

## NIELSEN NUMBERS FOR ROOTS OF MAPS OF ASPHERICAL MANIFOLDS

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Let  $f : X \rightarrow Y$  be a map of closed orientable manifolds of the same dimension, and let  $a \in Y$ . The topological degree of  $f$  is an algebraic count of the number of solutions to  $f(x) = a$ , but not an actual count. The Nielsen number  $N(f, a)$  of roots is an actual lower bound for the number of solutions. We investigate conditions under which  $N(f, a) = |\text{degree } f|$ . Our question is analogous to the question in fixed point theory: when is the Lefschetz number equal to the fixed point Nielsen number? We find equality when  $X = Y$  is an aspherical manifold whose fundamental group satisfies the ascending chain condition on normal subgroups, or if  $X$  and  $Y$  are aspherical manifolds with virtually polycyclic fundamental groups. This includes infra-solvmanifolds. Similar results are obtained for nonorientable manifolds by considering their orientable double covers.

### 1. Introduction.

Suppose  $f : X \rightarrow Y$  a mapping of topological spaces, and  $a \in Y$ . A point  $x \in X$  is a root of  $(f, a)$  if  $f(x) = a$ . When  $X$  and  $Y$  are closed orientable manifolds, the *topological degree* of  $f$  is an algebraic count of the number of roots of  $(f, a)$ . However, neither it - nor its absolute value - provide an actual count of the number of roots. For example, map the Riemann sphere  $(\mathbb{C} \cup \infty)$  onto itself by  $f(z) = z^3$ . Then the degree of  $f$  is 3, but there is just one root of  $f(z) = 0$ .

An analogous situation exists in fixed point theory, where the *Lefschetz number*  $L(f)$  is an algebraic count, but not an actual count, of the number of fixed points of  $f$ . In fixed point theory, the (fixed point) *Nielsen number*,  $N(f)$ , is a homotopically invariant lower bound on the actual number of fixed points of  $f$ . Although, in general, the fixed point Nielsen number is not the same as the Lefschetz number, it was shown in [8] that for maps of an  $n$ -dimensional torus  $N(f) = |L(f)|$ . This result was generalized by Anosov [1] and Fadell and Husseini [9] to self-maps of nilmanifolds. Numerous extensions of the fixed point results have been made by Keppelman and McCord [13], Kwasik and Lee [15], Lee [16], and McCord [18, 19, 20, 21]

to generalizations of nilmanifolds: infranilmanifolds, solvmanifolds, and infrasolvmanifolds. McCord's paper [19] also applies to coincidences, i.e., solutions of the equation  $f(x) = g(x)$  where  $f, g : X \rightarrow Y$  are two mappings between, possibly, different spaces.

By analogy with fixed point theory, the Nielsen number  $N(f, a)$  of roots of  $f(x) = a$  was defined in [4 and 7] (there it is called the  $\Delta_2$  Nielsen number). The definition is recalled in Section 2 below (see [14, pp. 123-138] for a very readable account of the Nielsen theory of roots). It has for roots the same or analogous properties as the ordinary fixed point Nielsen number. In particular, it is a lower bound for the number of roots of  $(f, a)$  and is invariant under homotopies of  $f$ . In fact, for maps of closed manifolds, both Nielsen numbers are special cases of the coincidence Nielsen number  $N(f, g)$ , which is a lower bound on the number of solutions to  $f(x) = g(x)$ . For orientable closed manifolds, there is also a Lefschetz coincidence number  $L(f, g)$  that specializes to the degree of  $f$ , when  $g$  is a constant map, and the fixed point Lefschetz number when  $g$  is the identity map.

In this paper, we address the question for roots: When is the Nielsen number  $N(f, a)$  of a map  $f : X \rightarrow Y$  equal to the absolute value of the degree of  $f$ ? This is a simpler question than the corresponding question for coincidences or even for fixed points. As a consequence, we are able to use elementary methods and obtain sharper results.

A space  $X$  is *aspherical* iff it is connected and all its higher homotopy groups,  $\pi_n X$  for  $n > 1$ , are trivial. Equivalently, the universal cover of  $X$  is contractible. The space  $X$  is also called an *Eilenberg-MacLane space of type  $K(\pi, 1)$*  where  $\pi$  is the fundamental group of  $X$ . The homotopy of aspherical spaces and their maps depends entirely on their fundamental group homomorphisms. Because we are examining the relation between the Nielsen number (determined largely by fundamental group behavior), and the degree of a map (determined by homology in the top dimension), it is natural to limit our attention to these manifolds, where, at least in principle, both can be computed in terms of fundamental group homomorphisms.

The degree of  $f : X \rightarrow Y$  is defined only when  $X$  and  $Y$  are orientable. But, for any nonorientable manifold  $X$ , there is a two sheeted covering  $p : X^0 \rightarrow X$  by an orientable manifold  $X^0$  unique up to covering space isomorphism. This is called the *orientable double covering*. Following McCord [20], we shall use it to state results for nonorientable manifolds. (To define  $X^0$ , consider the action of the fundamental group  $\pi X$  on the universal covering space of  $X$ . The elements whose action preserves orientation form a subgroup of index two. Let  $p : X^0 \rightarrow X$  be the corresponding covering, i.e.,  $p_{\#} \pi X^0$  is the subgroup of  $\pi X$  whose action preserves orientation.)

The following theorem is proved in Section 2.

**Theorem 1.** *Suppose  $f : X \rightarrow Y$  a map of closed aspherical manifolds of the same dimension and  $a$  is any point in  $Y$ . Suppose further that the homomorphism  $f_{\#} : \pi X \rightarrow \pi Y$  induced on the fundamental groups is a monomorphism. Then the Nielsen number  $N(f, a) = |\text{coker } f_{\#}| > 0$ . Moreover, depending on orientability, we have one of the following three cases:*

1.  *$Y$  is orientable. Then  $X$  is also orientable and*

$$N(f, a) = |\text{degree } f|.$$

2.  *$X$  is orientable but  $Y$  is not. Let  $r : Y^0 \rightarrow Y$  be the orientable double covering. Then there is a lift  $f^0 : X \rightarrow Y^0$  of  $f$  through  $r$  and*

$$N(f, a) = 2 |\text{degree } f^0|.$$

3. *Neither  $X$  nor  $Y$  are orientable. Let  $p : X^0 \rightarrow X$  and  $r : Y^0 \rightarrow Y$  be the orientable double coverings. Then there is a lift  $f^0 : X^0 \rightarrow Y^0$  of  $f \circ p$  through  $r$  and*

$$N(f, a) = |\text{degree } f^0|.$$

The hypothesis that  $f_{\#} : \pi X \rightarrow \pi Y$  be monomorphic is very strong. It amounts to assuming that  $f : X \rightarrow Y$  is homotopy equivalent to a covering map. It is satisfied, however, in two important instances treated in Theorems 2 and 3 below.

A group  $G$  satisfies the *ascending chain condition on normal subgroups* iff every strictly increasing chain of normal subgroups of  $G$  has finite length.

**Theorem 2.** *Suppose  $G$  a torsion free group satisfying the ascending chain condition on normal subgroups. Then any endomorphism  $G \rightarrow G$  with finite cokernel is a monomorphism.*

This is proved in Section 3 below. The fundamental groups of finite dimensional aspherical spaces are torsion free. Also, in order for a map of closed manifolds to have non-zero Nielsen number, its induced fundamental group homomorphism must have finite cokernel. Thus Theorem 2 should apply to a large number of aspherical manifolds.

However, Theorem 2 only applies in the case of self maps  $f : X \rightarrow X$ . We also would like conditions that can be applied when domain and range are different. A group is *polycyclic* iff it has a normal series whose factor groups are all cyclic. It is *virtually polycyclic* iff it contains a polycyclic subgroup of finite index. By  $\text{cd}G$  we mean the cohomological dimension of  $G$ . In Section 4 we use the Hirsch number and results of Bieri [2, 3] to prove

**Theorem 3.** *Suppose  $G$  and  $H$  torsion free virtually polycyclic groups and  $\text{cd}G = \text{cd}H$ . Then any homomorphism  $G \rightarrow H$  with finite cokernel is a monomorphism.*

This theorem applies to an important class of aspherical manifolds: the *infrasolvmanifolds*. The following description is taken from [10, pp. 15-18]. A closed manifold is an *infrasolvmanifold* iff it is a double coset space  $\Gamma \backslash L / K$  where  $L$  is a Lie group which is both virtually connected and virtually solvable,  $K$  is a maximal compact subgroup of  $L$ , and  $\Gamma$  is a torsion free, co-compact, discrete subgroup of  $L$ . Infrasolvmanifolds are aspherical and have torsion free virtually polycyclic fundamental groups. Every torsion free virtually polycyclic group is the fundamental group of an infrasolvmanifold. Two infrasolvmanifolds are homeomorphic if they have the same fundamental groups. In fact, in dimensions other than three, it is known that every closed aspherical manifold with virtually polycyclic fundamental group is homeomorphic to an infrasolvmanifold. The class of infrasolvmanifolds includes solvmanifolds (polycyclic fundamental group), infranilmanifolds (virtually nilpotent fundamental group), and nilmanifolds (nilpotent fundamental group). The following theorem therefore applies at least to all infrasolvmanifolds.

**Theorem 4.** *Suppose  $f : X \rightarrow Y$  a map of closed aspherical manifolds of the same dimension, and either  $X = Y$  and  $\pi X$  satisfies the ascending chain condition on normal subgroups, or  $\pi X$  is virtually polycyclic. The following are equivalent:*

1.  $N(f, a) > 0$ ,
2.  $N(f, a) = |\text{Coker } f_{\#} : \pi X \rightarrow \pi Y|$ ,
3.  $|\text{Coker } f_{\#} : \pi X \rightarrow \pi Y| < \infty$ ,
4.  $f_{\#} : \pi X \rightarrow \pi Y$  is a monomorphism and therefore Theorem 1 can be applied to compute  $N(f, a)$  in terms of topological degree.

We prove this theorem in Section 5.

Both the Klein bottle and its orientable double cover, the torus, provide examples of infrasolvmanifolds. We use them in Section 6 below to illustrate Theorems 1 and 4.

Finally, we give an example to show that asphericity alone is not enough to imply  $|\text{degree } f| = N(f, a)$ , even for closed orientable manifolds. It is known that for closed orientable manifolds of the same dimension,  $\text{degree } f \neq 0$  iff  $N(f, a) > 0$ , and in this case  $N(f, a)$  divides  $\text{degree } f$  [14, p. 138]. The following theorem is proved in Section 7. It shows that except for these constraints, just about anything can happen.

**Theorem 5.** *Let  $T$  be a surface of genus one (an ordinary torus) and  $T_2$  a*

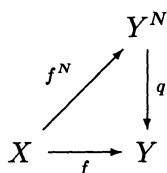
surface of genus two (a double torus). Both  $T$  and  $T_2$  are aspherical closed orientable two dimensional manifolds. For any integers  $k \neq 0$  and  $n > 0$  there is a map  $f : T_2 \rightarrow T$  with Nielsen number  $N(f, a) = n$  and degree  $= kn$ .

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**2. The Nielsen number  $N(f, a)$  and proof of Theorem 1.**

To define the Nielsen number of roots of  $f(x) = a$ , we first group the roots into so-called *root classes*, and then define an *essential* root class. The *Nielsen number* is then the number of essential root classes. The spaces we are dealing with are all connected, locally path connected, and locally simply connected. Therefore, instead of the usual definitions of root class and essential root class, [4, 7] or [14, pp. 125-126], we use an equivalent but more convenient characterization justified by Lemmas 1.1 and 1.2 in [5].

Let  $f : X \rightarrow Y$  be a map and  $a \in Y$ . Assume  $X$  and  $Y$  connected, locally path connected, and semi-locally simply connected. Then  $f$  induces a homomorphism  $f_{\#} : \pi X \rightarrow \pi Y$  of fundamental groups. Since the image of  $f_{\#}$  is a subgroup of  $\pi Y$  there is a covering  $q : Y^N \rightarrow Y$  such that  $q_{\#}$  takes  $\pi Y^N$  isomorphically onto  $f_{\#}\pi X$ . We may lift  $f$  through  $q$  to get a map  $f^N : X \rightarrow Y^N$  and the commutative diagram:



For each point  $a^N$  in the fibre  $q^{-1}(a)$  over  $a$  the set of points  $f^{N-1}(a^N)$  is mapped into  $a$  by  $f$ . Each such set of points is a *Nielsen root class* of  $(f, a)$ . Two points in the same class are *Nielsen equivalent*. A class  $f^{N-1}(a^N)$  is *essential* iff  $f_1^{N-1}(a^N)$  is non-empty for every map  $f_1^N : X \rightarrow Y^N$  homotopic to  $f^N$ . The number of essential root classes is the *Nielsen number* of  $(f, a)$  and is denoted  $N(f, a)$ .

We are interested in the case where  $Y$  is a manifold. Then either every root class is essential or none are [6]. Thus, if  $N(f, a) > 0$ , then the essential root classes are in one-one correspondence with  $q^{-1}(a)$ , which is also in one-one correspondence with  $\text{coker } q_{\#} = \text{coker } f_{\#}$ . Therefore, if  $N(f, a) > 0$ , then  $N(f, a) = |\text{Coker } f_{\#}|$ .

We turn to a proof of Theorem 1.

*Proof of Theorem 1.* Suppose, henceforth, that  $X$  and  $Y$  are both closed aspherical  $n$ -dimensional manifolds, and  $f_{\#} : \pi X \rightarrow \pi Y$  is a monomorphism. Then  $f_{\#}^N : \pi X \rightarrow \pi Y^N$  is an isomorphism, so since  $X$  and  $Y^N$  are both aspherical,  $f^N : X \rightarrow Y^N$  induces homology isomorphisms in all dimensions with all coefficient groups (in fact,  $f^N$  is a homotopy equivalence [10, p. 8]). Since  $X$  is compact, then  $H_n(X; \mathbb{Z}_2) \cong \mathbb{Z}_2$ , so  $H_n(Y^N; \mathbb{Z}_2) \cong \mathbb{Z}_2$ , and therefore  $Y^N$  is also compact (Otherwise  $H_n(Y^N; \mathbb{Z}_2) = 0$ ).

Since  $Y^N$  is compact, then for any point  $y \in Y^N$ , the inclusion  $i : Y^N \subset (Y^N, Y^N - y)$  induces an isomorphism

$$i_n : H_n(Y^N; \mathbb{Z}_2) \cong h_n(Y^N, Y^N - y; \mathbb{Z}_2).$$

Thus

$$(i \circ f^N) : H_n(X; \mathbb{Z}_2) \cong H_n(Y^N, Y^N - y; \mathbb{Z}_2)$$

is non-zero. But this means  $y \in f^N(X)$ , for otherwise we could factor  $i \circ f^N$  through  $(Y^N - y, Y^N - y)$ , and since

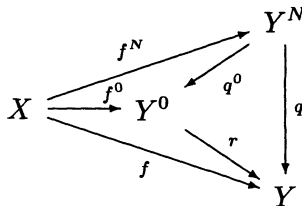
$$H_n(Y^N - y, Y^N - y) = 0,$$

we would have  $(i \circ f^N)_n = 0$ . Therefore  $f^N$  and every map homotopic to  $f^N$  is surjective. Consequently, every root class is essential so

$$N(f, a) = |\text{coker } f_{\#}| > 0.$$

Case 1.  $Y$  is orientable. Then  $Y^N$ , as a covering of an orientable manifold, must also be orientable. Now  $f^N : X \rightarrow Y^N$  induces a homology isomorphism  $H_n(X) \cong H_n(Y^N) \cong \mathbb{Z}$ , using integer coefficients, so  $X$  must also be orientable and  $\text{degree } f^N = \pm 1$ . Since  $f = q \circ f^N$ , we have  $\text{degree } f = (\text{degree } q) \cdot (\text{degree } f^N) = \pm \text{degree } q$ . Since  $q$  is a covering, its degree is  $N(f, a)$ , the cardinality of  $q^{-1}(a)$ . Therefore  $N(f, a) = |\text{degree } f|$ .

Case 2.  $X$  is orientable, but  $Y$  is not. Let  $r : Y^0 \rightarrow Y$  be the orientable double covering. Since  $H_n(Y^N) \cong H_n(X)$ ,  $Y^N$  is also orientable. The covering  $q : Y^N \rightarrow Y$  may therefore be lifted through  $r : Y^0 \rightarrow Y$  to a covering  $q^0 : Y^N \rightarrow Y^0$  resulting in the commutative diagram

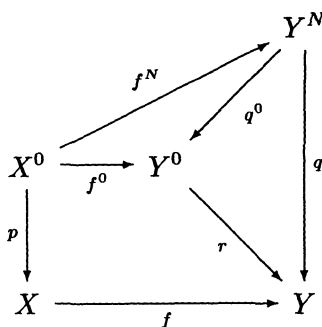


where  $f^0$  is defined to be  $q^0 \circ f^N$ .

Now  $r^{-1}(a)$  consists of two points,  $a_1^0$  and  $a_2^0$ , say. Since  $\text{image } q_{\#}^0 = \text{image } f_{\#}^0$ , then we may use  $q^0 : Y^N \rightarrow Y^0$  to compute  $N(f^0, a_1^0)$  and  $N(f^0, a_2^0)$ . It follows that

$$\begin{aligned} N(f^0, a_1^0) + N(f^0, a_2^0) &= |q^{0^{-1}}(a_1^0)| + |q^{0^{-1}}(a_2^0)| \\ &= |q^{-1}(a)| \\ &= N(f, a). \end{aligned}$$

Case 3. Neither  $X$  nor  $Y$  is orientable. Let  $p : X^0 \rightarrow X$  be the orientable double covering of  $X$ . Now treat  $f \circ p$  in the same way as  $f$  in Case 2 to obtain the commutative diagram



From Case 2,  $N(f \circ p, a) = 2|\text{degree } f^0|$ . But each root class of  $f$  is covered by two root classes of  $f \circ p$  so  $N(f, a) = |\text{degree } f^0|$ . □

### 3. Proof of Theorem 2.

Suppose  $h : G \rightarrow G$  an endomorphism with finite cokernel, and  $G$  is a torsion-free group satisfying the ascending chain condition on normal subgroups. We show that  $h$  is a monomorphism.

Assume, to the contrary, that  $\ker h \neq 1$ . We show first that  $h^2 = h \circ h$  also has a finite cokernel, and second that  $\ker h^2$  properly contains  $\ker h$ . We may then apply the same proof to  $h^2$  and then to  $h^4$ , and so on, thereby creating a properly ascending infinite sequence  $\ker h \subset \ker h^2 \subset \ker h^4 \subset \dots$  of normal subgroups of  $G$  that contradicts the ascending chain assumption.

Because  $h$  induces a surjection of  $G/hG$  onto  $hG/h^2G$ , we have  $[hG : h^2G] \leq [G : hG]$ . Thus,

$$[G : h^2G] = [G : hG] \cdot [hG : h^2G] \leq [G : hG]^2 < \infty.$$

So  $h^2$  has finite cokernel.

It remains to show that  $\ker h^2$  properly contains  $\ker h$ . We need to find an  $x$  in  $\ker h^2$  that is not in  $\ker h$ . Let  $y \neq 1$  and  $y \in \ker h$ . Since  $|G/hG| < \infty$ , we have  $y^n \in hG$  for some  $n$ . Choose  $x$  with  $hx = y^n$ . Since  $G$  is torsion free and  $y \neq 1$ , then  $y^n \neq 1$ . Thus,  $x \notin \ker h$ . On the other hand,  $y \in \ker h$ , so  $h^2x = hy^n = 1$ , and therefore  $x \in \ker h^2$ .

**4. The Hirsch number and a proof of Theorem 3.**

For an arbitrary group  $G$  define its *generalized Hirsch number*,  $\text{gh}G$ , as follows: For any integer  $n$ ,  $\text{gh}G \geq n$  iff there is a normal series

$$1 = G_0 < G_1 < \dots < G_{p-1} < G_p = G$$

for  $G$  in which at least  $n$  factors  $G_i/G_{i-1}$  are infinite. We define  $\text{gh}G = \infty$  if  $\text{gh}G \geq n$  for all integers  $n$ . Otherwise,  $\text{gh}G$  is the largest integer  $n$  for which  $\text{gh}G \geq n$ .

Obviously

**Proposition 4.1.**  $\text{gh}G = 0$  iff  $G$  is finite.

The most important property of  $\text{gh}$  is the following.

**Proposition 4.2.** Suppose that  $1 \rightarrow K \xrightarrow{f} G \xrightarrow{g} Q \rightarrow 1$  is a short exact sequence of groups. Then  $\text{gh}G = \text{gh}K + \text{gh}Q$ .

*Proof.* We first show that  $\text{gh}G \geq \text{gh}K + \text{gh}Q$ . Let  $1 = K_0 < \dots < K_p = K$  and  $1 = Q_0 < \dots < Q_q = Q$  be normal series for  $K$  and  $Q$ . Then  $1 = fK_0 < \dots < fK_p = g^{-1}Q_0 < \dots < g^{-1}Q_q = G$  is a normal series for  $G$ . Since  $f$  is injective, its first  $p$  factors are isomorphic to those in the sequence for  $K$ . Since  $g$  is surjective, then  $g^{-1}Q_i \rightarrow Q_i \rightarrow Q_i/Q_{i-1}$  is surjective with kernel  $g^{-1}Q_{i-1}$ , so  $g^{-1}Q_i/g^{-1}Q_{i-1} \cong Q_i/Q_{i-1}$ . Thus the last  $q$  factors are isomorphic to those for  $Q$ . Therefore the sequence for  $G$  has as many infinite factors as the number for  $K$  plus the number for  $Q$ .

We now show that  $\text{gh}G \leq \text{gh}K + \text{gh}Q$ . Let  $1 = G_0 < \dots < G_p = G$  be a normal series for  $G$ . Define normal series for  $K$  and  $Q$  by  $K_i = f^{-1}G_i$  and  $Q_i = gG_i$  for  $i = 1, \dots, p$ . We claim that for every  $i = 1, \dots, p$

$$|G_i/G_{i-1}| = |K_i/K_{i-1}| \cdot |Q_i/Q_{i-1}|.$$

If the claim is true, then whenever a factor group in the series for  $G$  is infinite, then at least one of the corresponding ones for  $K$  or for  $Q$  must also be infinite. This implies that the number of infinite factors in the series for  $K$ , plus the number in the series for  $Q$ , must be at least as great as the



number in the series for  $G$ , so  $ghG \leq ghK + ghQ$ . It remains to prove the claim.

For each  $i = 0, \dots, p$  there is a short exact sequence  $1 \rightarrow K_i \xrightarrow{f_i} G_i \xrightarrow{g_i} Q_i \rightarrow 1$ , where  $f_i$  and  $g_i$  are restrictions of  $f$  and  $g$ . The composition  $K_i \rightarrow G_i \rightarrow G_i/G_{i-1}$  has kernel  $f^{-1}G_{i-1} = K_{i-1}$ , and therefore induces a monomorphism  $f' : K_i/K_{i-1} \rightarrow G_i/G_{i-1}$ , where  $f'(xK_{i-1}) = f(x)G_{i-1}$ . The composition  $G_i \rightarrow Q_i \rightarrow Q_i/Q_{i-1}$  is a surjection whose kernel  $g_i^{-1}Q_{i-1}$  contains  $G_{i-1}$ , so  $g_i$  induces a surjection  $g' : G_i/G_{i-1} \rightarrow Q_i/Q_{i-1}$  where  $g'(xG_{i-1}) = g(x)Q_{i-1}$ . It suffices to show that the sequence  $1 \rightarrow K_i/K_{i-1} \rightarrow G_i/G_{i-1} \rightarrow Q_i/Q_{i-1} \rightarrow 1$  is exact, for then, by Lagrange's Theorem, we have  $|G_i/G_{i-1}| = |K_i/K_{i-1}| \cdot |Q_i/Q_{i-1}|$ . Since  $f'$  injects and  $g'$  surjects, we have exactness at  $K_i/K_{i-1}$  and  $Q_i/Q_{i-1}$ . Also, for any  $x \in K_i$  we have  $f(g(x)) = 1$ , so  $g'(f'(xK_{i-1})) = g'(f(x)G_{i-1}) = g(f(x))Q_{i-1} = Q_{i-1} = 1 \in Q_i/Q_{i-1}$ . Thus  $\text{image } f' \subset \text{kernel } g'$ . It remains to show  $\text{kernel } g' \subset \text{image } f'$ . Suppose  $xG_{i-1} \in \text{kernel } g'$ . Then  $g(x) \in Q_{i-1}$ . There is therefore a  $y \in G_{i-1}$  with  $g(y) = g(x)$ . Then  $g(xy^{-1}) = 1$ , so  $xy^{-1} = f(z)$  for some  $z \in K_i$ . Then  $f'(zK_{i-1}) = f(z)G_{i-1} = xy^{-1}G_{i-1} = xG_{i-1}$ , since  $y^{-1} \in G_{i-1}$ . Thus  $xG_{i-1} \in \text{image } f'$ .  $\square$

**Proposition 4.3.** *Suppose  $H$  a subgroup of finite index in  $G$ . Then  $ghH = ghG$ .*

*Proof.* Since  $H$  has finite index, it has only a finite number of conjugates in  $G$ , so the intersection  $\text{core } G$  of these conjugates also has finite index, and is normal in both  $H$  and  $G$ . By Proposition 4.1,  $gh(G/\text{core } H) = gh(H/\text{core } H) = 0$ . Applying Proposition 4.2 to the exact sequences  $1 \rightarrow \text{core } H \rightarrow H \rightarrow (H/\text{core } H) \rightarrow 1$  and  $1 \rightarrow \text{core } H \rightarrow G \rightarrow (G/\text{core } H) \rightarrow 1$  we get  $ghH = gh(\text{core } H) + 0 = ghG$ .  $\square$

**Proposition 4.4.** *Suppose  $ghG = ghH < \infty$ ,  $f : G \rightarrow H$  has finite cokernel, and  $G$  is torsion free. Then  $f$  is a monomorphism.*

*Proof.* From Proposition 4.3,  $gh(\text{image } f) = ghH$ . From Proposition 4.2 applied to the exact sequence  $1 \rightarrow \text{ker } f \rightarrow G \rightarrow \text{image } f \rightarrow 1$  we have  $gh(\text{ker } f) = ghG - gh(\text{image } f)$ . Thus  $gh(\text{ker } f) = 0$ . By Proposition 4.1,  $\text{ker } f$  is finite. But  $G$  is torsion free, so this implies  $\text{ker } f = 1$ .  $\square$

In 1938 Karl Hirsch showed [12] that when  $G$  is polycyclic, and  $1 = G_0 < \dots < G_p = G$  is a normal series with cyclic factor groups, then the number of infinite cyclic factors does not depend on the choice of series (see also [22, pp. 150-154]). This number is now [11, p. 149] called the *Hirsch number* of  $G$ , denoted by  $hG$ .

**Proposition 4.5.**  $ghG = hG$  for any polycyclic group  $G$ .

*Proof.* Suppose  $G$  polycyclic, and let  $1 = G_0 < \cdots < G_p = G$  be a normal series in which each factor  $G_i/G_{i-1}$  is cyclic. Clearly  $hG_0 = 0 = ghG_0$ . Assume  $0 < i \leq p$  and  $ghG_{i-1} = hG_{i-1}$ . By definition of  $h$ ,  $hG_i = hG_{i-1}$  if  $G_i/G_{i-1}$  is finite cyclic, and  $hG_i = hG_{i-1} + 1$  if  $G_i/G_{i-1}$  is infinite cyclic. On the other hand, if  $H$  is infinite cyclic, then  $ghH = 1$ , since every normal series for  $H$  has exactly one infinite factor, so by exactness of  $1 \rightarrow G_{i-1} \rightarrow G_i \rightarrow G_i/G_{i-1} \rightarrow 1$ , and Propositions 4.1 and 4.2, we also have  $ghG_i = ghG_{i-1}$  if  $G_i/G_{i-1}$  is finite cyclic, and  $ghG_i = ghG_{i-1} + 1$  if  $G_i/G_{i-1}$  is infinite cyclic. Thus  $ghG_i = hG_i$ , so by induction,  $ghG = hG$ .  $\square$

In [2, p. 390] Bieri uses the Lyndon-Hochschild-Serre spectral sequence to show that every polycyclic group is a Poincaré duality group with cohomological dimension equal to its Hirsch number. He also shows [2, p. 384] that a torsion free extension of a Poincaré duality group by a finite factor group is a Poincaré duality group of the same cohomological dimension (see also [3] for these results). Combining Bieri's results with Propositions 4.3 and 4.5 gives

**Proposition 4.6.** *Every torsion free virtually polycyclic group  $G$  is a Poincaré duality group with  $cdG = ghG$ .*

Theorem 3 follows immediately from Proposition 4.4 and 4.6.

## 5. Proof of Theorem 4.

We show  $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$ .

$1 \Rightarrow 2 \Rightarrow 3$ . If  $N(f, a) > 0$ , then [6, p. 725]  $N(f, a) = |\text{coker } f_\#|$ . But [14, p. 126]  $N(f, a) < \infty$ , so  $|\text{coker } f_\#| < \infty$ .

$3 \Rightarrow 4$ . Suppose  $|\text{coker } f_\#| < \infty$ . Since  $X$  and  $Y$  are both aspherical manifolds, the groups  $\pi X$  and  $\pi Y$  are torsion free. If  $X = Y$  and  $\pi X$  satisfies the ascending chain condition, then Theorem 2 implies that  $f_\#$  is a monomorphism. In the other case, if  $\pi X$  is virtually polycyclic, then so is its image  $f_\#\pi X$ . But, since  $f_\#\pi X$  has finite index in  $\pi Y$ , this implies  $\pi Y$  is also virtually polycyclic, so Theorem 3 implies that  $f_\# : \pi X \rightarrow \pi Y$  is a monomorphism.

$4 \Rightarrow 1$ . This follows from Theorem 1.

## 6. Maps of the torus and Klein bottle.

In this section we illustrate Theorems 1 and 4 by computing the Nielsen number for maps of the two-dimensional torus  $T$  and Klein bottle  $K$ . Both

spaces are two-dimensional closed infrasolvmanifolds, so Theorems 1 and 4 apply. Since the torus is orientable and the Klein bottle is not, we may use these spaces to illustrate all three cases of Theorem 1.

Build  $K$  as a  $CW$  complex by attaching two one-cells  $B$  and  $C$  by their ends to a single zero-cell, forming a figure-eight, or “bouquet of two circles”. To this attach a two-cell by its boundary, so the boundary traces out the curve  $BCB^{-1}C$ . Then in terms of generators and relations,

$$\pi K = \langle B, C | BCB^{-1}C = 1 \rangle.$$

Similarly, build  $T$  by attaching two one-cells  $D_1$  and  $D_2$  to a single zero-cell forming another bouquet of two circles, and then attach a two-cell so its boundary traces out the curve  $D_1D_2D_1^{-1}D_2^{-1}$ . Then

$$\pi T = \langle D_1, D_2 | D_1D_2D_1^{-1}D_2^{-1} \rangle = 1.$$

Because  $\pi T$  is free abelian on the two generators  $D_1$  and  $D_2$ , the homomorphism  $f_{\#} : \pi T \rightarrow \pi T$  induced by a map has a matrix relative to the basis  $D_1$  and  $D_2$ . We make repeated use of the fact that the degree of  $f$  is given by the determinant of this matrix. (Sketch of proof: The generator of the top cohomology group  $H^2(T)$  is the cup-product  $D_1^* \vee D_2^*$  of the cocycles dual to  $D_1$  and  $D_2$ , so  $f^2(D_1^* \vee D_2^*) = (\det A)(D_1^* \vee D_2^*)$ .)

Although  $\pi X$  is not abelian, the relation  $BCB^{-1}C = 1$  may still be used to express any element in  $\pi K$  uniquely in the form  $B^xC^y$ . The relation also implies that  $C^yB^x = B^xC^y$  if  $x$  is even, and  $C^yB^x = B^xC^{-y}$  if  $x$  is odd. In particular,  $(B^xC^y)^2 = B^{2x}$  whenever  $x$  is odd. It also implies  $B^2$  and  $C$  commute, so together they freely generate an abelian subgroup of  $\pi K$  of index two. The monomorphism  $q_{\#} : \pi T \rightarrow \pi K$  defined by  $q_{\#}D_1 = B^2$  and  $q_{\#}D_2 = C$  takes  $\pi T$  isomorphically onto this subgroup. In fact,  $q_{\#}$  is induced by the orientable double covering  $q : T \rightarrow K$ .

The following four propositions give formulas for the Nielsen number for maps  $T \rightarrow T$ ,  $T \rightarrow K$ ,  $K \rightarrow T$ , and  $K \rightarrow K$ . In the  $K \rightarrow T$  case, the Nielsen number is always zero. In the other cases, the formulas are given in terms of the images of the generators of the fundamental groups under the induced homomorphisms.

**Proposition 6.1.** *Suppose  $f : T \rightarrow T$  a map and let  $A$  be the matrix of  $f_{\#} : \pi T \rightarrow \pi T$  relative to the basis  $C_1$  and  $C_2$ . Then  $N(f, a) = |\det A|$ .*

*Proof.* If  $\det A = 0$ , then  $f_{\#}$  is not a monomorphism, so Theorem 4 implies  $N(f, a) = 0$ . If  $\det A \neq 0$ , then  $f_{\#}$  is a monomorphism so Case 1 of Theorem 1 implies  $N(f, a) = |\text{degree } f| = |\det A|$ . □

**Proposition 6.2.** *Let  $f : T \rightarrow K$  be a map and  $f_{\#} : \pi T \rightarrow \pi K$  its induced homomorphism. Let*

$$A = \begin{pmatrix} b & c \\ d & e \end{pmatrix}$$

where  $b, c, d,$  and  $e$  are characterized by  $f_{\#}D_1 = B^bC^d$  and  $f_{\#}D_2 = B^cC^e$ . If both  $b$  and  $c$  are even, then  $N(f, a) = |\det A|$ . Otherwise,  $f_{\#}$  is not monomorphic, and  $N(f, a) = 0$ .

*Proof.* If  $f_{\#}$  is not monomorphic, then Theorem 4 implies that  $N(f, a) = 0$ . Suppose henceforth that  $f_{\#}$  is monomorphic. Then, from Case 2 of Theorem 1, there is a lift  $f^0 : T \rightarrow T$  of  $f$  through  $q$ , so  $q_{\#} \circ f^0_{\#} = f_{\#}$ . Then  $f^0_{\#}D_1 = D_1^x D_2^y$  for some integers  $x$  and  $y$ . Thus  $B^bC^d = q_{\#} \circ f^0_{\#}D_1 = q_{\#}D_1^x D_2^y = B^{2x}C^y$ , so  $f^0_{\#}D_1 = D_1^{b/2}C_2^d$  and  $b$  is even. Similarly  $f^0_{\#}D_2 = D_1^{c/2}D_2^e$  and  $c$  is even. From Case 2 of Theorem 1,  $N(f, a) = 2|\text{degree } f^0|$ . But

$$\text{degree } f^0 = \det \begin{pmatrix} b/2 & c/2 \\ d & e \end{pmatrix} = \frac{1}{2} \det A.$$

Therefore,  $N(f, a) = \det A$ . □

**Proposition 6.3.** *For any map  $f : K \rightarrow T$ ,  $N(f, a) = 0$ .*

*Proof.* Since  $T$  is orientable and  $K$  is not, Case 1 of Theorem 1 implies that  $f_{\#}$  cannot be a monomorphism. (This also follows from the fact that  $\pi K$  is not abelian, but  $\pi T$  is.) Therefore Theorem 4 implies that  $N(f, a) = 0$ . □

**Proposition 6.4.** *Let  $f : K \rightarrow K$  be a map. Then there are integers  $b, d,$  and  $e$  such that  $f_{\#}B = B^bC^d$ ,  $f_{\#}C = C^e$ , and either  $b$  is odd or  $e = 0$ . In either event,  $N(f, a) = |be|$ .*

*Proof.* Clearly  $f_{\#}B = B^bC^d$  and  $f_{\#}C = B^cC^e$  for some integers  $b, c, d$  and  $e$ . We need to show that  $c = 0$ , and that either  $b$  is odd or  $e = 0$ .

One can show that the commutator subgroup of  $\pi K$  is  $\langle C^2 \rangle$ , so  $f_{\#}C^2 \in \langle C^2 \rangle$ . But the relation  $BCB^{-1}C = 1$  implies  $f_{\#}C^2 = B^{2c}C^x$  (where  $x = (-1)^c e + e$ ), so  $2c = 0$ , and therefore  $c = 0$ .

One computes  $1 = f_{\#}BCB^{-1}C = B^bC^dC^eC^{-d}B^{-b}C^e = C^x$  where  $x = (-1)^b e + e$ . Since  $x = 0$ , either  $b$  is odd or  $e = 0$ .

If either  $b = 0$  or  $e = 0$ , then  $f_{\#}$  is not monomorphic, so Theorem 4 implies that  $N(f, a) = |be| = 0$ . Suppose now that both  $b \neq 0$  and  $e \neq 0$ . Then  $f_{\#}$  is monomorphic so we apply Case 3 of Theorem 1. To do so we need to find  $f^0_{\#} : \pi T \rightarrow \pi T$  so that  $f_{\#} \circ q_{\#} = q_{\#} \circ f^0_{\#}$ .

Now  $f_{\#} \circ q_{\#}D_1 = f_{\#}B^2 = (B^bC^d)^2$ . But  $b$  is odd, since  $e \neq 0$ , so  $(B^bC^d)^2 = B^{2b}$ . Hence  $f_{\#} \circ q_{\#}D_1 = B^{2b} = q_{\#}D_1^b$ . Also,  $f_{\#} \circ q_{\#}D_2 =$

$f_{\#}C = C^e = q_{\#}D_2^e$ . So define  $f_{\#}^0$  by  $f_{\#}^0D_1 = D_1^b$ , and  $f_{\#}D_2 = D_2^e$ , to get  $f_{\#} \circ q_{\#} = q_{\#} \circ f_{\#}^0$ . Since the matrix of  $f_{\#}^0 : \pi T \rightarrow \pi T$  has determinant  $be$ , the degree of  $f^0 : T \rightarrow T$  is  $be$ . By Case 3 of Theorem 1,  $N(f, a) = |\text{degree} f^0| = |be|$ . □

### 7. Maps from the double torus to the ordinary torus.

The ordinary two dimensional torus and double torus  $T_2$  are well known to be closed, orientable, and aspherical. In this section we classify the maps from  $T_2$  into  $T$ . In particular, for any integers  $n > 0$  and  $k \neq 0$ , we exhibit a map with  $N(f, a) = n$  and  $\text{degree} f = nk$ , thus proving Theorem 5.

We use the usual  $CW$  structure for  $T_2$ : Four one-cells  $C_1, C_2, C_3$ , and  $C_4$  have their ends attached to a single zero-cell to form a ‘‘bouquet of four circles.’’ To this, the boundary of a single two-cell is attached so that the boundary traces out the curve

$$C_1C_2C_1^{-1}C_2^{-1}C_3C_4C_3^{-1}C_4^{-1}.$$

We use the same  $CW$  structure for  $T$  as in the previous section. Then

$$\begin{aligned} \pi T_2 &= \langle C_1, C_2, C_3, C_4 | C_1C_2C_1^{-1}C_2^{-1}C_3C_4C_3^{-1}C_4^{-1} = 1 \rangle \\ \pi T &= \langle D_1, D_2 | D_1D_2D_1^{-1}D_2^{-1} = 1 \rangle. \end{aligned}$$

Since  $T_2$  and  $T$  are aspherical, every homomorphism  $\pi T_2 \rightarrow \pi T$  is induced by a map  $T_2 \rightarrow T$ , unique up to homotopy. Since  $\pi T$  is abelian, the homomorphisms  $\pi T_2 \rightarrow \pi T$  are in one to one correspondence with their abelianizations  $H_1T_2 \rightarrow H_1T$  on the first homology groups. Thus, given any homomorphism  $H_1T_2 \rightarrow H_1T$ , there is a unique (up to homotopy) map  $T_2 \rightarrow T$  inducing it. We therefore concentrate on homology.

Now  $H_1T_2$  is free abelian on the four generators  $C_1, C_2, C_3$ , and  $C_4$ , and  $H_1T$  is free abelian on the two generators  $D_1$  and  $D_2$ . So for any map  $f : T_2 \rightarrow T$ , there is a  $2 \times 4$  integer matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}$$

for  $f_1 : H_1T_2 \rightarrow H_1T$ , relative to the bases  $\{C_1, \dots, C_4\}$  and  $\{D_1, D_2\}$ , and any such matrix determines a map  $f : T_2 \rightarrow T$  (up to homotopy).

In terms of this matrix,  $|\text{coker} f_{\#}| = |\text{coker} f_1|$  is the greatest common divisor of all the determinants of  $2 \times 2$  submatrices of  $A$  [4, p. 39]. Thus, if  $\text{degree} f \neq 0$ , then

$$N(f, a) = \text{g. c. d.} \{ \det B | B \text{ a } 2 \times 2 \text{ submatrix of } A \}.$$

In order to compute the degree of  $f$ , we use the cohomology rings  $H^*T_2$  and  $H^*T$ .

**Proposition 7.1.** *Let  $C_1^*, \dots, C_2^*$  and  $D_1^*, D_2^*$  be the 1-cocycles dual to  $C_1, \dots, C_4$ , and  $D_1, D_2$ . Then*

1.  $D_1^* \vee D_2^* = -D_2^* \vee D_1^* = V^*$ , where  $V^*$  generates  $H^2T$ , and  $D_1^* \vee D_1^* = D_2^* \vee D_2^* = 0$ .
2.  $C_1^* \vee C_2^* = -C_2^* \vee C_1^* = C_3^* \vee C_4^* = -C_4^* \vee C_3^* = U^*$ , where  $U^*$  generates  $H^2T_2$ , and  $C_i^* \vee C_j^* = 0$  for all other  $i, j$ .

*Proof.* For a proof of 1 see [17, p. 348]. We use 1 to sketch a proof of 2.

First, define a homomorphism  $g_\# : \pi T_2 \rightarrow \pi T$  by  $g_\# C_1 = D_1$ ,  $g_\# C_2 = D_2$  and  $g_\# C_3 = g_\# C_4 = 1$ . Then  $g_\#$  may be induced by a map  $g : T_2 \rightarrow T$  that is a relative homeomorphism of the single two-cell of  $T_2$  (relative to the boundary) onto the single two-cell of  $T$ , and maps the path  $C_1 C_2 C_1^{-1} C_2^{-1}$  onto the boundary  $D_1 D_2 D_1^{-1} D_2^{-1}$  of the two-cell of  $T$ , and takes the rest of the boundary,  $C_3 C_4 C_3^{-1} C_4^{-1}$ , to the single zero-cell of  $T$ . Since  $g$  is a relative homeomorphism of the two-cell of  $T_2$  onto the two-cell of  $T$ , it induces homology and cohomology isomorphisms in dimension two. Therefore  $g^*(V^*)$  is a generator of  $H^2(T_2)$ . Let  $U^* = g^*(V^*)$ . Then  $U^* = g^*(V^*) = g^*(D_1^* \vee D_2^*) = g^*(D_1^*) \vee g^*(D_2^*) = C_1^* \vee C_2^*$ . A similar construction shows that  $U^* = C_3^* \vee C_4^*$ .

To compute  $C_1^* \vee C_3^* = 0$ , redefine  $g_\# : \pi T_2 \rightarrow \pi T$  by  $g_\# C_1 = D_1$ ,  $g_\# C_3 = D_2$ , and  $g_\# C_2 = g_\# C_4 = 1$ . Then  $g_\#$  is induced by a free group homomorphism  $\langle C_1, \dots, C_4 \rangle \rightarrow \langle D_1, D_2 \rangle$  that takes the relator  $C_1 C_2 C_1^{-1} C_2^{-1} C_3 C_4 C_3^{-1} C_4^{-1}$  to  $D_1 D_1^{-1} D_2 D_2^{-1} = 1 \in \langle D_1, D_2 \rangle$ . But this implies that  $g_\#$  may be factored through the free group  $\langle D_1, D_2 \rangle$ . Since this is the group of the figure-eight space (bouquet of two circles), any map  $g : T_2 \rightarrow T$  inducing  $g_\#$  is homotopic to a map that may be factored through the figure-eight. The figure-eight is one-dimensional, so  $g^*(H^2T_2) = 0$ . Therefore  $C_1^* \vee C_3^* = g^*(D_1^* \vee D_2^*) = 0$ . Similarly  $C_1^* \vee C_4^* = C_2^* \vee C_3^* = C_2^* \vee C_4^* = 0$ .

The other products may be computed from these by antisymmetry. □

The matrix for  $f^1 : H^1T_2 \rightarrow H^1T$  relative to the dual bases is the transpose of  $A$ . Therefore

$$\begin{aligned}
 f^2(V^*) &= f^2(D_1^* \vee D_2^*) \\
 &= f^1(D_1^*) \vee f^1(D_2^*) \\
 &= \sum_j a_{1,j} C_j^* \vee \sum_j a_{2,j} C_j^* \\
 &= (a_{11} a_{22} - a_{12} a_{21}) U^* + (a_{13} a_{24} - a_{14} a_{23}) U^*
 \end{aligned}$$

so

$$\text{degree } f = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{13} & a_{14} \\ a_{23} & a_{24} \end{vmatrix}.$$

Finally, let  $k$  and  $n$  be any integers with  $k \neq 0$  and  $n > 0$ . Choose  $f : T_2 \rightarrow T$  so that  $f_1 : H_1 T^2 \rightarrow H_1 T$  has the matrix

$$\begin{pmatrix} nk & 0 & n & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Then

$$\text{degree } f = nk + 0 = nk,$$

and, since  $\text{degree } f \neq 0$ ,

$$N(f, a) = \text{g. c. d. } \{nk, nk, 0, -n, 0, 0\} = n$$

which proves Theorem 5.

### References

- [1] D. Anosov, *The Nielsen number of maps of nil-manifolds*, Russian Math. Surveys, **40** (1985), 149-150.
- [2] R. Bieri, *Gruppen mit Poincaré-Dualität*, Comment. Math. Helv., **47** (1972), 373-396.
- [3] \_\_\_\_\_, *Homological dimension of discrete groups*, Queen Mary College Mathematics Notes, London, 1976.
- [4] R. Brooks, *Coincidences, roots and fixed points*, Ph. D. dissertation, University of California, Los Angeles, CA (1967).
- [5] \_\_\_\_\_, *On the sharpness of the  $\Delta_2$  and  $\Delta_1$  Nielsen numbers*, J. Reine Angew. Math., **259** (1973), 101-108.
- [6] \_\_\_\_\_, *Certain subgroups of the fundamental group and the number of roots of  $f(x) = a$* , Amer. J. Math., **95** (1973), 720-728.
- [7] R. Brooks and R. Brown, *A lower bound for the  $\Delta$ -Nielsen number*, Trans. Amer. Math. Soc., **143** (1969), 555-564.
- [8] R. Brooks, R. Brown, J. Pak and D. Taylor, *Nielsen numbers of maps of tori*, Proc. Amer. Math. Soc., **52** (1975), 398-400.
- [9] E. Fadell and S. Husseini, *On a theorem of Anosov on Nielsen numbers for nil-manifolds*, Nonlinear Functional Analysis and its applications (Maratea, 1985), 47-53 NATO Adv. Sci. Inst. Ser. C: Math. Phys Sci., **173**, Reidel, Dordrecht-Boston, Mass., 1986.
- [10] F.T. Farrell and L.E. Jones, *Classical Aspherical Manifolds*, CBMS Lecture Notes, #75, Amer. Math. Soc., Providence, RI 1990.
- [11] K. Gruenberg, *Cohomological topics in group theory*, *Lectures in Math.*, Springer-Verlag, Berlin-Heidelberg-New York, 1970.
- [12] K. Hirsch, *On infinite soluble groups I*, Proc. London Math. Soc., **44** (1938), 53-60.

- [13] E. Keppelman and C. McCord, *The Anosov theorem for exponential solvmanifolds*, to appear in Pacific Journal of Math.
- [14] Kiang Tsai-han, *The theory of fixed point classes*, Springer-Verlag, Berlin-Heidelberg-New York, 1987.
- [15] S. Kwasik and K.B. Lee, *The Nielsen numbers of homotopically periodic maps of infranilmanifolds*, J. London Math. Soc., (2), **38** (1988), 544-554.
- [16] K. Lee, *Nielsen numbers of periodic maps on solvmanifolds*, To appear in J. London Math. Soc.
- [17] W. Massey, *A basic course in algebraic topology*, Springer-Verlag, Berlin-Heidelberg-New York, 1991.
- [18] C. McCord, *Nielsen numbers and Lefschetz numbers on solvmanifolds*, Pac. J. Math., **147** (1991), 153-164.
- [19] \_\_\_\_\_, *Lefschetz and Nielsen coincidence numbers on nilmanifolds and solvmanifolds*, Topology Appl., **43** (1992), 249-261.
- [20] \_\_\_\_\_, *Estimating Nielsen numbers of infrasolvmanifolds*, Pac. J. Math.. **154** (1992), 345-368.
- [21] \_\_\_\_\_, *Nielsen numbers of homotopically periodic maps on infrasolvmanifolds*, Preprint.
- [22] W.R. Scott, *Group Theory*, Dover Publications, New York, 1987. (Originally published by Prentice-Hall, Englewood Cliffs, NJ, 1964.)

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