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(A_2)-CONDITIONS AND CARLESON INEQUALITIES IN BERGMAN SPACES

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Let ν and μ be finite positive measures on the open unit disk D. We say that ν and μ satisfy the (ν, μ)-Carleson inequality, if there is a constant $C > 0$ such that

$$\int_D |f|^2 d\,\nu \leq C \int_D |f|^2 d\,\mu$$

for all analytic polynomials f. In this paper, we study the necessary and sufficient condition for the (ν, μ)-Carleson inequality. We establish it when ν or μ is an absolutely continuous measure with respect to the Lebesgue area mesure which satisfy the (A_2)-condition. Moreover, many concrete examples of such measures are given.

§1. Introduction.

Let D denote the open unit disk in the complex plane. For $1 \leq p \leq \infty$, let L^p denote the Lebesgue space on D with respect to the normalized Lebesgue area measure m, and $\|\cdot\|_p$ represents the usual L^p-norm. For $1 \leq p < \infty$, let L^p_a be the collection of analytic functions f on D such that $\|f\|_p$ is finite, which are so called the Bergman spaces. For any z in D, let ϕ_z be the Möbius function on D, that is

$$\phi_z(w) = \frac{z - w}{1 - \bar{z}w} \qquad (w \in D),$$

and put,

$$\beta(z, w) = 1/2\ \log(1 + |\phi_z(w)|)(1 - |\phi_z(w)|)^{-1} \quad (z, w \in D).$$

For $0 < r < \infty$ and z in D, set

$$D_r(z) = \{w \in D; \beta(z, w) < r\}$$

be the Bergman disk with "center" z and "radius" r, and we define an average of a finite positive measure μ on $D_r(a)$ by

$$\hat{\mu}_r(a) = \frac{1}{m(D_r(a))} \int_{D_r(a)} d\,\mu \qquad (a \in D),$$

and if there exists a non-negative function u in L^1 such that $d\,\mu = ud\,m$, then we may write it $\hat{u}_r$, instead of $\hat{\mu}_r$.

Let ν and μ be finite positive measures on D, and let P be the set of all analytic polynomials. We say that ν and μ satisfy the (ν,μ)-Carleson inequality, if there is a constant $C > 0$ such that

$$\int_D |f|^2 d\,\nu \leq C \int_D |f|^2 d\,\mu$$

for all f in P. Our purpose of this paper is to study conditions on ν and μ so that the (ν,μ)-Carleson inequality is satisfied. If $\nu \leq C\mu$ on D, then the (ν,μ)-Carleson inequality is true. However it is clear that this sufficient condition for the (ν,μ)-Carleson inequality is too strong. A reasonable and natural condition is the following: there exist $r > 0$ and $\gamma > 0$ such that

$$(*) \qquad \hat{\nu}_r(a) \leq \gamma \hat{\mu}_r(a) \qquad (a \in D).$$

The average $\hat{\mu}_r(a)$ are sometimes computable. If $\mu = m$, then $\hat{\mu}_r(a) = 1$ on D. If $d\,\mu = (1-|z|^2)^\alpha d\,m$ for $\alpha > -1$, then $\hat{\mu}_r(a)$ is equivalent to $(1-|a|^2)^\alpha$ on D.

When $d\,\mu = (1-|z|^2)^\alpha d\,m$ for $\alpha > -1$, Oleinik-Pavlov [**7**], Hastings [**2**], or Sitegenga [**8**] showed that ν and μ satisfy the Carleson inequality if and only if they satisfy (*). In §3 of this paper, when $d\,\mu = ud\,m$ and u satisfies the $(A_2)_\partial$-condition (the definition is in §3), we obtain that the (ν,μ)-Carleson inequality is satisfied if and only if they satisfy (*). We show that if both u and u^{-1} are in $B\ M\ O_\partial$ (see [**9**, p. 127]), then u satisfies the $(A_2)_\partial$-condition. We give some concrete examples which satisfy the $(A_2)_\partial$-condition.

When $\nu = m$ and $d\,\mu = \chi_G d\,m$, where χ_G is a characteristic function of a measurable subset G of D, Luecking [**4**] showed the equivalence between the (ν,μ)-Carleson inequality and the condition (*). If we do not put any hypotheses on μ, the problem is very difficult. The equivalence between the (ν,μ)-Carleson inequality and the condition (*) is not known even if $\nu = m$. Luecking [**5**] showed the following:

(1) If there exists $\gamma > 0$ such that $\hat{m}_r(a) \leq \gamma\hat{\mu}_r(a)$ for all $r > 0$ and a in D, then the (m,μ)-Carleson inequality is satisfied.

(2) Suppose the (μ, m)-Carleson inequality is valid (equivalently $\hat{\mu}_r$ is bounded on D). Then the (m,μ)-Carleson inequality implies the condition (*).

In §2 of this paper, we give a sufficient condition (close to that of (1)) for the (ν,μ)-Carleson inequality when ν is not necessarily m. Moreover, using the idea of Luecking's proof of (2), a generalization of (2) is given. In §4, when $d\,\nu = vd\,m$ and v satisfies the (A_2)-condition (the definition is in

§3), we establish a more natural extension of (2) under some condition of a quantity $\varepsilon_r(\nu)$ (the definition is in §2), that is $\varepsilon_r(\nu) \to 0$ as $r \to \infty$. The (A_2)-condition is weaker than the $(A_2)_\partial$ -condition. We give some concrete examples which satisfy the (A_2)-condition or the above condition of $\varepsilon_r(\nu)$.

§2. (ν, μ)-Carleson inequality.

Let G be a measurable subset of D and u be a non-negative function in L^1, and put

$$(u_G^{-1})_r^\wedge(a) = \frac{1}{m(D_r(a))} \int_{D_r(a)} u^{-1}\chi_G d\, m.$$

Particular, when $G = D$, we will omit the letter D in the above notation. The following Proposition 1 gives a general sufficient condition on ν and μ which satisfy the (ν, μ)-Carleson inequality. In order to prove it we use ideas in [**5**] and [**9**, p. 109]. Since $(u^{-1})_r^\wedge(a)^{-1} \leq \hat{u}_r(a)$ for all a in D, Proposition 1 is also related with (1) of §1 (cf. [**5**, Theorem 4.2]).

Proposition 1. *Suppose that $d\,\mu = ud\,m$. Put $E_r = \{z \in D$; there is a $w \in \operatorname{supp}\nu$ such that $\beta(z,w) < r/2\}$. If there exist $r > 0$ and $\gamma > 0$ such that $u > 0$ a.e. on $E = E_r$, and $\hat{\nu}(a) \times (u_E^{-1})_r^\wedge(a) \leq \gamma$ for all a in D, then there is a constant $C > 0$ such that*

$$\int_D |f|^2 d\,\nu \leq C \int_E |f|^2 d\,\mu$$

for all f in P.

Proof. Suppose that $\hat{\nu}_{2r}(a) \times (u_E^{-1})_{2r}^\wedge(a) \leq \gamma$ for all a in D, and put $E = \{z \in D$; there is a $w \in \operatorname{supp}\nu$ such that $\beta(z,w) < r\}$. By an elementary theory for Bergman disks, there is a positive integer $N = N_r$ such that there exists $\{\lambda_n\} \subset D$ satisfying that $D = \cup D_r(\lambda_n)$ and any z in D belongs to at most N of the sets $D_{2r}(\lambda_n)$ (cf. [**9**, p. 62] therefore

$$\begin{aligned}\int_{\operatorname{supp}\nu} |f|^2 d\,\nu &\leq \sum \int_{D_r(\lambda_n)\cap \operatorname{supp}\nu} |f|^2 d\,\nu \\ &\leq \sum \nu(D_r(\lambda_n)) \times \sup\{|f(z)|^2;\ z \in D_r(\lambda_n) \cap \operatorname{supp}\nu\}.\end{aligned}$$

By Proposition 4.3.8 in [**9**, p. 62], there is a constant $C = C_r > 0$ such that

$$|f(z)| \leq \frac{C}{m(D_r(z))} \int_{D_r(z)} |f(w)| d\,m(w)$$

for all f analytic, z in D. If z in $D_r(\lambda_n) \cap \operatorname{supp}\nu$, then $D_r(z)$ is contained in $D_{2r}(\lambda_n) \cap E$, and there exists a constant $K = K_r > 0$ such that $m(D_{2r}(\lambda_n)) \leq$

$Km(D_r(z))$ for all $n \geq 1$ (cf. [**9**, p. 61]). Hence the Cauchy-Schwarz's inequality implies that

$$\begin{aligned}\int_D |f|^2 d\,\nu \leq& \sum \nu(D_r(\lambda_n)) \times \left(\frac{KC}{(m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n)\cap E} |f| d\,m\right)^2 \\ \leq& \sum \nu(D_r(\lambda_n)) \times K^2C^2 \\ &\times \left(\frac{1}{m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n)} |f|^2 u\chi_E d\,m\right) \\ &\times \left(\frac{1}{m(D_{2r}(\lambda_n))} \int_{D_{2r}(\lambda_n)} u^{-1}\chi_E d\,m\right) \\ \leq& K^2C^2 \sum \hat{\nu}_{2r}(\lambda_n) \times (u_E^{-1})^{\wedge}_{2r}(\lambda_n) \\ &\times \left(\int_{D_{2r}(\lambda_n)\cap E} |f|^2 u d\,m\right).\end{aligned}$$

By the hypothesis and a choice of disks, it follows that

$$\int_D |f|^2 d\,\nu \leq K^2C^2\gamma N \int_E |f|^2 d\,\mu.$$

This completes the proof. □

Let μ be a finite nonzero positive measure on D. For any a in D, put

$$k_a(z) = (1-|a|^2)/(1-\bar{a}z)^2 \qquad (z \in D),$$

and a function $\tilde{\mu}$ on D is defined by

$$\tilde{\mu}(a) = \int_D |k_a|^2 d\,\mu.$$

Moreover, for any fixed $r < \infty$, put

$$\varepsilon_r(\mu) = \sup_{a\in D} \left(\int_{D\setminus D_r(a)} |k_a|^2 d\,\mu\right) \times \left(\int_D |k_a|^2 d\,\mu\right)^{-1}.$$

If there exists a non-negative function u in L^1 such that $d\,\mu = ud\,m$, then making a change of variable, it is easy to see that

$$\varepsilon_r(\mu) = \sup_{a\in D} \left(\int_{D\setminus D_r(0)} u\circ\phi_a d\,m\right) \times \left(\int_D u\circ\phi_a d\,m\right)^{-1}.$$

In general $0 < \varepsilon_r(\mu) \leq 1$. In this section and §4, this quantity ε_r is important. The following Proposition 2 gives two general necessary conditions on ν

and μ which satisfy the (ν, μ)-Carleson inequality. In order to prove (2) of Proposition 2 we use ideas in [**5**, Theorem 4.3]. Since $\varepsilon_r(m) < 1$ and $\varepsilon_r(m) \to 0$ $(r \to \infty)$, (2) of Proposition 2 is related with (2) of §1.

Lemma 1. *Let μ be a finite positive measure on D and $0 < r < \infty$, then the following* (1) $\sim$ (3) *are equivalent.*

(1) $\varepsilon_r(\mu) < 1$.

(2) *There is a $\delta = \delta_r < \infty$ such that*

$$\int_{D \setminus D_r(a)} |k_a|^2 d\,\mu \leq \delta \int_{D_r(a)} |k_a|^2 d\,\mu$$

for all a in D.

(3) *There is a $\rho = \rho_r < \infty$ such that*

$$\tilde{\mu}(a) \leq \rho \hat{\mu}_r(a)$$

for all a in D

Proof. The implication (1) $\Rightarrow$ (2) is trivial. (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) follow from Lemma 4.3.3 in [**9**, p. 60]. In fact, by Lemma 4.3.3, there exist $L = L_r > 0$ and $M = M_r > 0$ such that

$$L \leq m(D_r(a)) \times \inf\{|k_a(z)|^2; z \in D_r(a)\}$$

and

$$m(D_r, (a)) \times \sup\{|k_a(z)|^2; z \in D_r(a)\} \leq M$$

for all a in D. Thus remainder implications are obtained. □

Proposition 2. *Suppose that ν and μ satisfy the (ν, μ)-Carleson inequality, then the following are true.*

(1) *If there exists $r < \infty$ such that $\varepsilon_r(\mu) < 1$, then there exists $\gamma > 0$ such that $\hat{\nu}_r(a) \leq \gamma \hat{\mu}_r(a)$ for all a in D.*

(2) *If $d\,\nu = vd\,m$, $v > 0$ a.e. on D, $\varepsilon_t(\nu) \to 0$ $(t \to \infty)$, and there are $l > 0$ and $\gamma' > 0$ such that $\hat{\mu}_l(a) \times (v^{-1})_l^\wedge(a) \leq \gamma'$ for all a in D, then there are $r > 0$ and $\gamma = \gamma_r > 0$ such that $\hat{\nu}_r(a) < \gamma \hat{\mu}_r(a)$ for all a in D.*

Proof. Since $k_a(z)$ is uniformly approximated by polynomials, the inequality is valid for $f = k_a$, that is

$$\int_D |k_a|^2 d\,\nu \leq C \int_D |k_a|^2 d\,\mu.$$

Firstly, we show that (1) is true. The above inequality and Lemma 1 imply that

$$\tilde{\nu}(a) \leq C\tilde{\mu}(a) \leq C\rho\hat{\mu}_r(a)$$

for all a in D. Moreover, by Lemma 4.3.3 in [**9**, p. 60], there exists a constant $L > 0$ such that

$$\hat{\nu}_r(a) \leq L^{-1}\tilde{\nu}(a)$$

for all a in D. Hence we have that

$$\hat{\nu}_r(a) \leq C\rho L^{-1}\hat{\mu}_r(a).$$

Next, we prove that (2) is true. For any a in D and $r \geq l$, put $d\ \mu_{a,r} = (1 - \chi_{D_r(a)})d\ \mu$. By the latter half of the hypothesis in (2), we have that

$$(\mu_{a,r})_l^\wedge(\lambda) \times (v^{-1})_l^\wedge(\lambda) \leq \gamma'$$

for all a, λ in D, and $r \geq l$. Set $E_{a,r,l} = \{z \in D$; there is a w in $\operatorname{supp}\mu_{a,r}$, such that $\beta(z,w) < l/2\}$. By Proposition 1, there exists a constant $C' > 0$ such that

$$\int_{D\backslash D_r(a)} |f|^2 d\ \mu \leq C' \int_{E_{a,r,l}} |f|^2 d\ \nu$$

for all a in D, $r \geq l$ and f in P. Here we claim that $E_{a,r,l}$ is contained in $D\backslash D_{r/2}(a)$. In fact, since $D\backslash D_r(a)$ contains $\operatorname{supp}\mu_{a,r}$ and $r \geq l$, if z belongs to $E_{a,r,l}$ then there exists w in D such that $\beta(w,a) \geq r$ and $\beta(w,z) < r/2$. Therefore,

$$r \leq \beta(w,a) \leq \beta(w,z) + \beta(z,a) < r/2 + \beta(z,a),$$

thus we have that z is contained in $D\backslash D_{r/2}(a)$. Particularly put $f = k_a$ in the above inequality, then

$$\int_{D\backslash D_r(a)} |k_a|^2 d\ \mu \leq C' \int_{D\backslash D_{r/2}(a)} |k_a|^2 d\ \nu$$

for all a in D and $r \geq l$. It follows that

$$\begin{aligned}\int_{D_r(a)} |k_a|^2 d\ \mu &= \int_D |k_a|^2 d\ \mu - \int_{D\backslash D_r(a)} |k_a|^2 d\ \mu \\ &\geq C^{-1} \int_D |k_a|^2 d\ \nu - C' \int_{D\backslash D_{r/2}(a)} |k_a|^2 d\ \nu.\end{aligned}$$

By the definition of $\varepsilon_t(\nu)$, the above inequality implies that

$$\int_{D_r(a)} |k_a|^2 d\ \mu \geq (C^{\ 1} - C'\varepsilon_{r/2}(\nu)) \int_D |k_a|^2 d\ \nu$$

for all a in D and $r \geq l$. Here let r be sufficiently large, then by the hypothesis on $\varepsilon_r(\nu), C^{-1} - C'\varepsilon_{r/2}(\nu) > 0$, and by Lemma 4.3.3 in [**9**, p. 60], we conclude that

$$\hat{\mu}_r(a) \geq [M^{-1}(C^{-1}C'\varepsilon_{r/2}(\nu))L]\hat{\nu}_r(a)$$

for all a in D. $\square$

§3. (A_2)-condition.

For a complex measure μ on D, recall that a function $\tilde{\mu}$ on D is defined by

$$\tilde{\mu}(a) = \int_D |k|^2 d\,\mu.$$

Particularly, if there exists a complex valued L^1-function u such that $d\,\mu = ud\,m$, then we denote the function by $\tilde{u}$ instead of $\tilde{\mu}$, and say that $\tilde{u}$ is the Berezin transform of the function u.

Let v and u be non-negative functions in L^1, put $d\,\nu = vd\,m$ and $d\,\mu = ud\,m$. Suppose that there is a constant $\gamma > 0$ such that

$$\tilde{v}(a) \times (u^{-1})^{\sim}(a) \leq \gamma$$

for all a in D, then Lemma 4.3.3 in [**9**, p. 60] implies that there exist $r > 0$ and $\gamma' > 0$ such that

$$\hat{v}_r(a) \times (u^{-1})^{\wedge}_r(a) \leq \gamma'$$

for all a in D, and hence by Proposition 1, we obtain that the (ν, μ)-Carleson inequality is satisfied. In the above two inequalities, if we put $u = v$, then such a function u is interesting for us.

A non-negative function u in L^1 is said to satisfy an $(A_2)_\partial$-condition, if there exists a constant $A > 0$ such that

$$\tilde{u}(a) \times (u^{-1})^{\sim}(a) \leq A$$

for all a in D. If there exist $r > 0$ and $A_r > 0$ such that

$$\hat{u}_r(a) \times (u^{-1})^{\wedge}_r(a) \leq A_r$$

for all a in D, then we say that u satisfies an (A_2)-condition. In [**6**], the (A_2)-condition is called Condition C_2 . It is known that u satisfies the (A_2)-condition for some $0 < r < \infty$ if and only if u satisfies the (A_2)-condition for all $0 < r < \infty$ [**6**]. Hence it shows that the definition of the (A_2)-condition is independent of r. In general, Lemma 4.3.3 in [**9**, p. 60] and the familiar inequality between the harmonic and arithmetic means imply that for any $0 < r < \infty$ there exists a constant $M = M_r > 0$ such that $M^{-1}(u^{-1})^{\sim -1} \leq (u^{-1})^{\wedge\ -1}_r \leq \hat{u}_r \leq M\tilde{u}$. Therefore, if u satisfies the (A_2)-condition, then $(u^{-1})^{\sim -1}, (u^{-1})^{\wedge -1}_r, \hat{u}_r$, and $\tilde{u}$ are equivalent. Similarly, if u satisfies the (A_2)-condition, then $(u^{-1})^{\wedge -1}_r$, and $\hat{u}_r$, are equivalent. When u is in $L^1(\partial D)(L^1$ is a usual Lebesgue space on the unit circle and $k_a(z)$ is a normalized reproducing kernel of a Hardy space), the $(A_2)_\partial$-condition has been studied in [**3**, (c) of Theorem 2].

The following Theorem 3 gives a necessary and sufficient condition in order to satisfy the (ν, μ)-Carleson inequality when $d\,\mu = u\,d\,m$ and u satisfies the $(A_2)_\partial$-condition.

Theorem 3. *Suppose that u satisfies the $(A_2)_\partial$-condition, then the following are equivalent.*

(1) *There is a constant $C > 0$ such that*

$$\int_D |f|^2 d\,\nu \leq C \int_D |f|^2 u\,d\,m$$

for all f in P.

(2) *There exist $r > 0$ and $\gamma > 0$ such that*

$$\hat{\nu}_r(a) \leq \gamma \hat{u}_r(a)$$

for all a in D.

(3) *For any $r > 0$, there exists $\gamma = \gamma_r > 0$ such that*

$$\hat{\nu}_r(a) \leq \gamma \hat{u}_r(a)$$

for all a in D.

Proof. Suppose that (1) holds. Since u satisfies the $(A_2)_\partial$-condition, by (1) of Proposition 8, u satisfies a relation in (3) of Lemma 1 for all $r > 0$. Therefore, (3) follows from (1) of Proposition 2. The implication (3) $\Rightarrow$ (2) is obvious. We will show that (2) $\Rightarrow$ (1). Since u satisfies the $(A_2)_\partial$-condition, u^{-1} is integrable, hence $u > 0$ a.e. on D. Moreover, by (5) of Proposition 4, u satisfies the (A_2)-condition for all $r > 0$ and therefore (2) implies that

$$\hat{\nu}_r(a) \times (u^{-1})_r^\wedge(a) \leq A_r \gamma$$

for all a in D. In the statement of Proposition 1, put $E = D$, then the above fact shows that the inequality in (1) is satisfied. This completes the proof. $\square$

For any u in L^2, a in D, we put

$$MO(u)(a) = \{|u|^{2\sim}(a) - |\tilde{u}(a)|^2\}^{1/2},$$

and let BMO_∂ be the space of functions u such that $MO(u)(a)$ is bounded on D (cf. [**9**, p. 127]). We give several simple sufficient conditions.

Proposition 4. *Let u be a non-negative function in L^1, then the following are true.*

(1) *If both* $\tilde{u}$ *and* $(u^{-1})^{\sim}$ *are in* L^{∞}, *then* u *satisfies the* $(A_2)_{\partial}$*-condition.*

(2) *If both* u *and* u^{-1} *are in* BMO_{∂}, *then* u *satisfies the* $(A_2)_{\partial}$*-condition.*

(3) *Let* $1 < p, q < \infty$ *and* $1/p + 1/q = 1$. *If* u_1^p *and* u_2^q *satisfy the* $(A_2)_{\partial}$*-condition, then* $u = u_1 u_2$ *satisfies the* $(A_2)_{\partial}$*-condition.*

(4) *Suppose that* f *is a complex valued function in* L^1 *such that* $f \neq 0$ *on* D, f^{-1} *is in* $L^1, \tilde{f} \times (f^{-1})^{\sim}$ *is in* L^{∞}, *and* $|\arg f| \leq \pi/2 - \varepsilon$ *for some* $\varepsilon > 0$. *If* $u = |f|$, *then* u *satisfies the* $(A_2)_{\partial}$*-condition.*

(5) *If* u *satisfies the* $(A_2)_{\partial}$*-condition, then* u *satisfies the* (A_2)*-condition.*

Proof. (1) is trivial. By Proposition 6.1.7 in [**9**, p. 108], we have that

$$\tilde{u}(a) \times (u^{-1})^{\sim}(a) \leq MO(u)(a) \times MO(u^{-1})(a) + 1.$$

This implies that (2) is true. The Hölder's inequality implies that (3) is true. (5) follows from Lemma 4.3.3 in [**9**, p. 60].

We show that (4) is true. Suppose that $u = |f|$ and there exists $\varepsilon > 0$ such that $|\arg f| \leq \pi/2 - \varepsilon$ on D. Since $|\arg f| \leq \pi/2 - \varepsilon$ on D, there exists $\delta > 0$ such that $\cos(\arg f) \geq \delta$ on D . Therefore, we have that

$$\operatorname{Re} f = |f| \times \cos(\arg f) \geq |f| \cdot \delta = \delta u.$$

For any a in D, it follows that

$$\delta \tilde{u}(a) \leq \int \operatorname{Re} f \cdot |k_a|^2 d\ m \leq |\tilde{f}(a)|.$$

Similarly, we have that

$$\delta (u^{-1})^{\sim}(a) \leq |(f^{-1})^{\sim}(a)|.$$

Thus,

$$\tilde{u}(a) \times (u^{-1})^{\sim}(a) \leq \delta^{-2} \times |\tilde{f}(a)| \times |(f^{-1})^{\sim}(a)|$$

for all a in D, and hence (4) follows. □

We exhibit some concrete examples which satisfy the $(A_2)_{\partial}$-condition.

Proposition 5. *If* u *is a function that is given by* (1), (2), *or* (3), *then* u *satisfies the* $(A_2)_{\partial}$*-condition.*

(1) *For any* $-1 < \alpha < 1$, *put* $u(z) = (1 - |z|^2)^{\alpha}$.

(2) *Let* $\{b_j\}$ *be a finite sequence of complex numbers in* $D \cup \partial D$ *with* $b_i \neq b_j (i \neq j)$, *and let* $0 \leq \alpha(j) < 2$ *for all* j *or* $-2 < \alpha(j) \leq 0$ *for all* j. *Put* $u = \prod p_j^{\alpha(j)}$ *where* $p_j(z) = |z - b_j|$.

(3) *Let* $\{b_j\}, \{p_j\}$ *as in* (2) *and* $-1 < \alpha(j) < 1$ *for all* j. *Put* $u = \prod p_j^{\alpha(j)}$.

Proof. We suppose that u has the form of (1). For any a in D, making a change of variable, we have that

$$\begin{aligned}\tilde{u}(a) \times (u^{-1})^{\sim}(a) &= \int (1-|a|^2)^{\alpha}(1-|z|^2)^{\alpha}|1-\bar{a}z|^{2\alpha} d\, m(z)\\ &\quad \times \int (1-|a|^2)^{-\alpha}(1-|z|^2)^{-\alpha}|1-\bar{a}z|^{2\alpha} d\, m(z)\\ &= \int (1-|z|^2)^{\alpha}|1-\bar{a}z|^{-2\alpha} d\, m(z)\\ &\quad \times \int (1-|z|^2)^{-\alpha}|1-\bar{a}z|^{2\alpha} d\, m(z).\end{aligned}$$

Since $-1 < \alpha < 1$, Rudin's lemma (cf. [**9**, p. 53]) implies that both factors of the right hand side in the above equality are bounded. Hence satisfies the $(A_2)_\partial$-condition.

We show that u satisfies the $(A_2)_\partial$-condition when u has the form of (2). Let α be a real number such that $0 < \alpha < 2$. For any fixed b in D, put $p(z) = |z-b|$. Firstly, we show that the Berezin transform of $p^{-\alpha}$ is bounded. In fact, making a change of variable, elementary calculations show that

$$(p^{-\alpha})^{\sim}(a) \le |1-\bar{a}b|^{-\alpha} \cdot \|1-\bar{a}z\|_{\infty}^{a} \times \int |\phi_a(b) - z|^{-\alpha} d\, m(z).$$

Since $\phi_a(b) - z$ lies in $2D = \{2z; z \in D\}$ for any a, z in D and an area measure is translation invariant, we have that

$$(p^{-\alpha})^{\sim}(a) \le (1-|b|)^{-\alpha} \cdot \|1-\bar{a}z\|_{\infty}^{\alpha} \times \int_{2D} |w|^{-\alpha} d\, m(w)$$

for all a in D. Hence we obtain that the Berezin transform of $p^{-\alpha}$ is bounded. Next, let b be in ∂D and put $p(z) = |z-b|$. Then, as in the proof of the above case, we have that

$$(p^{\alpha})^{\sim}(a) \le |a-b|^{\alpha} \cdot \|\phi_a(b) - z\|_{\infty}^{\alpha} \times \int |1-\bar{a}z|^{-\alpha} d\, m(z),$$

and

$$(p^{-\alpha})^{\sim}(a) \le |a-b|^{-\alpha} \cdot \|1-\bar{a}z\|_{\infty}^{\alpha} \times \int_{2D} |w|^{-\alpha} d\, m(w).$$

Therefore, Rudin's lemma implies that p^{α} satisfies the $(A_2)_\partial$-condition. For any b_1 in D and b_2 in ∂D, put $p_1(z) = |z-b_1|$ and $p_2(z) = |z-b_2|$. Fix $0 < \alpha(j) < 2$ for $j = 1, 2$ and $\varepsilon > 0$. Because $b_1 = b_2$, there exist measurable subsets B_j of D such that $B_1 \cap B_2 = \phi$ and $p_j \geq \varepsilon$ on B_j^c for $j = 1, 2$. Set $B_0 = D \backslash B_1 \cup B_2$, then

$$\begin{aligned}
&(p_1^{\alpha(1)} \cdot p_1^{\alpha(2)})^{\sim}(a) \times (p_1^{-\alpha(1)} \cdot p_2^{-\alpha(2)})^{\sim}(a) \\
&\quad \leq (p_1^{\alpha(1)} \cdot p_2^{\alpha(2)})^{\sim}(a) \times \Big(\varepsilon^{-\alpha(1)-\alpha(2)} \int_{B_0} |k_a|^2 d\, m \\
&\qquad + \varepsilon^{-\alpha(2)} \int_{B_1} p_1^{-\alpha(1)} |k_a|^2 d\, m \\
&\qquad + \varepsilon^{-\alpha(1)} \int_{B_2} p_2^{-\alpha(2)} |k_a|^2 d\, m\Big) \\
&\quad \leq M_0 \times \varepsilon^{-\alpha(1)-\alpha(2)} + M_0 \times \varepsilon^{-\alpha(2)} \cdot (p_1^{-\alpha(1)})^{\sim}(a) \\
&\qquad + M_1 \times \varepsilon^{-\alpha(1)} \cdot (p_2^{\alpha(2)})^{\sim}(a) \cdot (p_2^{-\alpha(2)})^{\sim}(a),
\end{aligned}$$

where $M_0 = \|p_1^{\alpha(1)} \cdot p_2^{\alpha(2)}\|_\infty$ and $M_1 = \|p_1^{\alpha(1)}\|_\infty$. Hence we have that $p_1^{\alpha(1)} \cdot p_2^{\alpha(2)}$ satisfies the $(A_2)_\partial$-condition. If u has the form of (2), then applying the sarne argument for finitely many factors of u and u^{-1}, we obtain that u satisfies $(A_2)_\partial$-condition.

Apparently, (3) follows from (2) of this proposition and (3) of Proposition 4. In fact, we let $-1 < \alpha(j) < 1$ for all j, and set

$$j(+) = \{j;\ \alpha(j) \geq 0\}, \quad j(-) = \{j;\ \alpha(j) < 0\}.$$

Put $u_1 = \prod_{j(+)} p_j^{\alpha(j)}$ and $u_2 = \prod_{j(-)} p_j^{\alpha(j)}$, then u_1^2 and u_2^2 satisfy the $(A_2)_\partial$ -condltion. Hence, (3) of Proposition 4 implies that $u = u_1 \times u_2$ satisfies the $(A_2)_\partial$-condition. □

Corollary 1 is a partial result of [**2**], [**7**] and [**8**].

Corollary 1, Oleinik-Pavlov-Hastings-Stegenga. *Let ν be a finite positive measure on D. For any $-1 < \alpha < 1$, there is a constant $C > 0$ such that*

$$\int_D |f|^2 d\, \nu \leq C \int_D |f|^2 (1 - |z|^2)^\alpha d\, m$$

for all f in P if and only if there exist $r > 0$ and $\gamma > 0$ such that

$$\hat{\nu}_r(a) \leq \gamma(1 - |a|^2)^\alpha$$

for all a in D.

Proof. Since $[(1 - |z|^2)^\alpha]_r^\wedge(a)$ is comparable to $(1 - |a|^2)^\alpha$, by Theorem 3 and (1) of Proposition 5 the corollary follows. □

Lemma 2. *Let $\{b_j\}$ be a finite sequence of complex numbers in $D \cup \partial D$ with $b_i \neq b_j (i \neq j)$, and let $\{\alpha(j)\}$ be a finite sequence of real numbers such that $-2 < \alpha(j)$ when j is in Λ^c (the definition of Λ is below). Put $p_j(z) = |z - b_j|$ and $u = \prod p_j^{\alpha(j)}$, and let $0 < r < \infty$, then there are constants $\gamma_1 > 0$ and $\gamma_2 > 0$ such that*

$$\gamma_1 \hat{u}_r(a) \leq \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)} \leq \gamma_2 \hat{u}_r(a)$$

for all a in D, here $\Lambda = \{j; b_j \text{ is in } \partial D\}$.

Proof. For any fixed $0 < r < \infty$, in general, Lemma 4.3.3 in [**9**, p. 60] implies that there are constants $L > 0$ and $M > 0$ such that

$$L\hat{u}_r(a) \leq \int_{D_r(0)} u \circ \phi_a d\, m \leq M\hat{u}_r(a)$$

for all a in D, where u is a non-negative integrable function on D. Let $u = \prod |z - b_j|^{\alpha(j)}$, $\{b_j\} \subset D \cup \partial D$, $b_i \neq b_j (i \neq j)$, and $\alpha(j)$ be real numbers. Then, by the same calculations in the proof of (2) of Proposition 5, we have that

$$\int_{D_r(0)} u \circ \phi_a d\, m$$
$$= \prod |1 - \bar{a} b_j|^{\alpha(j)} \int_{D_r(0)} \prod |\phi_a(b_j) - z|^{\alpha(j)} \cdot |1 - \bar{a} z|^{-\Sigma \alpha(j)} d\, m(z).$$

Put

$$I(a) = \int_{D_r(0)} \prod |\phi_a(b_j) - z|^{\alpha(j)} d\, m(z),$$

then it is easy to see that $\int_{D_r(0)} u \circ \phi_a d\, m$ is equivalent to

$$I(a) \times \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)}.$$

Firstly, we show that the lemma is true when $0 \leq \alpha(j)$ for all j. By the above facts, it is enough to prove that the integration

$$I(a) = \int_{D_r(0)} \prod |\phi_a(b_j) - z|^{\alpha(j)} d\, m(z)$$

is bounded below for all a in D, because $0 \leq \alpha(j)$. Conversely, suppose that there exists $\{a_n\} \subset D$ such that $I(a_n) < 1/n$. Here we can choose a subsequence $\{a_k\} \subset \{a_n\}$ such that $a_k \to a' (k \to \infty)$, where a' may be in $D \cup \partial D$. Therefore, Fatou's lemma implies that $I(a') = 0$, thus it follows that $\prod |\phi_{a'}(b_j) - z|^{\alpha(j)} = 0$ on $D_r(0)$. This contradiction implies that the assertion is true when $0 \leq \alpha(j)$ for all j.

Next, we prove that the lemma is true when $-2 < \alpha(j) < 0$ for all j in Λ^c and $-\infty < \alpha(j) < 0$ for all j in Λ. In fact, we claim that $I(a)$ is bounded for all a in D. If j is in Λ, then $|\phi_a(b_j)| = 1$ for all a in D, therefore $|\phi_a(b_j) - z|^{-1}$ is bounded, because z belongs to $D_r(0)$. Analogously, if j is in Λ^c, then $|\phi_a(b_j)| \to 1$ $(|a| \to 1)$, therefore $|\phi_a(b_j) - z|^{-1}$ is bounded when a is nearby ∂D, because z belongs to $D_r(0)$. Thus, it is sufficient to prove that

$$J(a) = \int_{D_r(0)} \prod_{j \in \Lambda^c} |\phi_a(b_j) - z|^{\alpha(j)} d\, m(z)$$

is bounded for all a in $U_\eta(0) = \{a \in D; |a| \leq \eta\}$, where $0 < \eta < 1$ is a constant which is close to 1. Put

$$\Phi_{i,j}(a) = |\phi_a(b_i) - \phi_a(b_j)| \quad (i, j \in \Lambda^c,\ a \in U_\eta(0)).$$

For any fixed $i, j \in \Lambda^c$, since $\Phi_{i,j}$ is a continuous function on $U_\eta(0)$ and Möbius functions are one-to-one correspondence on D, there exists $\varepsilon(i,j) > 0$ such that $\Phi_{i,j}(a) \geq \varepsilon(i,j)$ for all a in $U_\eta(0)$ when $i \neq j$. Put $\varepsilon = \min\{\varepsilon(i,j)/2;\ i, j \in \Lambda^c \text{ such that } i \neq j\}$,

$$B_j(a) = \{z \in D_r(0); |\phi_a(b_j) - z| < \varepsilon\}$$

and $B_0(a) = D_r(0) \setminus \cup B_j(a)$. For any j in $\Lambda^c \cup \{0\}$, since $|\phi_a(b_i) - z| \geq \varepsilon$ when z belongs to $B_j(a)$ and i belongs to Λ^c such that $i \neq j$, therefore we have that

$$\begin{aligned} J(a) &\leq \sum_{j \in \Lambda^c} \varepsilon^{\alpha - \alpha(j)} \int_{B_j(a)} |\phi_a(b_j) - z|^{\alpha(j)} d\, m(z) + \varepsilon^{\alpha} \int_{B_0(a)} d\, m(z) \\ &\leq \sum_{j \in \Lambda^c} \varepsilon^{\alpha - \alpha(j)} \int_{2D} |w|^{\alpha(j)} d\, m(w) + \varepsilon^{\alpha} \end{aligned}$$

where

$$\alpha = \sum_{j \in \Lambda^c} \alpha(j).$$

Therefore, J is bounded on $D_\eta(0)$, and hence we obtain that I is bounded on D.

Using the above facts, we can show that the assertion is true when u has the general form of the statement of this lemma. Let $\{\alpha(j)\}$ be a finite sequence of real numbers such that $-2 < \alpha(j) < \infty$ when j is in Λ^c and $-\infty < \alpha(j) < \infty$ when j is in Λ. As in the proof of Proposition 5, set

$j(+) = \{j;\ \alpha(j) \geq 0\}$ and $j(-) = \{j;\ \alpha(j) < 0\}$, then we have that

$$I(a) \leq 2^{\Sigma_{j(+)}\alpha(j)} \int_{D_r(0)} \prod_{j(-)} |\phi_a(b_j) - z|^{\alpha(j)} d\, m(z)$$

and

$$I(a) \geq 2^{\Sigma_{j(-)}\alpha(j)} \int_{D_r(0)} \prod_{j(+)} |\phi_a(b_j) - z|^{\alpha(j)} d\, m(z).$$

Therefore, we obtain that I is bounded and bounded below on D. Hence, this completes the proof. □

Corollary 2. *Let u be a non-negative function in L^1 that is given by (2), or (3) of Proposition 5 and ν be a finite positive measure on D, then there is a constant $C > 0$ such that*

$$\int_D |f|^2 d\, \nu \leq C \int_D |f|^2 u\, d\, m$$

for all f in P if and only if there exist $r > 0$ and $\gamma = \gamma_r > 0$ such that

$$\hat{\nu}_r(a) \leq \gamma \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)}$$

for all a in D, here $\Lambda = \{j;\ b_j \text{ is in } \partial D\}$.

Proof. The corollary follows from Theorem 3, Proposition 5 and Lemma 2. □

We give a characterization of u which satisfies the (A_2)-condition or the $(A_2)_\partial$-condition when u is a modulus of a rational function or a modulus of a polynomial, respectively. Let u be a non-negative integrable function on D, then it is easy to see that if u satisfies the $(A_2)_\partial$-condition then u^{-1} is integrable on D. But, we claim that the converse is true, when u is a modulus of a polynomial. As the result, we show that the $(A_2)_\partial$-condition is properly contained in the (A_2)-condition. The essential part of the following theorem is proved in Proposition 5 and Lemma 2.

Theorem 6. *Let $\{b_j\}$ be a finite sequence of complex numbers such that $b_i \neq b_j (i \neq j)$ and $\{\alpha(j)\}$ be a finite sequence of real numbers. Put $p_j(z) = |z - b_j|$ and $u = \prod p_j^{\alpha(j)}$, then the following are true.*

(1) *If $\alpha(j) \geq 0$ for all j or $\alpha(j) \leq 0$ for all j, then u satisfies the $(A_2)_\partial$-condition if and only if $\alpha(j) < 2$ or $\alpha(j) > -2$ when b_j is in $D \cup \partial D$ respectively.*

(2) *u satisfies the (A_2)-condition if and only if $-2 < \alpha(j) < 2$ when b_j is in D.*

Proof. (1) By (2) of Proposition 5 and the remark above this theorem, it is enough to prove that u^{-1} is not integrable on D when $\alpha(j) \geq 2$ for some b_j in $D \cup \partial D$. Suppose that there is a j such that b_j in $D \cup \partial D$ and $\alpha(j) \geq 2$, then there exists a L^∞-function h such that $u(z) = |z - b_j|^2 \cdot h(z)$. It is easy to see that u^{-1} is not integrable on $U = \{z \in D; |z - b| < \operatorname{dist}(b_j, \partial D)\}$ when b_j is in D, therefore we consider the case when $b_j = 1$. Put $M_2 = \|h\|_\infty$, then

$$\begin{aligned} \int u^{-1} d\, m &\geq M_2^{-1} \int_0^1 2r \int_0^{2\pi} |1 - re^{i\theta}|^2 d\,\theta/2\pi d\, r \\ &= M_2^{-1} \int_0^1 2r(1-r^2)^{-1}\, d\, r = M_2^{-1} \int_0^1 t^{-1}\, d\, t. \end{aligned}$$

Hence we obtain that u^{-1} is not integrable.

(2) Suppose that $-2 < \alpha(j) < 2$ when b_j is in D, then apparently Lemma 2 implies that u satisfies the (A_2)-condition. Conversely, suppose that there exist $r > 0$ and $A_r > 0$ such that

$$\hat{u}_r(a) \times (u^{-1})_r^\wedge(a) \leq A_r$$

for all a in D. Since $\hat{u}_r$ is non-zero on D, therefore $(u^{-1})_r^\wedge(a) < \infty$ for all a in D. By the same argument in (1), we have that $\alpha(j)$ must be less than 2 when b_j is in D. In fact, if $\alpha(j) \geq 2$ for some b_j in D, then there exists a function h such that $u(z) = |z - b_j|^2 \cdot h(z)$. Put

$$\varepsilon = \min\{\operatorname{dist}(b_i, b_j)/2;\ i \neq j\}$$

and

$$U(j) = \{z \in D; |z - b_j| < \varepsilon\},$$

then obviously h is bounded on $U(j)$. Since there exists a_j such that a center of the Bergman disk $D_r(a_j)$ is just equal to b_j, therefore we have that u^{-1} is not integrable on $D_r(a_j) \cap U(j)$, and thus, it follows that the average of u^{-1} on $D_r(a_j)$ is infinite. This contradicts the above fact. Consequently, we obtain that $\alpha(j)$ must lie in $(-\infty, 2)$ when b_j is in D. Applying the same argument to u^{-1}, we have that $\alpha(j)$ must lie in $(-2, \infty)$ when b_j is in D. Therefore, we conclude that $-2 < \alpha(j) < 2$ when b_j is in D. $\square$

§4. Uniformly absolutely continuous.

Recall that

$$\varepsilon_r(\mu) = \sup_{a\in D}\left(\int_{D\setminus D_r(a)} |k_a|^2 \, d\,\mu\right) \times \left(\int_D |k_a|^2 \, d\,\mu\right)^{-1},$$

where μ is a finite positive measure on D (see Lemma 1 and Proposition 2). Using the quantity ε_r we give a necessary condition on ν and μ which satisfy the (ν,μ)-Carleson inequality.

Theorem 7. *Suppose that $d\,\nu = vd\,m$, $\varepsilon_t(\nu) \to 0$ $(t \to \infty)$, and that v satisfies the (A_2)-condition, furthermore μ and ν satisfy the (μ,ν)-Carleson inequality. If there is a constant $C > 0$ such that*

$$\int_D |f|^2 d\,\nu \le C \int_D |f|^2 d\,\mu$$

for all f in P, then there exist $r > 0$ and $\gamma > 0$ such that

$$\hat{\nu}_r(a) \le \gamma \hat{\mu}_r(a)$$

for all a in D.

Proof. By hypotheses on ν and Lemma 1, there exist $t > 0, \rho > 0$ and $A > 0$ such that

$$\tilde{\nu} \le \rho \cdot \hat{\nu}_t \le A\rho \cdot (v^{-1})_t^{\wedge -1}.$$

Moreover, Lemma 4.3.3 in [**9**, p. 60] and the (μ,ν)-Carleson inequality imply that there exist $L > 0$ and $C' > 0$ such that

$$L \cdot \hat{\mu}_t \le \tilde{\mu} \le C' \cdot \tilde{\nu}.$$

Thus, a desired result follows from (2) of Proposition 2. □

Luecking [**5**] shows the above theorem when ν is the Lebesgue area measure m. It is clear that $\varepsilon_r(m) \to 0$ $(r \to \infty)$ and m satisfies the (A_2)-condition. Now, we are interested in measures μ such that $\varepsilon_r(\mu) < 1$ or $\varepsilon_r(\mu) \to 0 (r \to \infty)$.

Proposition 8. *Suppose that $d\,\mu = ud\,m$, and u is a non-negative function in L^1. If u is the function such that* (1) *or* (2), *then there exists $0 < r < \infty$ such that $\varepsilon_r(\mu) < 1$.*

(1) *u satisfies the $(A_2)_\partial$-condition.*

(2) *$u(z) = (1 - |z|^2)^\alpha$ for some $1 \le \alpha < 2$.*

Proof. If u has the property in (1), then by the remark above Theorem 3, for any $r > 0$ there is a positive constant $\rho = \rho_r$ such that $\tilde{\mu}(a) \le \rho \hat{\mu}_r(a)$ for all a in D and hence $\varepsilon_r(\mu) < 1$ by Lemma 1. Suppose that u has the form of (2). For any fixed $1 \leq \alpha < 2$, put $u(z) = (1 - |z|^2)^\alpha$, Then, Rudin's lemma (cf. [**9**, p. 53]) shows that

$$\tilde{u}(a) = (1 - |a|^2)^\alpha \int_D (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} d\, m(z) \leq \gamma(1 - |a|^2)^\alpha,$$

where $\gamma > 0$ is finite. On the other hand, Lemma 4.3.3 in [**9**, p. 60] implies that

$$\begin{aligned}\hat{u}_r(a) &\geq M^{-1} \times (1 - |a|^2)^\alpha \int_{D_r(0)} (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} d\, m(z) \\ &\geq M^{-1} \times (1 - |z|^2)^\alpha (1 - \tanh^2 r)^\alpha \times 2^{-2\alpha},\end{aligned}$$

therefore, by (3) of Lemma 1, we obtain that $\varepsilon_r(\mu) < 1$. □

Proposition 9. *Suppose that $d\, \mu = ud\, m$, and u is a non-negative function in L^1. If u is one of the following functions* (1) $\sim$ (7), *then* $\varepsilon_r(\mu) \to 0 (r \to \infty)$.

(1) *There exists $\varepsilon_0 > 0$ such that $\tilde{u} \geq \varepsilon_0$ on D, and $\{u \circ \phi_a d\, m; a \in D\}$ is uniformly absolutely continuous with respect to the Lebesgue area measure m.*

(2) *There exists $\varepsilon_0 > 0$ such that $\tilde{u} \geq \varepsilon_0$ on D, and there is a constant $C > 0$ such that $(u^{1+\beta})^\sim \leq C$ on D for some $\beta > 0$.*

(3) *u is in L^∞, and there exist $r > 0$ and $\delta > 0$ such that $u \geq \delta$ on $D \backslash D_r(0)$.*

(4) *$u = |p|$, where p is an analytic polynomial which has no zeros on ∂D.*

(5) *$u(z) = (1 - |z|^2)^\alpha$ for some $-1 < \alpha \leq 1$.*

(6) *$u = \prod p_j^{\alpha(j)}$, where $p_j(z) = |z - \beta_j|$, $b_i \neq b_j (i \neq j)$, and $0 < \alpha(j) < 2$ for b_j in $D \cup \partial D$, or $-2 < \alpha(j) < 0$ for b_j in $D \cup \partial D$.*

(7) *$u = \prod p_j^{\alpha(j)}$ where $p_j(z) = |z - b_j|$, $b_i \neq b_j (i \neq j)$, and $-1 < \alpha(j) < 1$ for b_j in $D \cup \partial D$.*

Proof. Firstly, we show that the assertion is true when u has the property of (1). Since $\{u \circ \phi_a d\, m; a \in D\}$ is uniformly absolutely continuous, for any $\varepsilon > 0$ there exists $r > 0$ such that $\int_{D_r(0)^c} u \circ \phi_a d\, m < \varepsilon_0 \cdot \varepsilon$ for all a in D. Therefore, making a change of variable, let r be sufficiently large, then $\varepsilon_r(\mu) < \varepsilon_0^{-1} \cdot \varepsilon_0 \cdot \varepsilon = \varepsilon$. Hence, we obtain that $\varepsilon_r(\mu) \to 0 (r \to \infty)$.

Next, we prove the implications (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (2), and (4) $\Rightarrow$ (3). Then $\varepsilon_r(\mu) \to 0$ when u is a function such that (2), (3) or (4). In fact, suppose that there exists $\beta > 0$ such that the Berezin transform of the function $u^{1+\beta}$ is bounded, then a set of functions $\{u \circ \phi_a; a \in D\}$ is uniformly integrable (cf. [**1**, p. 120]), therefore it follows that $\{u \circ \phi_a d\, m; a \in D\}$ is uniformly absolutely continuous with respect to m. Hence, (2) implies (1). If there exist $r > 0$ and $\delta > 0$ such that $u \geq \delta$ on $D \backslash D_r(0)$, then

$$\tilde{u}(a) \geq \delta - \delta \int_{D_r(0)} |k_a|^2 d\, m = \delta[1 - m(D_r(a))] \geq \delta(1 - \tanh^2 r) > 0.$$

Hence (3) implies (2) because $(u^{1+\beta})^{\sim}(a) \leq \|u\|_\infty^{1+\beta}$ for all a in D and any $\beta > 0$. Next, let p be an analytic polynomial which has no zeros on ∂D, then there are $r > 0$ and $\delta > 0$ such that $u = |p| \geq \delta$ on $D \backslash D_r(0)$, therefore (4) $\Rightarrow$ (3).

We prove that the assertion is true when u has the form of (5). For any fixed $-1 < a \leq 1$, put $u(z) = (1 - |z|^2)^\alpha$ and making a change of variable, then

$$\begin{aligned} \varepsilon_r(\mu) = &\sup \left(\int_D (1 - |z|^2)^\alpha |1 - \bar{a}z|^{2\alpha} d\, m(z) \right) \\ &\times \left(\int_{D \backslash D_r(0)} (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} d\, m(z) \right). \end{aligned}$$

When $0 \leq \alpha \leq 1$, since $0 < 1 - |z|^2 \leq 1$, we have that

$$\int_D (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} d\, m \geq 2^{-2\alpha} \int_D (1 - |z|^2) d\, m = \text{constant}.$$

If $-1 < \alpha < 0$, then the familiar inequality between the harmonic and arithmetic means shows that

$$\begin{aligned} \int_D (1 - |z|^2)^\alpha |1 - \bar{a}z|^{-2\alpha} d\, m &\geq \left(\int_D (1 - |z|^2)^{-\alpha} |1 - \bar{a}z|^{2\alpha} d\, m \right)^{-1} \\ &\geq \text{constant}. \end{aligned}$$

Here, the last inequality follows from Rudin's lemma (cf. [**9**, p. 53]). Again using Rudin's lemma, since $-1 < \alpha \leq 1$, there exists $\beta > 0$ such that a set of functions $\{[(1 - |z|^2)^\alpha |1 - az|^{-2\alpha}]^{1+\beta}; a \in D\}$ is bounded in L^1. This implies that the set of these functions are uniformly integrable (cf. [**1**, p. 120]), therefore it follows that $\varepsilon_r(\mu) \to 0 (r \to \infty)$.

We show that $\varepsilon_r(\mu) \to 0$ when u has the form of (6). As in the proof of (2) of Proposition 5, we only prove that $\varepsilon_r(\mu) \to 0 (r \to \infty)$ when $u = p_1^{\alpha(1)} \cdot p_2^{\alpha(2)}$, where $p_1(z) = |z - b_1|$, $p_2(z) = |z - b_2|$, $0 < \alpha(1)$, $\alpha(2) < 2$, and b_1 is in D, b_2 is in ∂D. We suppose that B_j, M_1, and ε are as in the proof of (2) of Proposition 5. By the definition of $\varepsilon_r(\mu)$, we have that

$$\varepsilon_r(\mu) = \sup (u \chi_{D_r(a)^c})^{\sim}(a) \times \tilde{u}(a)^{-1}.$$

Moreover,

$$\begin{aligned}(u\chi_{D_r(a)^c})^{\sim}(a)\times\tilde{u}(a)^{-1}\leq&(u\chi_{D_r(a)^c})^{\sim}(a)\times(u^{-1})^{\sim}(a)\\ \leq&(u\chi_{D_r(a)^c})^{\sim}(a)\times\varepsilon^{-\alpha(1)-\alpha(2)}\int_{B_0}|k_a|^2d\,m\\ &+(u\chi_{D_r(a)^c})^{\sim}(a)\times\varepsilon^{-\alpha(2)}\cdot(p_1^{-\alpha(1)})^{\sim}(a)\\ &+M_1\times\varepsilon^{-\alpha(1)}\times C\int_{D\setminus D_r(0)}|1-\bar{a}z|^{-\alpha(2)}d\,m,\end{aligned}$$

where

$$C=\|\phi_a(b_2)-z\|_{\infty}^{\alpha(2)}\times\|1-\bar{a}z\|_{\infty}^{\alpha(2)}\times\int_{2D}|w|^{-\alpha(2)}d\,m.$$

Since u is bounded, therefore $\{u\circ\phi_a;a\in D\}$ is uniformly integrable (cf. [**1**, p. 120]), moreover applying the same argument in the proof of this proposition when u has the form of (5), Rudin's lemma implies that a set of functions $\{|1-\bar{a}z|^{-\alpha(2)};a\in D\}$ is also uniformly integrable, hence we conclude that $\varepsilon_r(\mu)\to 0(r\to\infty)$. The proof of the latter half of (6) of this proposition is similar that in the above.

If u has the form of (7), then by the similar arguments in the proof of (3) of Proposition 5, set $j(+)=\{j;\alpha(j)\geq 0\}$, $j(-)=\{j;\alpha(i)<0\}$. And put $u_1=\prod_{j(+)}p_j^{\alpha(j)}$, $u_2=\prod_{j(-)}p_j^{\alpha(j)}$, then

$$\begin{aligned}(u\chi_{D_r(a)^c})^{\sim}(a)\times\tilde{u}(a)^{-1}\leq&(u\chi_{D_r(a)^c})^{\sim}(a)\times(u^{-1})^{\sim}(a)\\ =&(u_1u_2\chi_{D_r(a)^c})^{\sim}(a)\times(u_1^{-1}u_2^{-1})^{\sim}(a).\end{aligned}$$

Therefore, the desired result follows from the Cauchy-Schwarz's inequality and (6) of this proposition. □

Corollary 3. *Suppose that $d\,\nu=vd\,m$ and there is a consrant $C>0$ such that*

$$\int_D|f|^2d\,\nu\leq C\int_D|f|^2d\,\mu$$

for all a in D, then the following are true.

(1) *If $v(z)=(1-|z|^2)^{\alpha}$ for some $-1<\alpha\leq 1$, and there exist $l>0$ and $\gamma'=\gamma'_l>0$ such that*

$$\hat{\mu}_l(a)\leq\gamma'(1-|a|^2)^{\alpha}$$

for all a in D, then there exist $r>0$ and $\gamma=\gamma_r>0$ such that

$$(1-|a|^2)^{\alpha}\leq\gamma\hat{\mu}_r(a)$$

for all a in D.

(2) *If* $v = \prod p_j^{\alpha(j)}$, *where* $p_j(z) = |z - b_j|$, $b_i \neq b_j (i \neq j)$, *and* $0 < \alpha(j) < 2$ *for* b_j *in* $D \cup \partial D$ *or* $-2 < \alpha(j) < 0$ *for* b_j *in* $D \cup \partial D$, *and if there exist* $l > 0$ *and* $\gamma' = \gamma_l' > 0$ *such that*

$$\hat{\mu}_l(a) \leq \gamma' \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)}$$

for all a *in* D, *then there exist* $r > 0$ *and* $\gamma = \gamma_r > 0$ *such that*

$$\prod_{j \in \Lambda} |a - b_j|^{\alpha(j)} \leq \gamma \hat{\mu}_r(a)$$

for all a *in* D, *where* $\Lambda = \{j; b_j \text{ is in } \partial D\}$.

(3) *If* $v = \prod p_j^{\alpha(j)}$ *where* $p_j(z) = |z - b_j|$, $b_i \neq b_j (i \neq j)$, *and* $-1 < \alpha(j) < 1$ *for* b_j *in* $D \cup \partial D$, *and if there exist* $l > 0$ *and* $\gamma = \gamma_l' > 0$ *such that*

$$\hat{\mu}_l(a) \leq \gamma' \prod_{j \in \Lambda} |a - b_j|^{\alpha(j)}$$

for all a *in* D, *then there exist* $r > 0$ *and* $\gamma = \gamma_r > 0$ *such that*

$$\prod_{j \in \Lambda} |a - b_j|^{\alpha(j)} \leq \gamma \hat{\mu}_r(a)$$

for all a *in* D, *where* $\Lambda = \{j; b_j \text{ is in } \partial D\}$.

Proof. We show that (1) is true. By the fact in the proof of Corollary 1, and the fact that $u(z) = (1 - |z|^2)^\alpha$ satisfies the (A_2)-condition for all $\alpha > -1$ (see [**6**]), the hypothesis in (1) of the Corollary and Proposition 1 imply the (μ, ν)-Carleson inequality. Hence, Theorem 7 and Proposition 9 show that the assertion is true.

Similarly, (2) and (3) follow from Proposition 1, Lemma 2, (5) of Proposition 4, Theorem 6, Theorem 7, and Proposition 9. □

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