, \pi(Y) = ie^t$. By Lemma 2.3, for each $k, j \geq 0$, there are constants $M_{k,j}$ and $N_{k,j}$ and elements $V_{k,j}, W_{k,j}$ in $\mathbf{U}(\mathfrak{g}_c)$ such that

$$\text{(i) } \left\| e^{ks}e^{jt}e^{(t-s)/2}f(s,t) \right\|_\infty \leq M_{k,j} \left\| \pi(V_{k,j}) f \right\|_{\mathcal{H}_l}, \text{ and}$$

$$\text{(ii) } \left\| e^{ks}e^{jt}e^{(t-s)/2}e^{s-t}\partial_t f(s,t) \in C_0(H) \right\|_\infty \leq N_{k,j} \left\| \pi(W_{k,j}) f \right\|_{\mathcal{H}_l}.$$

Let $U(++) = \{(s,t) : s > 0, t > s\}$, $U(+-) = \{(s,t) : s > 0, t < s\}$, $U(-+) = \{(s,t) : s < 0, t > s\}$, $U(--) = \{(s,t) : s < 0, t < s\}$. Using (i) and (ii) and the fundamental theorem of calculus we get (Lemma 2.4)

$$\begin{aligned}\text{(iii) } \sup\{|e^{ks}f(s,t)| : (s,t) \in U(+-) \cup U(--)\} \\ \leq M_{k,0} \left\|\pi(V_{k,0}) f\right\|_{\mathcal{H}_l} + 2N_{k,0} \left\|\pi(W_{k,0}) f\right\|, k \geq 0.\end{aligned}$$

For each of the above four subsets $U$ of $H$, we write $\bar{f}(s,t)e^s e^{(t-s)/2}$ as a product of a function $\phi$ for which one of the above estimates (i), (ii) or (iii) holds, and a function $\Psi$ which is absolutely integrable over $U$.

$$\begin{aligned}
&U(++) : \phi(s,t) = f(s,t)\, e^{2s}e^t e^{(t-s)/2}, \Psi(s,t) = e^{-s}e^{-t}, \\
&U(+-) : \phi(s,t) = f(s,t)\, e^{2s}, \Psi(s,t) = e^{-s}e^{(t-s)/2}, \\
&U(-+) : \phi(s,t) = f(s,t)\, e^t e^{(t-s)/2}, \Psi(s,t) = e^{s-t}, \\
&U(--) : \phi(s,t) = f(s,t), \Psi(s,t) = e^s e^{(t-s)/2}.
\end{aligned}$$

Now in a manner similar to example (1) (cf. also the proof of Theorem 2.2) we see that (4.2) is absolutely convergent and defines a generalized vector for $\pi$.

The cross-section for $H$-orbits in $\mathfrak{h}^\perp \cap \Omega$ is

$$\Sigma = \{(\varepsilon_1, \varepsilon_2, \mu, 0, 0) : \mu \in \mathbb{R}, \varepsilon_1 = \pm 1, \varepsilon_2 = \pm 1, \} = \bigcup_\varepsilon \Sigma_\varepsilon$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2)$ runs through $\{-1, 1\}^2$. Here the Plancherel measure is given on each $\Sigma_\varepsilon$ by $(2\pi)^{-3}\, d\mu$. For $\omega \in \mathcal{D}(G)$ and $\phi = \tau(\omega)\alpha_\tau$, we have

$$\begin{aligned}
\langle \pi_l(\omega)\beta_l, \beta_l \rangle &= \langle \pi_{\varepsilon_1,\varepsilon_2,\mu}(\omega)\beta_{\varepsilon_1,\varepsilon_2,\mu}, \beta_{\varepsilon_1,\varepsilon_2,\mu} \rangle \\
&= \iint_H \iiint_B \phi\left(\exp(e^s zZ)\exp(e^{-t}yY)\exp\left(Ad^{-1}(s,t)\, w\right)\right) \\
&\qquad \times e^{iw(\varepsilon_2/\varepsilon_1)\mu} e^{-i[y\varepsilon_2 + z\varepsilon_1]} e^{w/2}\, dz dy dw dt ds,
\end{aligned}$$

and a computation like that of Example (1) shows that

$$(2\pi)^{-3} \sum_\varepsilon \int \langle \pi_{\varepsilon_1,\varepsilon_2,\mu}(\omega)\beta_{\varepsilon_1,\varepsilon_2,\mu}, \beta_{\varepsilon_1,\varepsilon_2,\mu} \rangle\, d\mu = \phi(e) = \langle \tau(\omega)\alpha_\tau, \alpha_\tau \rangle.$$

## 5. Concluding Remark.

Suppose that $G$ is any connected, simply connected nilpotent Lie group, $H$ a closed connected subgroup, and $\tau$ an associated finite multiplicity monomial representation. Given choices of appropriate basis for $\mathfrak{g}$ and $\mathfrak{h}$, there is a unique construction of a flat cross-section $\Sigma$ for generic $H$-orbits in $\mathfrak{h}^{\perp}$. Attaching the Vergne polarizations to each $l \in \Sigma$ one has a natural, explicit algorithm for deriving Fujiwara's Plancherel fromula, and hence for constructing an explicit, smooth decomposition of $\tau$ over $\Sigma$, as described in the intriduction. Can one give an explicit description of the Plancherel measure on $\Sigma$? One could even hope that this decomposition diagonalizes the differential operators on $H\backslash G$ that commute with $\tau$, in the manner of [**5**]. Of course one could also entertain such questions for $G$ completely solvable, once the technical difficulties surrounding the construction of the generalized vectors $\beta_l$ are overcome.

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