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REPRODUCING KERNELS AND COMPOSITION SERIES FOR SPACES OF VECTOR-VALUED HOLOMORPHIC FUNCTIONS

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We calculate the norm of each K-type in a vector-valued Hilbert space of holomorphic functions on a tube domain of type I. As a consequence we get composition series of the analytic continuation of certain holomorphic discrete series and an expansion relative to K of the matrix-valued reproducing kernel.

Introduction.

The theory of unitary highest weight modules for semisimple Lie groups is by now very well developed. Questions of classification, intertwining operators and primitive ideals have been settled by algebraic means, see [**1**, **2**, **3**] and [**8**] and reference there. However, there remain some open problems on the analytic side, in particular to find analytical proofs and expressions for the unitarity.

The problem we consider here is to calculate by purely analytical means the invariant Hermitian form in a Harish-Chandra module of highest weight. At the same time, we find explicitly the K-types in a composition series at reducible values of the parameter for the module. This is of interest for example in finding explicit intertwining differential operators giving the various subquotients.

The problem of composition series and expansion of reproducing kernels for analytic continuations of the holomorphic discrete series has been studied extensively. In [**15**] this is done for the type I domain of tube type and for the scalar-valued Bergman spaces by calculating the norm of each K-type. Recall the simplest case, where $G = SU(1,1)$ and the expansion amounts to the binomial formula

$$(1 - z\bar{w})^{-\lambda} = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{k!} z^k \bar{w}^k.$$

Here the monomials z^k, $k \geq 0$, of the complex variable z in the unit disk give the K-types of the corresponding highest weight module, and the coefficients in the expansion give the reciprocals of the invariant Hermitian form for the module. Faraut and Koranyi [**4**] recently solved this problem for a general bounded symmetric domain by using the generalized Gamma-function. The similar problem for vector-valued function spaces of holomorphic functions has been around for some time, see [**17**]. However, it seems to us that even for the group $SU(2,2)$ with the "smallest" representation of $U(2)$ on $\mathbb{C}^2$ as fiber no explicit result is known. Generally speaking, this problem involves the explicit decomposition of tensor products of representations of compact groups, which according [**20**], is *prohibitive.*

In this paper we give the expansion of the reproducing kernel of a vector-valued Bergman space of holomorphic functions on a tube domain of type I. We consider here the first simple nontrivial representation of the compact group $U(n)$ (or rather $S(U(n), U(n))$), namely its defining representation on $\mathbb{C}^n$. The K-irreducible decomposition of the space of $\mathbb{C}^n$-valued polynomials can be then read off abstractly. We find the norm of each K-space and give the expansion of the matrix-valued reproducing kernel.

As an application we can then read off from our formula the composition series in the analytic continuation and also the unitarizablity of the quotients in the composition series. Since the arguments are similar to those in [**4**] and [**15**] we only sketch the proofs.

The calculation of the tensor product decompositions in this paper is elementary. We can certainly use the spherical functions on compact groups and Harish-Chandra c-functions to simplify our result. On the other hand our calculation gives more information about the decomposition. In a subsequent paper we will use the result in this paper to study the vector-valued function spaces of holomorphic functions on some other bounded symmetric domains.

The main results are Theorems 2, 3, and Corollary 1, 2 but also Lemma 1 and Theorem 1 might be of independent interest giving detailed structure of the tensor products with $\mathbb{C}^n$.

§1. Decomposition of tensor products.

In this section we give the decomposition of tensor products of holomorphic representations of $U(n)$ with its defining representation on $\mathbb{C}^n$. The result will be used in the next section to calculate the norm of each K-type in Hilbert space of vector-valued holomorphic functions on

$$D = SU(n,n)/S(U(n), U(n)).$$

Before going into the calculation we briefly recall here the root system for the compact group $U(n)$ and fix some notations. See [**20**].

The Lie algebra $u(n)$ of $U(n)$ has complixification $u(n)_{\mathbb{C}} = gl(n, \mathbb{C})$, the Lie algebra of $GL(n, \mathbb{C})$, which has the standard realization as $n \times n$ complex matrices. Denote E_i the diagonal matrix with ith part being 1 and the rest 0, and $E_{i,j}$, $i \neq j$, the matrix with (i,j) entry being 1 and the rest 0. The elements $\{E_i, i = 1, \ldots, n\}$ span a Cartan algebra of $u(n)_{\mathbb{C}}$. We denote ε_i the linear functional on the Cartan algebra defined by $\varepsilon_i(E_j) = \delta_{i,j}$. The root spaces and roots are $\mathbb{C}E_{i,j}$ and $\varepsilon_{i,j} = \varepsilon_i - \varepsilon_j$. respectively. The postive and positive simple roots are $\varepsilon_i - \varepsilon_j$, $i < j$ and $\varepsilon_i - \varepsilon_{i+1}$, $i = 1, \ldots, n-1$ respectively.

Let $(V^{\underline{\mathbf{m}}}, \underline{\mathbf{m}})$ be a holomorphic representation of $U(n)$ with highest weight $\underline{\mathbf{m}} = (m_1, \ldots, m_n) = m_1\varepsilon_1 + \cdots + m_n\varepsilon_n$, where $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. Denote $v_{\underline{\mathbf{m}}}$ its highest weight vector with $\|v_{\underline{\mathbf{m}}}\| = 1$. When $\underline{\mathbf{m}} = (1, 0, \ldots, 0) = \varepsilon_1$, $V^{\underline{\mathbf{m}}} = \mathbb{C}^n$ is the defining representation of $U(n)$. We denote $\{e_i\}$ its orthonormal basis $e_1 = (1, 0, \ldots, 0)$, ..., $e_n = (0, 0, \ldots, 1)$. Hence e_1 is the highest weight vector. We normalize so that $\|e_1\| = 1$.

It follows that ([**20**])

$$V^{\underline{\mathbf{m}}} \otimes \mathbb{C}^n = \sum_{i=1}^{n} V^{\underline{\mathbf{m}}+\varepsilon_i}, \tag{1.1}$$

where the ith term will not appear if $\underline{\mathbf{m}} + \varepsilon_i$ violates the condition for highest weight, that is if $m_{i-1} = m_i$. We let as above $v_{\underline{\mathbf{m}}+\varepsilon_i}$ be the highest weight vector of $V^{\underline{\mathbf{m}}+\varepsilon_1}$. Our objective in this section is to write explicitly the vectors $v_{\underline{\mathbf{m}}} \otimes e_i$ as sum of their components in $V^{\underline{\mathbf{m}}+\varepsilon_j}$ according to the above decomposition.

Lemma 1. *We have the following formulae for the highest weight vectors $v_{\underline{\mathbf{m}}+\varepsilon_i}$ for $i = 1, 2$*

$$\begin{aligned} v_{\underline{\mathbf{m}}+\varepsilon_1} &= v_{\underline{\mathbf{m}}} \otimes e_1, \\ v_{\underline{\mathbf{m}}+\varepsilon_2} &= f \otimes e_1 - E_{1,2}f \otimes e_2, \end{aligned}$$

where $f \in E_{2,1}V^{\underline{\mathbf{m}}}$ is such that $\|f\|^2 + \|E_{1,2}f\|^2 = 1$. If $i \geq 3$, they are of the form

$$v_{\underline{\mathbf{m}}+\varepsilon_i} = f \otimes e_1 - E_{1,2}f \otimes e_2 - \cdots - E_{1,i}f \otimes e_i,$$

where $f \in V^{\underline{\mathbf{m}}}$ is a weight vector of weight $\underline{\mathbf{m}} - \varepsilon_{1,i}$ and $E_{2,3}f = \cdots = E_{i-1,i}f = 0$.

Proof. Since $v_{\underline{\mathbf{m}}}$ is a highest weight vector of $V^{\underline{\mathbf{m}}}$ of weight $\underline{\mathbf{m}}$ we see that the vector $v_{\underline{\mathbf{m}}} \otimes e_1$ is of weight $\underline{\mathbf{m}} + \varepsilon_1$. It is annihilated by the operators $E_{i,i+1}$:

$$E_{i,i+1}(v_{\underline{\mathbf{m}}} \otimes e_1) = (E_{i,i+1}v_{\underline{\mathbf{m}}}) \otimes e_1 + v_{\underline{\mathbf{m}}} \otimes (E_{i,i+1}e_1) = 0.$$

Moreover $\|v_{\underline{\mathbf{m}}} \otimes e_1\| = 1$ so we can take $v_{\underline{\mathbf{m}}+\varepsilon_1} = v_{\underline{\mathbf{m}}} \otimes e_1$.

The element $f \otimes e_1 - E_{1,2}f \otimes e_2$ for $f \in E_{2,1}V^{\underline{\mathbf{m}}}$ is of weight $\underline{\mathbf{m}} + \varepsilon_2$. We need only to prove that it is annihilated by $E_{i,i+1}$. Since the weight space with weight $\underline{\mathbf{m}} + \varepsilon_{1,2}$ is the zero space, we know that $E_{1,2}E_{1,2}f = 0$. Clearly

$$E_{1,2}(f \otimes e_1 - E_{1,2}f \otimes e_2) = (E_{1,2}f) \otimes e_1 - E_{1,2}f \otimes (e_1) = 0.$$

If $j \geq 2$, then $E_{j,j+1}f = 0$ since it is of weight $\underline{\mathbf{m}} - \varepsilon_{1,2} + \varepsilon_{j,j+1}$, whose weight space is $\{0\}$ because $\varepsilon_{1,2}$ and $\varepsilon_{j,j+1}$ are simple positive. Therefore $f \otimes e_1 - E_{1,2}f \otimes e_2$ is a highest weight vector and is of norm 1. This proves the formula for $v_{\underline{\mathbf{m}}+\varepsilon_2}$.

Now we prove the formula for $v_{\underline{\mathbf{m}}+\varepsilon_i}$, $i \geq 3$. Let $f_1, \dots, f_n$ in $V^{\underline{\mathbf{m}}}$ be such that

$$v_{\underline{\mathbf{m}}+\varepsilon_i} = f_1 \otimes e_1 + \cdots + f_n \otimes e_n$$

if $V^{\underline{\mathbf{m}}+\varepsilon_i}$ exists. By computing weights in both sides we see that f_j is of weight $\underline{\mathbf{m}} - \varepsilon_{j,i}$. However $-\varepsilon_{j,i}$ is a positive root if $j > i$, whose weight space is $\{0\}$. So $f_j = 0$ if $j > i$. That is

$$v_{\underline{\mathbf{m}}+\varepsilon_i} = f_1 \otimes e_1 + \cdots + f_i \otimes e_i,$$

and f_i is a constant multiple of the highest weight vector $v_{\underline{\mathbf{m}}}$ by the above weight argument.

The condition $E_{j,j+1}v_{\underline{\mathbf{m}}+\varepsilon_i} = 0$, $j = 1, 2, \dots, i-1$ implies that $E_{j,j+1}f_1 = 0$ and

$$f_2 = -E_{1,2}f_1, \ \dots \ f_i = -E_{i-1,i}f_{i-1}.$$

Hence

$$f_3 = -E_{2,3}f_2 = E_{2,3}E_{1,2}f_1 = [E_{2,3}, E_{1,2}]f_1 = -E_{1,3}f_1,$$

where the third equality is obtained since $E_{2,3}f_1 = 0$. Continuing similar calculation we get $f_j = -E_{1,j}f_1$, $j = 2, \dots, i$. This finishes the proof of the Lemma. □

Lemma 2. *The vectors* $v_{\underline{\mathbf{m}}} \otimes e_i$ *have the following decomposition according to* (1.1)

$$\begin{aligned}
v_{\underline{\mathbf{m}}} \otimes e_1 &= a_{1,1}P_1^{(1)}, \\
v_{\underline{\mathbf{m}}} \otimes e_2 &= a_{2,1}P_1^{(2)} + a_{2,2}P_2^{(2)} \\
&\vdots \\
v_{\underline{\mathbf{m}}} \otimes e_n &= a_{n,1}P_1^{(n)} + a_{n,2}P_2^{(n)} + \cdots + a_{n,n-1}P_{n-1}^{(n)} + a_{n,n}P_n^{(n)},
\end{aligned}$$

where $P_i^{(j)} \in V^{\underline{\mathbf{m}}+\varepsilon_i}$ and $\|P_i^{(j)}\| = 1$. That is, each vector $v_{\underline{\mathbf{m}}} \otimes e_i$ has no projection in the space $V^{\underline{\mathbf{m}}+\varepsilon_j}$ if $j > i$.

Proof. This can be seen by using the similar weight argument as in the proof of Lemma 1. □

Theorem 1. *We have the following formula for entries $a_{k,i}$, $k \geq i$, in the above (lower triangular) matrix*

$$|a_{k,i}|^2 = \frac{\prod_{j=1,j\neq i}^{k-1}(m_i - m_j - i + j + 1)}{\prod_{j=1,j\neq i}^{k}(m_i - m_j - i + j)},$$

where $|a_{1,1}|^2 = 1$.

To prove the Theorem, we need an identity.

Lemma 3. *The following identity holds*

$$\sum_{i=1}^{k} \frac{\prod_{j=1,j\neq i}^{k-1}(m_i - m_j - i + j + 1)}{\prod_{j=1,j\neq i}^{k}(m_i - m_j - i + j)} = 1.$$

Proof. Consider the following partial fractional expansion

$$\prod_{i=1}^{k-1}\left(1 + \frac{1}{z - n_i}\right) = 1 + \sum_{i=1}^{k-1} \frac{r_i}{z - n_i}. \tag{1.2}$$

A easy calculation shows that

$$r_i = \frac{\prod_{j=1,j\neq i}^{k-1}(n_i - n_j + 1)}{\prod_{j=1,j\neq i}^{k-1}(n_i - n_j)}.$$

Put $z = 0$ in (1.2). We get

$$\sum_{i=1}^{k-1} \frac{r_i}{n_i} + \prod_{i=1}^{k-1}\left(1 - \frac{1}{n_i}\right) = 1.$$

Substituting $n_j = m_j - m_k - j + k$ we then get the identity in the Lemma. □

For formulas of this kind see further [**14**].

Proof of Theorem 1. We will prove the Theorem by induction. The idea is simply to use the *branching rule* by restriction of $U(n)$ to its subgroup $U(n-1)$. It is clear that the result is true for $n = 1$. Suppose the result is true for $n-1$, that is the above formula is true for the tensor product of a holomorphic representation of $U(n-1)$ with its defining representation on $\mathbb{C}^{n-1}$.

Now let $V^{\underline{\mathbf{m}}}$ be a representation of $U(n)$ with highest weight $\underline{\mathbf{m}} = (m_1, \ldots, m_n)$, $m_1 \geq \cdots \geq m_n \geq 0$. It is easy to see that $V^{\underline{\mathbf{m}}} = V^{(m_1-m_n,\ldots,m_{n-1}-m_n,0)} \otimes V^{(m_n,\ldots,m_n)}$ and $V^{(m_n,\ldots,m_n)}$ is one-dimensional ([**7**]). So by tensoring everything for the formulas about $(m_1 - m_n, \ldots, m_{n-1} - m_n, 0)$ by $V^{(m_n,\ldots,m_n)}$ we get the result about $\underline{\mathbf{m}}$. So we can assume $m_n = 0$. We also notice that the formula for $|a_{k,j}|^2$ stays the same if we replace m_i by $m_i - m_n$.

We consider the natural imbedding of $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$ by identifying the first $n-1$ basis vectors $e_1, \ldots, e_{n-1}$ of $\mathbb{C}^{n-1}$ and of $\mathbb{C}^n$. The group $U(n-1)$ is then a subgroup of $U(n)$ under this identification. The space $V^{\underline{\mathbf{m}}}$, as a $U(n-1)$-module, is then decomposed into irreducibles, and the multiplicities of each is one. See [**20**]. One of these is of highest weight $\underline{\mathbf{m}}' = (m_1, \ldots, m_{n-1})$. With abuse of notation we denote this module by $V^{\underline{\mathbf{m}}'}$. Under the $U(n-1)$-action, the vector $v_{\underline{\mathbf{m}}}$ is of weight $\underline{\mathbf{m}}'$ and is annihilated by $E_{i,i+1}$. By the multiplicity one property we see that $v_{\underline{\mathbf{m}}} \in V^{\underline{\mathbf{m}}}$ is also in the space $V^{\underline{\mathbf{m}}'}$ and is the highest weight vector of $V^{\underline{\mathbf{m}}'}$.

The tensor product $V^{\underline{\mathbf{m}}'} \otimes \mathbb{C}^{n-1}$, as $U(n-1)$-module, is decomposed as follows,

$$V^{\underline{\mathbf{m}}'} \otimes \mathbb{C}^{n-1} = \sum_{i=1}^{n-1} V^{\underline{\mathbf{m}}'+\varepsilon_i},$$

with the same convention as that of (1.1). Let $v_{\underline{\mathbf{m}}'+\varepsilon_i}$, $i = 1, \ldots, n-1$ be the highest weight vectors of $V^{\underline{\mathbf{m}}'+\varepsilon_i}$ We claim that

$$v_{\underline{\mathbf{m}}'+\varepsilon_i} \in V^{\underline{\mathbf{m}}+\varepsilon_i},$$

the subspace in (1.1) of $U(n)$-decomposition. In fact, it is clear that $v_{\underline{\mathbf{m}}'+\varepsilon_i}$ has the weight $\underline{\mathbf{m}}+\varepsilon_i$ under $U(n)$. It is annihilated by $E_{j,j+1}$, $j = 1, \ldots, n-2$. We need therefore only to prove that $E_{n-1,n}v_{\underline{\mathbf{m}}'+\varepsilon_i} = 0$. From Lemma 1 we know that

$$v_{\underline{\mathbf{m}}'+\varepsilon_i} = f \otimes e_1 - E_{1,2}f \otimes e_2 - \cdots - E_{1,i-1}f \otimes e_{i-1} - E_{1,i}f \otimes e_i,$$

with $f \in V^{\underline{\mathbf{m}}'}$ is of the weight $\underline{\mathbf{m}}' - \varepsilon_{1,i}$. Now

$$\begin{aligned} E_{n-1,n}v_{\underline{\mathbf{m}}'+\varepsilon_i} &= E_{n-1,n}f \otimes e_1 - E_{n-1,n}E_{1,2}f \otimes e_2 \ldots \\ &\quad - E_{n-1,n}E_{1,i-1}f \otimes e_{i-1} - E_{n-1,n}E_{1,i}f \otimes e_i. \end{aligned}$$

The vector $E_{n-1,n}f$ under $U(n)$ is then of weight $\underline{\mathbf{m}} - \varepsilon_{1,i} + \varepsilon_{n-1,n} = \underline{\mathbf{m}} - \sum_{j=1}^{i-1} \varepsilon_{j,j+1} + \varepsilon_{n-1,n}$, which is zero by the weight theory. (Every weight appearing in the module $V^{\underline{\mathbf{m}}}$ is of the form $\underline{\mathbf{m}} - \sum_{j=1}^{n-1} n_j\varepsilon_{j,j+1}$. This expression is unique since $\varepsilon_{j,j+1}$ are linearly independent.) Similarly, we get

$$E_{n-1,n}E_{1,2}f = 0, \ldots, E_{n-1,n}E_{1,i-1}f = 0.$$

The vector $E_{1,i}f$ is of weight $\underline{\mathbf{m}}$ therefore $E_{n-1,n}E_{1,i}f = 0$. That is $E_{n-1,n}v_{\underline{\mathbf{m}}'+\varepsilon_i} = 0$. So $v_{\underline{\mathbf{m}}'+\varepsilon_i}$ is in $V^{\underline{\mathbf{m}}+\varepsilon_i}$. Consequently we have

$$V^{\underline{\mathbf{m}}'+\varepsilon_i} \subset V^{\underline{\mathbf{m}}+\varepsilon_i}.$$

So we have a decomposition table of $v_{\underline{\mathbf{m}}} \otimes e_i$, $i = 1, \ldots, e_{n-1}$, with the same $P_j^{(i)}$ as they are viewed as $U(n)$-vectors. By our induction assumption, we have

$$v_{\underline{\mathbf{m}}} \otimes e_{n-1} = a_{n-1,1}P_1^{(n-1)} + \cdots + a_{n-1,n-1}P_{n-1}^{(n-1)}. \tag{1.3}$$

Here we have used the simple observation that the formulae of $|a_{k,i}|^2$, $k, i \leq n-1$ for the tensor product $V^{\underline{\mathbf{m}}'} \otimes \mathbb{C}^{n-1}$ of $U(n-1)$ are the same as of $V^{\underline{\mathbf{m}}} \otimes \mathbb{C}^n$ of $U(n)$.

Now we consider the decomposition of $v_{\underline{\mathbf{m}}} \otimes e_n$,

$$v_{\underline{\mathbf{m}}} \otimes e_n = a_{n-1,1}P_1^{(n)} + \cdots + a_{n,n-1}P_{n-1}^{(n)} + a_{n,n}P_n^{(n)}. \tag{1.4}$$

We observe further that

$$a_{n,k}E_{n-1,n}P_k^{(n)} = a_{n-1,k}P_k^{(n-1)}, \ k = 1, \ldots, n-1$$

by operating $E_{n-1,n}$ on (1.4) and comparing with (1.3). Taking norm in this identity we see that

$$|a_{n,k}|^2 = \|E_{n-1,n}P_k^{(n)}\|^{-2}|a_{n-1,k}|^2. \tag{1.5}$$

Below we will calculate $\|E_{n-1,n}P_k^{(n)}\|^{-2}$. By calculating weights in both sides, we see that $P_k^{(n)}$ is of weight $\underline{\mathbf{m}} + \varepsilon_n$, $k = 1, \ldots, n-1$ and $P_n^{(n)}$ is a highest weight vector of $V^{\underline{\mathbf{m}}+\varepsilon_n}$. Moreover, all those vectors are annihilated by $E_{j,j+1}$, $j = 1, \ldots, n-2$. Therefore $P_k^{(n)} \in V^{\underline{\mathbf{m}}+\varepsilon_k}$ is the highest weight vector for the submodule of $V^{\underline{\mathbf{m}}+\varepsilon_k}$ with highest weight $\underline{\mathbf{m}}' + \varepsilon_n$ in the decomposition of $V^{\underline{\mathbf{m}}+\varepsilon_k}$ under $U(n-1)$, again by the multiplicity one result.

We now recall the Gelfand-Cetlin orthonormal basis and tableaus. See **[20]** for further details. For any $U(n)$-module with highest weight $m_{1,n}\varepsilon_1 + m_{2,n}\varepsilon_2 + \cdots + m_{n,n}\varepsilon_n$, we restrict $V^{\underline{\mathbf{m}}}$ to its subgroup $U(n-1)$. Under $U(n-1)$ it is decomposed into irreducibles with highest weights $m_{1,n-1}\varepsilon_1 + m_{2,n-1}\varepsilon_2 + \cdots + m_{n-1,n-1}\varepsilon_{n-1}$ as indicated above. We decompose further those spaces succesively under $U(n-2)$ and so on. In the last step we get one-dimensional spaces of $U(1)$. Normalize the nonzero vectors in these spaces we get the so called Gelfand-Cetlin orthonormal basis. Each such vector corresponds to a tableau

$$\begin{pmatrix} m_{1,n} & \cdots & m_{n,n} \\ m_{1,n-1} & \cdots & m_{n-1,n-1} \\ \cdots & \cdots & \cdots \\ & m_{1,2}\, m_{2,2} & \\ & m_{1,1} & \end{pmatrix}.$$

It follows from our observation above that $P_k^{(n)}$ is a normalized Gelfand-Cetlin basis vector with the corresponding weight tableaus with entries $m_{k,n} = m_k + k$ and $m_{i,j} = m_i$ if $i \neq k$. It follows further from the Theorem 7 in [**20**, p. 205] that

$$\begin{aligned} &\|E_{n-1,n}P_k^{(n)}\|^2 \\ &= \frac{\prod_{i=1,i\neq k}^{n}(m_i - i - (m_k - k))}{\prod_{i=1}^{n-1}(m_i - i - (m_k - k) - 1)} \frac{\prod_{i=1}^{n-2}(m_i - i - (m_k - k) - 1)}{\prod_{i=1,i\neq k}^{n-1}(m_i - i - (m_k - k))} \\ &= \frac{m_n - m_k - n + k}{m_{n-1} - m_k - n + k} \\ &= \frac{m_k - m_n - k + n}{m_k - m_{n-1} - k + n}. \end{aligned}$$

(Notice that only the term $j = k$ is nonzero in the formula in Theorem 7 there.)

From this, (1.5) and the induction assumption we see that

$$\begin{aligned} |a_{n,k}|^2 &= \frac{(m_k - m_{n-1} - k + n)}{(m_k - m_n - k + n)}|a_{n-1,k}|^2 \\ &= \frac{(m_k - m_{n-1} - k + n)}{(m_k - m_n - k + n)} \frac{\prod_{j=1,j\neq k}^{n-2}(m_k - m_j - k + j + 1)}{\prod_{j=1,j\neq k}^{n-1}(m_k - m_j - k + j)} \\ &= \frac{\prod_{j=1,j\neq k}^{n-1}(m_k - m_j - k + j + 1)}{\prod_{j=1,j\neq k}^{n}(m_k - m_j - k + j)} \end{aligned}$$

for $i = 1, \ldots, n-1$. So the theorem is true for $a_{k,i}$, $k, i \leq n-1$.

Since $\sum_{i=1}^{n} |a_{n,i}|^2 = 1$, by Lemma 3 we know that the Theorem is true for $a_{n,n}$. This finishes the proof. □

Lemma 4. *Let $a_{k,i}$ be the entries in Theorem 1. The following identity holds*

$$\sum_{k=i}^{n} |a_{k,i}|^2 = \frac{\prod_{j=1,j\neq i}^{n}(m_i - m_j - i + j + 1)}{\prod_{j=1,j\neq i}^{n}(m_i - m_j - i + j)}.$$

Proof. We prove this by induction on the number n. If $n = 1$ there is nothing to prove. If $n = 2$ the above claim reads

$$1 + \frac{1}{m_1 - m_2 - 1 + 2} = \frac{m_1 - m_2 - 1 + 2 + 1}{m_1 - m_2 - 1 + 2}$$

for $i = 1$, which is clearly true; for $i = 2$ it is just an obvious identity.

Suppose this is true for all $n-1$ tuples of integers $m_1 \geq \cdots \geq m_{n-1}$. We prove this for n. For $i \leq n$ we have

$$\begin{aligned}
&\sum_{k=i}^{n} |a_{k,i}|^2 \\
&= \sum_{k=i}^{n-1} |a_{k,i}|^2 + |a_{n,i}|^2 \\
&= \frac{\prod_{j=1,j\neq i}^{n-1}(m_i - m_j - i + j + 1)}{\prod_{j=1,j\neq i}^{n-1}(m_i - m_j - i + j)} + \frac{\prod_{j=1,j\neq i}^{n-1}(m_i - m_j - i + j + 1)}{\prod_{j=1,j\neq i}^{n}(m_i - m_j - i + j)} \\
&= \frac{\prod_{j=1,j\neq i}^{n-1}(m_i - m_j - i + j + 1)(m_i - m_n - i + n + 1)}{\prod_{j=1,j\neq i}^{n}(m_i - m_j - i + j)} \\
&= \frac{\prod_{j=1,j\neq i}^{n}(m_i - m_j - i + j + 1)}{\prod_{j=1,j\neq i}^{n}(m_i - m_j - i + j)}.
\end{aligned}$$

□

§2. The norm of K-types.

Let D be the tube domain of type I, that is,

$$D = \{Z \in M(n, \mathbb{C}) : 1 - Z^*Z > 0\}.$$

The domain D is the bounded realization of G/K, where $G = SU(n,n)$ and $K = S(U(n), U(n))$.

Let $(\mathbb{C}^n, \varepsilon_1)$ be the representation space of $U(n)$ with highest weight ε_1 as in §1. For $\lambda > 2n - 1$ we consider the Hilbert space H_λ of $\mathbb{C}^n$-valued holomorphic functions $f(Z)$ on D with the norm

(2.1)
$$\|f\|_\lambda^2 = d(\lambda) \int_D \Big\langle (1 - Z^*Z) f(Z), f(Z) \Big\rangle \det(1 - Z^*Z)^{\lambda - 2n} dZ < \infty.$$

Here $\langle \cdot, \cdot \rangle$ is the innner product in $\mathbb{C}^n$, dZ is the Lebesgue measure and

$$d(\lambda) = \pi^{-n^2} \frac{\lambda}{\lambda - n} \frac{\prod_{j=1}^{n} \Gamma(\lambda - j + 1)}{\prod_{j=1}^{n} \Gamma(\lambda - n - j + 1)}.$$

This normalization correponds to the constant vector-valued function e_1 has norm 1, i.e. $\|e_1\|_\lambda = 1$.

The group G acts on H_λ via the following

$$U_\lambda(g) : f(Z) \mapsto (\det(CZ+D))^{-\lambda}(CZ+D)^{-1}f(g^{-1}Z), g^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $g^{-1}Z = (AZ+B)(CZ+D)^{-1}$. Note that $\mathbb{C}^n$ here is also viewed as a representation space of K. So for $P(Z) \otimes v$, with $v \in \mathbb{C}^n$ and $P(Z)$ a polynomial, viewed as a $\mathbb{C}^n$ -valued polynomial, we have

$$(2.2)\quad U_\lambda(g) : P(Z) \otimes v \mapsto (\det(CZ+D))^{-\lambda}P(g^{-1}Z) \otimes (CZ+D)^{-1}v.$$

Denote $K_{\lambda,W}(Z) = K_\lambda(Z,W)$ the $\mathbb{C}^n \otimes \mathbb{C}^n$-valued reproducing kernel of H_λ in the sense that

$$f(W) = (f, K_{\lambda,W}(\cdot))_\lambda$$

for any polynomial f in H_λ. It follows from the transformation property of K_λ under the group G that

$$K_\lambda(Z,W) = (\det(1-W^*Z))^{-\lambda}(1-W^*Z)^{-1}.$$

Let $\mathcal{P}$ be the space of holomorphic polynomials on D. The space of K-finite functions in H_λ is then $M = \mathcal{P} \otimes \mathbb{C}^n$, as representations of K. As is well-known ([**15**]),

$$\mathcal{P} \cong \sum_{\underline{\mathbf{m}} \geq 0} V^{\underline{\mathbf{m}}} \otimes (V^{\underline{\mathbf{m}}})^*,$$

where $(V^{\underline{\mathbf{m}}})^*$ is the contragredient representation of $V^{\underline{\mathbf{m}}}$.

From this and (1.1) we have that

$$M = \mathcal{P} \otimes \mathbb{C}^n \cong \sum_{i=1}^n \sum_{\underline{\mathbf{m}} \geq 0} V^{\underline{\mathbf{m}}+\varepsilon_i} \otimes (V^{\underline{\mathbf{m}}})^*,$$

where we use the same convention of the summation as in (1.1). The intertwining operator from the right hand side to the left in the above is

$$\left(\sum_{i=1}^n g_i \otimes e_i\right) \otimes f \in V^{\underline{\mathbf{m}}+\varepsilon_i} \otimes (V^{\underline{\mathbf{m}}})^* \mapsto \sum_{i=1}^n (D^{\underline{\mathbf{m}}}(Z)g_i, f)e_i,\ Z \in D.$$

Here $D^{\underline{\mathbf{m}}}$ denotes the action in $V^{\underline{\mathbf{m}}}$ of $U(n)$ and its holomorphic continuation to the complexification. We will therefore identify the vectors in $V^{\underline{\mathbf{m}}+\varepsilon_i} \otimes (V^{\underline{\mathbf{m}}})^*$ with the corresponding $\mathbb{C}^n$-valued polynomials on D. Note that here we are identifying the space $(V^{\underline{\mathbf{m}}})^*$ with $V^{\underline{\mathbf{m}}}$.

Theorem 2. *The norm square of the highest weight vector* $v_{\underline{\mathbf{m}}+\varepsilon_i} \otimes v_{\underline{\mathbf{m}}}$ *in the K-type* $(\underline{\mathbf{m}} + \varepsilon_i) \otimes \underline{\mathbf{m}}^*$ *is given by*

$$\frac{\prod_{j=1}^n (m_j + n - j)!}{\prod_{j<k}(m_j - m_k - j + k)} \frac{\lambda \prod_{j=1}^n \Gamma(\lambda - j + 1)}{\Gamma(m_i + \lambda - i + 2) \prod_{j=1, j\neq i}^n \Gamma(m_j + \lambda - j + 1)}.$$

Proof.

$$v_{\underline{\mathbf{m}}+\varepsilon_i} = \sum_{j=1}^n f_j \otimes e_j,$$

with $\sum_{j=1}^n \|f_j\|^2 = 1$. We have

$$\begin{aligned}
&\left\langle (1 - Z^*Z)\, v_{\underline{\mathbf{m}}+\varepsilon_i} \otimes v_{\underline{\mathbf{m}}}, v_{\underline{\mathbf{m}}+\varepsilon_i} \otimes v_{\underline{\mathbf{m}}} \right\rangle \\
&\quad = \left\langle (1 - Z^*Z) \sum_{j=1}^n \left(D^{\underline{\mathbf{m}}}(Z)\, f_j, v_{\underline{\mathbf{m}}}\right) e_j, \sum_{j=1}^n \left(D^{\underline{\mathbf{m}}}(Z)\, f_j, v_{\underline{\mathbf{m}}}\right) e_j \right\rangle \\
&\quad = \sum_{j=1}^n \left|\left(D^{\underline{\mathbf{m}}}(Z)\, f_j, v_{\underline{\mathbf{m}}}\right)\right|^2 - \left\| \sum_{j=1}^n \left(D^{\underline{\mathbf{m}}}(Z)\, f_j, v_{\underline{\mathbf{m}}}\right) Z e_j \right\|^2 \\
&\quad = I_1 - I_2.
\end{aligned}$$

The integral of I_1 in the above formula over D with respect to the measure in (2.1), by the Schur lemma, is

$$\int I_1 = d(\lambda) \int_D \left|\left(D^{\underline{\mathbf{m}}}(Z)\, v_{\underline{\mathbf{m}}}, v_{\underline{\mathbf{m}}}\right)\right|^2 \det(1 - Z^*Z)^{\lambda - 2n}\, dZ,$$

which furthermore by the formula (13) in [**15**, p. 568], is

$$\frac{\lambda}{\lambda - n} \frac{\prod_{j=1}^n (m_j + n - j)!}{\prod_{j<k}(m_j - m_k - j + k)} \frac{\prod_{j=1}^n \Gamma(\lambda - j + 1)}{\prod_{j=1}^n \Gamma(m_j + \lambda - j + 1)}.$$

The second term I_2 is

$$\begin{aligned}
I_2 &= \left\| \sum_{j=1}^n \left(D^{\underline{\mathbf{m}}}(Z)\, f_j, v_{\underline{\mathbf{m}}}\right) Z e_j \right\|^2 \\
&= \sum_{k=1}^n \left| \sum_{j=1}^n \left(D^{\underline{\mathbf{m}}}(Z)\, f_j, v_{\underline{\mathbf{m}}}\right) \langle Z e_j, e_k \rangle \right|^2
\end{aligned}$$

$$= \sum_{k=1}^{n} \left| \sum_{j=1}^{n} (D^{\underline{\mathbf{m}}\otimes\varepsilon_1}(Z) f_j \otimes e_j, v_{\underline{\mathbf{m}}} \otimes e_k) \right|^2$$

$$= \sum_{k=1}^{n} \left| (D^{\underline{\mathbf{m}}\otimes\varepsilon_1}(Z) v_{\underline{\mathbf{m}}+\varepsilon_i}, v_{\underline{\mathbf{m}}} \otimes e_k) \right|^2$$

$$= \sum_{k=1}^{n} \left| (D^{\underline{\mathbf{m}}+\varepsilon_i}(Z) v_{\underline{\mathbf{m}}+\varepsilon_i}, v_{\underline{\mathbf{m}}} \otimes e_k) \right|^2$$

$$= \sum_{k=i}^{n} \left| (D^{\underline{\mathbf{m}}+\varepsilon_i}(Z) v_{\underline{\mathbf{m}}+\varepsilon_i}, v_{\underline{\mathbf{m}}} \otimes e_k) \right|^2,$$

where in the last equality we use the fact that $v_{\underline{\mathbf{m}}} \otimes e_k$ only has projections in $V^{\underline{\mathbf{m}}+\varepsilon_i}$ for $k \geq i$ as proved in Lemma 2.

Let $v_{\underline{\mathbf{m}}} \otimes e_k = a_{k,1}P_1^{(k)} + \cdots + a_{k,k}P_k^{(k)}$ as in Lemma 2. So

$$I_2 = \sum_{k=i}^{n} |a_{k,i}|^2 \left| (D^{\underline{\mathbf{m}}+\varepsilon_i}(Z) v_{\underline{\mathbf{m}}+\varepsilon_i}, P_i^{(k)}) \right|^2.$$

Its integral is,

$$\int I_2 = \sum_{k=i}^{n} |a_{k,i}|^2 \int \left| \left(D^{\underline{\mathbf{m}}+\varepsilon_i}(Z) v_{\underline{\mathbf{m}}+\varepsilon_i}, P_i^{(k)}\right) \right|^2$$
$$= \sum_{k=i}^{n} |a_{k,i}|^2 \int \left| (D^{\underline{\mathbf{m}}+\varepsilon_i}(Z) v_{\underline{\mathbf{m}}+\varepsilon_i}, v_{\underline{\mathbf{m}}+\varepsilon_i}) \right|.$$

Here we have used the Schur Lemma since $\|P_i^{(k)}\| = \|v_{\underline{\mathbf{m}}+\varepsilon_i}\| = 1$. Now by Lemma 4 above and Proposition 1 in [**15**] we have

$$\int I_2 = \sum_{k=i}^{n} |a_{k,i}|^2 d(\lambda) \int_D \left| (D^{\underline{\mathbf{m}}+\varepsilon_i}(Z) v_{\underline{\mathbf{m}}+\varepsilon_i}, v_{\underline{\mathbf{m}}+\varepsilon}) \right|^2 \det(1 - Z^*Z)^{\lambda-2n} dZ$$
$$= \frac{\prod_{j=1,j\neq i}^{n}(m_i - m_j - i + j + 1)}{\prod_{j=1,j\neq i}^{n}(m_i - m_j - i + j)} \times$$
$$\times \frac{\lambda}{\lambda - n} \frac{\prod_{j=1}^{n}(m_j + n - j)!}{\prod_{j<k}(m_j - m_k - j + k)} \frac{\prod_{j=1}^{n}\Gamma(\lambda - j + 1)}{\prod_{j\neq i}^{n}\Gamma(m_j + \lambda - j + 1)\Gamma(m_i + \lambda - i + 2)}.$$

Finally we get the norm of $v_{\underline{\mathbf{m}}} \otimes v_{\underline{\mathbf{m}}+\varepsilon_i}$, after a long compution, from

$$\int I_1 - \int I_2$$
$$= \frac{\prod_{j=1}^{n}(m_j + n - j)!}{\prod_{j<k}(m_j - m_k - j + k)} \frac{\lambda \prod_{j=1}^{n}\Gamma(\lambda - j + 1)}{\Gamma(m_i + \lambda - i + 2)\prod_{j=1,j\neq i}^{n}\Gamma(m_j + \lambda - j + 1)}.$$

This completes the proof. □

Remark 1. To write the above expression further in a compact form we recall briefly here the Gindikin Gamma-function, see [**4**]. For a bounded symmetric space $D = G/K$ with the corresponding strongly orthogonal roots $\gamma_1, \ldots, \gamma_r$. Let a be the (common) multiplicity of the root space in $\mathfrak{k}^*$ of $\frac{\gamma_i - \gamma_j}{2}$, $i \neq j$. (See [**4**] for precise definition.) The Gindikin Gamma-function is defined by

$$\Gamma(\underline{\mathbf{m}}) = \prod_{i=1}^{r} \Gamma\left(m_i - \frac{a}{2}(i-1)\right).$$

One defines similarly then the Pochhammer's symbol

$$(\lambda)_{\underline{\mathbf{m}}} = \frac{\prod_{i=1}^{r} \Gamma(\lambda + m_i - \frac{a}{2}(i-1))}{\prod_{i=1}^{r} \Gamma(\lambda - \frac{a}{2}(i-1))}.$$

In our case $a = 2$. In terms of the Gindikin's Gamma-function (see [**4**]) the formula in Theorem 2 can written as

$$\frac{\Gamma(\underline{\mathbf{m}} + n)}{\prod_{j<k}(m_j - m_k - j + k)} \frac{\lambda}{(\lambda)_{\underline{\mathbf{m}}+\varepsilon_i}}.$$

Note that we may write $\prod_{j<k}(m_j - m_k - j + k)$ as $(2n-2)^{\frac{n(n-1)}{2}} \prod_\alpha(\underline{\mathbf{m}} + \underline{\nu}, \alpha)$, where the product is over all the positive roots of $u(n)_{\mathbb{C}}$ and $\underline{\nu} = (-1, -2, \ldots, -n)$ and $(\cdot, \cdot)$ is the Killing form. This expression of norm is quite analogous to the correponding formula for the K-types in the scalar-valued case ([**4**, Theorem 3.6]). This indicates that presumably there is a general formula for the vector-valued case. See our forthcoming preprint on type II and III domains.

Let $K^{(i)}_{\underline{\mathbf{m}}}(Z, W)$ be the reproducing kernel of each K-space $V^{\underline{\mathbf{m}}+\varepsilon_i} \otimes (V^{\underline{\mathbf{m}}})^*$, normalized as above so that $v_{\underline{\mathbf{m}}}$ and e_1 have norm 1. Denote $d_{\underline{\mathbf{m}},i}$ the dimension of the the space $V^{\underline{\mathbf{m}}+\varepsilon_i} \otimes (V^{\underline{\mathbf{m}}})^*$, see [**7**] and [**18**].

Corollary 1. *For $\lambda \in \mathbb{C}$ we have the following expansion*

(2.3)
$$K_\lambda(Z, W) = \sum_{i=1}^{n} \sum_{\underline{\mathbf{m}} \geq 0} d^{-1}_{\underline{\mathbf{m}},i} \frac{\prod_{j<k}(m_j - m_k - j + k)}{\Gamma(\underline{\mathbf{m}} + n)} \frac{(\lambda)_{\underline{\mathbf{m}}+\varepsilon_i}}{\lambda} K^{(i)}_{\underline{\mathbf{m}}}(Z, W).$$

This series converges uniformly on compact sets of $D \times D$.

Proof. The case when $\lambda > 2n - 1$ follows from the standard argument about reproducing kernels, see [**5**] for the scalar case. The case for general $\lambda \in \mathbb{C}$ follows from similar arguments as in [**4**]. □

Example. We take $n = 2$. Let $W = I$ and $Z = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$ in the above expansion. The (1, 1) entry of this matrix equality is the following

$$\begin{aligned}(1-s)^{-\lambda-1}(1-t)^{-\lambda} &= \sum_{l=0,\, m=0}^{\infty} \frac{\Gamma(m+1)\Gamma(l+m+2)}{(\lambda-1)_m(\lambda+1)_{l+m}}(st)^m \left(\sum_{i=0}^{l}(l+1-i)s^{l-i}t^i\right) + \\ &+ \sum_{l=1,\, m=0}^{\infty} \frac{\Gamma(m+1)\Gamma(l+m+2)}{(\lambda-1)_{m+1}(\lambda+1)_{m+l-1}}(st)^m \left(\sum_{i=1}^{l} i\, s^{l-i}t^i\right).\end{aligned}$$

This might be of interest from the combinatorics point of view.

§3. The composition series.

In this section we will study the analytic continuation in the parameter λ of the expansion of the reproducing kernel obtained above. In particular, we will find the Wallach set, namely the set of those λ's at which we get a unitary representation on the module generated by the reproducing kernel, and we will find the composition series when M is reducible. We will only briefly indicate the results since the argument is the much same as in [**4**] and [**15**].

Lemma 5. *For $\lambda \in \mathbb{C}$ the function $K_\lambda(Z, W)$ is positive definite on $D \times D$ if and only if $\lambda > n-1$ or $\lambda = n-1, \dots, 1$.*

Proof. It can be proved similarly as Lemma 5.1 in [**4**] that $K_\lambda(Z, W)$ is positive definite if and only if the coefficients in the expansion of $K_\lambda(Z, W)$ are positive. The Lemma follows then by simple calculation. □

Remark 2. The Wallach set in this case is therefore $(n-1, \infty) \cup \{1, \dots, n-1\}$. We note here that at the last point in the Wallach set, i.e., at $\lambda = 1$ the module M_0 is infinite dimensional, whereas in the scalar case the last point of the Wallach set is $\lambda = 0$ and it corresponds to the trivial representation.

For $\lambda \in \mathbb{C}$ let $q = q(\lambda)$ be the number of nonnegative integers among the n complex numbers $\lambda + 1$, $\lambda - 1$, $\dots$, $\lambda - n + 1$. For fixed $0 \le j \le q$ let

$$M_j = \sum_i \sum_{\underline{\mathbf{m}}} V^{\underline{\mathbf{m}}+\varepsilon_i} \otimes (V^{\underline{\mathbf{m}}})^*$$

where the summation is over the set of $\underline{\mathbf{m}}$ and i for which the function $\lambda \to \dfrac{(\lambda)_{\underline{\mathbf{m}}+\varepsilon_i}}{\lambda}$ has zero of at most multiplicity j, and with the same convention as in (1.1). We can therefore read off from our Theorem 2 the zeros of this function and get

Theorem 3. *For $\lambda \in \mathbb{C}$, M has a composition series*

$$M_0 \subset M_1 \subset \cdots \subset M_q = M$$

of length q. $q = q(\lambda) > 0$ if and only if $\lambda \le n-1$ and is an integer. Explicitly those modules are given by (to mimize the notations we use highest weight to indicate a module)

(1) *if $\lambda = n-k$, $k = 1, \ldots, n-1$ then $q(\lambda) = k$ and*

$$M_l = \sum_{i \neq n-k+l+1}^{n} \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_i) \otimes \underline{\mathbf{m}}^*, m_{n-k+l+1} \le l\}$$
$$\oplus \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_{n-k+l+1}) \otimes \underline{\mathbf{m}}^*, m_{n-k+l+1} \le l-1\},$$

for $0 \le l \le k$;

(2) *if $\lambda = 0$, then $q = n-1$ and*

$$M_l = \sum_{i \neq l+2}^{n} \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_i) \otimes \underline{\mathbf{m}}^*, m_{l+2} \le l+1\}$$
$$\oplus \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_{l+2}) \otimes \underline{\mathbf{m}}^*, m_{l+2} \le l\}$$

for $0 \le l \le n-1$;

(3) *if λ is a negative integer then $q = n$ and*

$$M_l = \sum_{i \neq l+1}^{n} \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_i) \otimes \underline{\mathbf{m}}^*, m_{l+1} \le -\lambda + l\}$$
$$\oplus \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_{l+1}) \otimes \underline{\mathbf{m}}^*, m_{l+1} \le -\lambda + l - 1\}$$

for $0 \le \lambda \le n$.

Note here M_0, the most interesting part, has the following form at $\lambda = n-k$, $k = 1, \ldots, n-1$,

$$M_0 = \sum_{i=1}^{n-k} \sum_{\underline{\mathbf{m}}} \{(\underline{\mathbf{m}} + \varepsilon_i) \otimes \underline{\mathbf{m}}^*, m_{n-k+1} = \cdots = m_n = 0\}.$$

Corollary 2. *M_0 is unitarizable precisely when $\lambda > n-1$ or $\lambda = 1, \ldots, n-1$. M_0 is finite dimensional if and only if λ is a negative integer. M_j/M_{j-1},*

$j \geq 1$, is unitarizable if and only if $j = q$ and $\lambda \leq n-1$ is an integer. In that case it is isomorphic to the Harish-Chandra module of polynomials in $H_{2n-\lambda}$.

Proof. The first two parts of the Corollary are direct consequences of Lemma 5 and Theorem 3 and some computations, keeping track of the signs of the coefficients in the expansion, see **[15]** and **[4]**. We now prove the third part. We consider here the case $\lambda = n - k$ the general case is just the same.

In the realization of the the group G as above we let F_i be the elements of $2n \times 2n$ diagonal matrices

$$F_i = \begin{pmatrix} E_i & 0 \\ 0 & -E_i \end{pmatrix}$$

where E_i is the $n \times n$ matrix we used in §1. We choose a maximal abelian subspace of the Lie algebra $\mathfrak{k}_{\mathbb{C}}$ containing F_i, where $\mathfrak{k}_{\mathbb{C}}$ is the complexification of the Lie algebra of K. We define linear functional γ_i on the Cartan algebra by the relation $\gamma_i(F_j) = 2\delta_{i,j}$ and γ_i is 0 on the orthogonal component (respect to the Killing form) of the linear span of F_j's. Those are the Harish-Chandra strongly orthogonal roots, as in **[18]** for example. We choose a compatible ordering for the compact roots so that the space $\mathbb{C}^n$ is a representation of K with highest weight $-\frac{\gamma_1}{2}$ and e_1 is a highest weight vector. The positive and negative noncompact root spaces of $\mathfrak{g}_{\mathbb{C}}$, the complexification of the Lie algebra of G, will be denoted by $\mathfrak{p}_+$, $\mathfrak{p}_-$ respectively. $\mathfrak{p}_+$ is simply the spaces of the matrices of the form

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

where x is a $n \times n$ matrix.

Now the module H_λ for $\lambda \in \mathbb{C}$ is a highest weight module of $\mathfrak{g}_{\mathbb{C}}$ with highest weight $-(\lambda)(\sum_{i=1}^n \gamma_i) - \frac{1}{2}\gamma_1$ with highest weight vector $1 \otimes e_1$, i.e., the constant vector-valued function. See **[19]** for the general case and **[4]** for the scalar case.

We fix $\lambda = n - k$, so $q = k$. We claim that M_k/M_{k-1} is a highest weight module with highest weight $-(\lambda+2k)\sum_{i=1}^n \gamma_i - \frac{1}{2}\gamma_1$ and highest weight vector $(\det Z)^k \otimes e_1$. In fact it is clear that $(\det Z)^k \otimes e_1$ is of the stated weight. Moreover this vector is annihilated by the compact positive root vectors, since generally, a tensor product of highest weight vectors is a highest weight vector. Thus we need only to prove that it is annihilated by $\mathfrak{p}_+$. Let $X \in \mathfrak{p}_+$, and by (2.2) calculate

$$\begin{aligned} X\left((\det Z)^k \otimes e_1\right) &= X(\det Z)^k \otimes e_1 + (\det Z)^k \otimes X e_1 \\ &= X(\det Z)^k \otimes e_1, \end{aligned}$$

since X is upper triangular. However $X(\det Z)^k \otimes e_1$ is a polynomial of lower degree ([**4**]) so by Theorem 3 we know that $X(\det Z)^k \otimes e_1$ is in M_{k-1}. That is $X((\det Z)^k \otimes e_1) = 0$ modulo M_{k-1}. □

The argument above is as in [**4**]. One can also use the argument as in [**15**] to compare the invariant Hermitian form on M_q/M_{q-1} with that on $H_{2n-\lambda}$ to prove the Corollary.

From the above results we can conclude

Proposition. *The space M as a $\mathfrak{g}_{\mathbb{C}}$-module is reducible precisely when $\lambda \leq n-1$ is an integer.*

Remark 3. At $\lambda = -1$ in the analytic continuation of our expansion (2.3) we see that the left hand side is then $\mathrm{ad}(1 - W^*Z)$, the adjoint matrix of $(1 - W^*Z)$. The coefficients of $K_{\underline{\mathbf{m}}}^{(i)}(Z, W)$ in the right hand side are nonzero only for $\underline{\mathbf{m}} = \varepsilon_1 + \cdots + \varepsilon_{i-1} = (1, \ldots, 1, 0, \ldots, 0)$ with the first $i-1$ components being 1 and the rest 0. Therefore the formula (2.3), after a direct calculation, is

$$\mathrm{ad}(1 - W^*Z) = \sum_{i=1}^{n} (-1)^{i-1}\, i\, d^{-1}_{\varepsilon_1+\cdots+\varepsilon_{i-1},i} K_{\underline{\mathbf{m}}}^{(i)}(Z, W).$$

Here $d_{\varepsilon_1+\cdots+\varepsilon_{i-1},i}$ as we defined in §2 is the dimension of the representation with highest weight $(\varepsilon_1 + \cdots + \varepsilon_{i-1}) \otimes (\varepsilon_1 + \cdots + \varepsilon_{i-1} + \varepsilon_i)$. We thus obtain a representation theoretic interpretation of adjoint.

Consider finally the example $n = 2$, i.e., the conformal group. Here the last Wallach point is the well-known positive-spinor solutions to Dirac's equation. Indeed, one checks that the K-types satisfying this are exactly M_o above at $\lambda = 1$ with K-types $(\underline{\mathbf{m}} + \varepsilon_i) \otimes \underline{\mathbf{m}}^*$ with $i = 1$, $m_2 = 0$, and $m_1 \geq 0$. This of course, is in agreement with [**9**].

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