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APPROXIMATION BY NORMAL ELEMENTS WITH FINITE SPECTRA IN C^*-ALGEBRAS OF REAL RANK ZERO

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We study the problem when a normal element in a C^*-algebra of real rank zero can be approximated by normal elements with finite spectra. We show that all purely infinite simple C^*-algebras, irrational rotation algebras and some types of C^*-algebras of inductive limit of the form $C(X) \otimes M_n$ of real rank zero have the property weak (FN), i.e., a normal element x can be approximated by normal elements with finite spectra if and only if $\Gamma(x) = 0$ ($\lambda - x \in \operatorname{Inv}_0(A)$ for all $\lambda \notin \operatorname{sp}(x)$). For general C^*-algebras with real rank zero, we show that a normal element x with $\dim \operatorname{sp}(x) \leq 1$ can be approximated by normal elements with finite spectra if and only if $\Gamma(x) = 0$. One immediate application is that if A is a simple C^*-algebra with real rank zero which is an inductive limit of C^*-algebras of form $C(X_n) \otimes M_{m(n)}$, where each X_n is a compact subset of the plane, then A is an AF-algebra if and only if $K_1(A) = 0$.

1. Introduction.

The notion of real rank for C^*-algebras was introduced by L.G. Brown and Gert K. Pedersen ([**BP**]). At present, it seems that it is the notion of real rank zero that attracted most attention. (See [**BBEK**], [**BDR**], [**BP**], [**BKR**], [**CE**], [**Ell2**], [**EE**],[**G**], [**GL**], [**LZ**], [**Zh1-4**], [**Ph**], [**Lin4**], [**Lin5**], and other articles. When this revision is writing, many other articles on the subject are appearing.) It turns out that the class of C^*-algebras of real rank zero is fairly large. In fact, from the remarkable work of G.A. Elliott ([**Ell2**]), for any countable unperforated graded ordered group G with Riesz decomposition property, there exists a separable nuclear C^*-algebra A of real rank zero (and stable rank one) such that the graded ordered group $(K_0(A), K_1(A))$ is isomorphic to G. (See [**Ell3**, 1.2 and 5.1] for the definition of unperforated graded ordered group. Notice that Elliott's definition of unperforated is in fact weaker than that of weakly unperforated in [**Bl3**, 6.7.1].) If A has real rank zero, then A has the property (FS), i.e., every selfadjoint element can be approximated by selfadjoint elements with finite spectra (see [**BP**, 2.6]) (and converse is also true). What about unitaries? Does real rank zero imply the property (FU), i.e., every unitary can be approximated by unitaries with finite spectra? One quickly realizes (for example, from Elliott's classification theorem [**Ell2**]) that there are many C^*-algebras with real rank zero have nontrivial K_1-groups. It is clear that those unitaries which are not connected to the identity can not be approximated by unitaries with finite spectra. N. C. Phillips introduced the notion of weak (FU), i.e., every unitary in the connected component containing the identity can be approximated by unitaries with finite spectra ([**Ph1**]). Many C^*-algebras of real rank zero were proved to have the property weak (FU) ([**Ph1-2**], [**GL**]). Recently, we show that, in fact, every C^*-algebra with real rank zero has the property weak (FU)([**Lin5**]). This result has many applications in the study of the unitary groups and K_1-groups and is shown to have applications to C^*-algebra extension theory (see [**Lin3**], [**Lin4**] and Lemma 2.2). Next, of course, we will consider general normal elements. At this point we would like to mention the BDF-theory. It is known that the Calkin algebra $A \cong B(H)/K$, where H is the separable, infinite dimensional *Hilbert* space and K is the C^*-algebra of compact operators, has real rank zero. One crucial result in the BDF-theory is that a normal element x in the Calkin algebra can be approximated by normal elements with finite spectra if and only if the index of x, $\Gamma(x)$ is zero. All of these previous results lead us to the following question:

Q *Is it true that in a C^*-algebra with real rank zero, a normal element x can be approximated by normal elements with finite spectra if and only if $\Gamma(x) = 0$ (the index Γ will be defined later)?*

Our result for weak (FU) in [**Lin5**] shows that if we further assume that $\mathrm{sp}(x) = S^1$, then the answer to **Q** is affirmative. The BDF-theory shows that the answer to **Q** is affirmative if A is the Calkin algebra. Recent results in [**Lin4**] show that for many other corona algebras the answer to **Q** is also affirmative. If all of these give evidence that, in general, the answer to **Q** should be positive, we would like to remind the readers that the question for AF-algebras remains open, i.e., we do not know if every normal element in an AF-algebra can be approximated by normal elements with finite spectra, even though some progresses have been made (see [**Lin6**]).

In the present paper, we show that the answer to **Q** is affirmative for purely infinite simple C^*-algebras and for many other C^*-algebra of real rank zero such as the Bunce-Deddens algebras and irrational rotation algebras. For general C^*-algebras, we show that if we restrict to those normal elements x with dim $\mathrm{sp}(x) \leq 1$, the answer to **Q** is affirmative. But for more general normal elements, we have to work in the $A \otimes K$ (see Theorem 3.13). While the author still believes the answer to **Q** should be affirmative, Theorem 3.13 might be the best form we can have for the general case at present.

An affirmative answer to Q has some interesting implications. Let $X_{(n,i)}$ be a sequence of compact subsets of the plane and A be a C^*-inductive limit of C^*-algebras of the form

$$\sum \oplus_{i=1}^{k(n)} C(X_{(n,i)}) \otimes M_{m(n,i)}.$$

Some necessary and sufficient conditions for A having real rank zero are given in [**BEEK**]. From [**Ell2**], if every $X_{(n,i)}$ is either the unit circle or the unit interval, then such C^*-algebras of real rank zero are completely determined by their K_*. An immediate consequence of an affirmative answer to Q is that A is in fact an AF-algebra if and only if $K_1(A) = 0$. So in the case that $K_1(A) = 0$, these algebras will be completely determined by their K_0-groups ([**Ell1**]). It should not be hard, then, from an affirmative answer to Q, that in fact these C^*-algebras of real rank zero in general (without assuming their K_1-groups being trivial), are completely determined by their K_*-groups. In particular, these C^*-algebras are included in [**Ell2**].

The paper is organized as follows: Section 2 gives a generalized notion of index Γ for normal elements and some basic facts which we are needed in subsequent sections. In Section 3, we present some technical approximation results in $A \otimes K$ which are important for the rest of the paper. In Section 4, we show that for purely infinite simple C^*-algebras, irrational rotation algebras, the Bunce-Deddens algebras and many other C^*-algebras with real rank zero, the answer to **Q** is affirmative. In Section 5, we show that for general C^*-algebras with real rank zero, a normal element x with $\dim \mathrm{sp}(x) \leq 1$ can be approximated by normal elements with finite spectra if and only if $\Gamma(x) = 0$. Finally, in Section 6, we give some applications. We will show, for example, if A is a simple C^*-algebra of real rank zero which is an inductive limit of C^*-algebras of the from $C(X_n) \otimes M_{m(n)}$, where each X_n is a contractible compact subset of the plane, then A is an AF-algebra.

The following notations are used throughout this paper.

Let A be a C^*-algebra. We use the notation A^{**} for the enveloping W^*-algebra. If p is an open projection (of A) in A^{**}, $\mathrm{Her}(p)$ is the hereditary C^*-subalgebra $pA^{**}p \cap A$.

Let

$$A_1 \to A_2 \to \cdots \to A_n \to \cdots$$

be a sequence of C^*-algebras and $\phi_n : A_n \to A_{n+1}$ be the connecting homomorphisms. We will use the notation $\lim_{\to}(A_n, \phi_n)$ for the C^*-inductive limit and $\phi_\infty(A_n)$ for the image of A_n in the inductive limit.

Added in proof: This paper was written in 1992. Since then there are significant development in the study of C^*-algebras of real rank. We would only like to mention that, by the result that a pair of almost commuting selfadjoint matrices is close to a pair of commuting selfadjoint matrices proved by the author, every AF-algebras has (FN).

2. Preliminaries.

While 2.2 and 2.3 are certainly new, most results stated in this section are either routine or easy consequences of some known facts. Since many of these will be needed frequently in the subsequent sections, we present them here for reader's convenience. Some sketch of proofs are also presented.

Definition 2.1. Let A be a unital C^*-algebra and X be a compact Hausdorff space. Suppose that $\phi : C(X) \to A$ is a monomorphism and $\phi_* : K_1(C(X)) \to K_1(A)$ is the induced homomorphism. Let x be a normal element in A. Denote by B the C^*-subalgebra of A generated by x and the identity 1. Then $B \cong C(X)$, where $X = \mathrm{sp}(x)$. This gives a monomorphism $\phi : C(X) \to A$. We define

$$\Gamma(x) = \Gamma_A(x) = \phi_*.$$

If A is not unital, we define $\Gamma(x) = \Gamma_{\tilde{A}}(x)$.

Suppose that A is unital. We denote the unitary group of A by $U(A)$ and the path connected component containing the identity by $U_0(A)$. There is a homomorphism i from $U(A)/U_0(A)$ into $K_1(A)$. We will use the following lemma which is an application of our result for weak (FU) ([**Lin5**]).

Lemma 2.2. *Let A be a (unital) C^*-algebra of real rank zero. Then the map $i : U(A)/U_0(A) \to K_1(A)$ is injective.*

Proof. Suppose that $u \in U(A)$ and $V = \mathrm{diag}(1, 1, ..., 1, u)$ is in $U_0(M_k(A))$ for some integer k. It follows from [**BP**, 2.10] that $M_k(A)$ has real rank zero. So, by [**Lin5**], V can be approximated by unitaries in $M_k(A)$ with finite spectra. It follows from [**Lin5**, Lemma 3] that u can be approximated by unitaries in A with finite spectra. Since unitaries with finite spectra are connected by a path with the identity, we conclude that $u \in U_0(A)$. □

When X is a compact subset of the plane, it is well known that

$$K_1(C(X)) \cong U(C(X))/U_0(C(X)).$$

So $\Gamma(x)$ is a homomorphism from $U(C(\mathrm{sp}(x)))/U_0(C(\mathrm{sp}(x)))$ into $U(A)/U_0(A)$, when the map i is injective, in particular, when A has real rank zero. Let $\mathrm{Inv}(A)$ denote the group of invertible elements in A and $\mathrm{Inv}_0(A)$ denote the path connected component of $\mathrm{Inv}(A)$ containing the identity. It is well known that $\mathrm{Inv}(A)/\mathrm{Inv}_0(A) \cong U(A)/U_0(A)$. We also will use the notation $\pi^1(X)$ for the group $\mathrm{Inv}(C(X))/\mathrm{Inv}_0(C(X)) \cong U(C(X))/U_o(C(X))$.

Lemma 2.3. *Let A be a (unital) C^*algebra of real rank zero and let $p \in A$ be a projection. If $x \in pAp$ and $y \in (1-p)A(1-p)$ such that $x+y \in \mathrm{Inv}_o(A)$ and $y \in \mathrm{Inv}_0((1-p)A(1-p))$, then $x \in \mathrm{Inv}_0(pAp)$.*

Proof. Let u be the unitary part of the polar decomposition of $x+y$ in A. Then $u \in U_0(A)$. It follows from [**Lin5**] that A has the property weak (FU). So u can be approximated by unitaries in A with finite spectra. We can write $u = u_1 + u_2$, where $u_1 \in U(pAp)$ and $u_2 \in U((1-p)A(1-p))$. Since $y \in \mathrm{Inv}_0((1-p)A(1-p))$, $u_2 \in U_0((1-p)A(1-p))$. By [**Lin5**] again, u_2 can be approximated by unitaries in $(1-p)A(1-p)$ with finite spectra. Then, from [**Lin5**, Lemma 3], we conclude that u_1 can be approximated by unitaries in pAp with finite spectra. Hence $u \in U_0(pAp)$. This implies that $x \in \mathrm{Inv}_0(pAp)$. □

Proposition 2.4. *Let A be a (unital) C^*-algebra and let $x \in A$ be a normal element. Then $\Gamma(x) = 0$ if and only if for any $\lambda \notin \mathrm{sp}(x)$, $\lambda - x \in \mathrm{Inv}_0(A)$. Or equivalently, the unitary part of the polar decomposition of $\lambda - x$ is in $U_0(A)$ for all $\lambda \notin \mathrm{sp}(x)$.*

Proof. The "only if" part is trivial from the definition of Γ. For the "if" part, we know that $\pi^1(\mathrm{sp}(x))$ is the free abelian group with a generator for each bounded component of $C \setminus \mathrm{sp}(x)$. Morover, if Ω is such a component and $\lambda \in \Omega$, then the homotopy class containing the invertible function θ_λ, defined by

$$\theta_\lambda = \lambda - z \quad \text{for} \quad z \in \mathrm{sp}(x),$$

is a generator corresponding to Ω. Since

$$\phi_*([\theta_\lambda]) = 0,$$

we conclude that $\phi_* = 0$. So $\Gamma(x) = 0$. □

Lemma 2.5. *Let x and y be two elements in a unital C^*-algebra A. If x is invertible and if*

$$\|x - y\| < \|x^{-1}\|,$$

then y is invertible and $[x] = [y]$ in $\mathrm{Inv}(A)/\mathrm{Inv}_0(A)$.

Corollary 2.6. *If λ is in the unbounded component of $C \setminus \mathrm{sp}(x)$, then $\lambda - x \in \mathrm{Inv}_0(A)$.*

Lemma 2.7. *For any $d > 0$, any C^*-algebra A and normal element $x \in A$ and $\Gamma(x) = 0$, if $y \in A$ such that*

$$\|x - y\| < d,$$

then for any bounded component Ω of $C\backslash \mathrm{sp}(x)$ and $\lambda \in \Omega$ with $\mathrm{dis}(\lambda, \mathrm{sp}(x)) \geq d$, *$\lambda - y \in \mathrm{Inv}_0(A)$, moreover, the unitary part of the polar decomposition of $\lambda - y$ is in $U_0(A)$.*

Proof. Since x is normal, it is known that

$$\|(\lambda - x)^{-1}\| \leq 1/\mathrm{dis}(\lambda, \mathrm{sp}(x)) \leq 1/d.$$

If $\|x - y\| < d$, then $\|(\lambda - x) - (\lambda - y)\| < d$. By 2.5, $\lambda - y$ is invertible and

$$[\lambda - y] = [\lambda - x] = 0 \quad \text{in} \quad \mathrm{Inv}(A)/\mathrm{Inv}_0(A).$$

Therefore, $\lambda - y \in \mathrm{Inv}_0(A)$. □

Lemma 2.8. *For any $\epsilon > 0$, $b > 0$, an analytic function f on a region X and a continuous function g on X, there exists $\delta > 0$,*
(1) *if x and y are two normal elements in a (unital) C^*-algebra A with* $\mathrm{sp}(x)$, $\mathrm{sp}(y) \subset X$ *and $\|x\|$, $\|y\| \leq b$ such that $\|x - y\| < \delta$, then*

$$\|g(x) - g(y)\| < \epsilon;$$

(2) *if $p \in A$ is a projection and $\|px - xp\| < \delta$, then*

$$\|f(pxp) - pf(x)p\| < \epsilon.$$

(In the above, note that if $f = \lambda$ is a constant, $f(pxp) = \lambda p$.)

Proof. (1) is the same as [**Lin6**, 1].

(2) If δ is small enough, by [**Lin6**, 4], there is closed curve $\Gamma \in X$ such that

$$\mathrm{dist}(\Gamma, \mathrm{sp}(x) \cup \mathrm{sp}(pxp)) > d > 0,$$
$$pf(x)p = 1/2\pi i \int f(\lambda)p(\lambda - x)^{-1}p d\lambda$$

and

$$f(pxp) = 1/2\pi i \int f(\lambda)(\lambda p - pxp)^{-1} d\lambda.$$

If $\lambda \in \Gamma$, by [**Lin6**, 4], again,

$$\begin{aligned}
&\|p(\lambda - x)^{-1}p - (\lambda p - pxp)^{-1}\| \\
&= \left\| \left[p(\lambda - x)^{-1}p - (\lambda p - pxp)^{-1}\right] (\lambda - x)(\lambda - x)^{-1} \right\| \\
&\leq \|p(\lambda - x)^{-1}p[p(\lambda - x) - (\lambda - x)p](\lambda - x)^{-1}\| \\
&\quad + \|p(\lambda - x)^{-1}p[(\lambda - x)p - (\lambda p - pxp)^{-1}p(\lambda - x)p](\lambda - x)^{-1}\| \\
&\quad + \|(\lambda p - pxp)^{-1}[(\lambda - x)p - p(\lambda - x)](\lambda - x)^{-1}\| \\
&< \delta/d^2 + \delta/(d - \delta)d.
\end{aligned}$$

Therefore

$$\|f(pxp) - pf(x)p\| < \|f\| \cdot (\text{length}\,(\Gamma)) \cdot (1/2\pi)(1/d^2 + 1/(d-\delta)d)\delta.$$

$\square$

Remark 2.9. Lemma 2.8 (1) remains true if g is analytic, x and y are not assumed to be normal, but then δ may depend on x.

2.10. Let P be a polynomial of z and z^*. For any element x in a C^*-algebra A, we define $P(x)$ to be the corresponding linear combination of $x^n(x^m)^*$. Notice that we do not assume that $xx^* = x^*x$. In fact, we should view P as a linear combination of $z^n(z^*)^m$ (note that z^n appears before $(z^*)^m$). Similar to [**Ch**, 2], one can show that for any $\epsilon > 0$ and $b > 0$, there is a $\delta > 0$ such that

$$\|P(x) - P(y)\| < \epsilon$$

whenever $\|x - y\| < \delta$, $\|x\| \le b$ and $\|y\| \le b$.

Lemma 2.11. *Let f be a continuous function in $C(D)$ for some disk with the center at the origin and radius $r > 0$. For any $\epsilon > 0$, there exist $\delta > 0$ and a polynomial P of z and z^* such that for any C^*-algebra A and a normal element $x \in A$, if $\|x\| \le r, p$ is a projection in A with the property that* $\mathrm{sp}(pxp) \subset D$ *and*

$$\|px - xp\| < \delta,$$

then

$$\|pf(x)p - P(pxp)\| < \epsilon.$$

Proof. From the Stone-Weierstrass theorem, there is a polynomial P of z and z^* such that

$$\|f - P\|_D < \epsilon/2.$$

It is routine and standard that if δ is small enough,

$$\|pP(x)p - P(pxp)\| < \epsilon/2.$$

Therefore

$$\|pf(x)p - P(pxp)\| \le \|pf(x)p - pP(x)p\| + \|pP(x)p - P(pxp)\| < \epsilon.$$

$\square$

3. Approximation in $A \otimes \mathbf{K}$.

The main result in this section is Theorem 3.13. We first consider the case that the spectrum of x is a square. In this case, y and z are constructed by "cutting" the spectrum of x into small pieces (3.1, 3.3, 3.4). Then, by using conforming mappings, we can deal with the case that $\mathrm{sp}(x)$ is homeomorphic to a square (3.5). For the case that $\mathrm{sp}(x)$ is an annulus, the method is somewhat different (3.6, 3.7, 3.8 and 3.9). Then we deal with the case that $\mathrm{sp}(x)$ is a square with finitely many holes. This is done by "cutting" the spectrum into several pieces each of which is homeomorphic to an annulus. The general idea of "cutting" spectrum comes from [**BD**]. However, techniques used in this section come from other sources such as [**BDF**], [**GL**] and [**Ph2**].

Some of the statements in this section are somewhat complicated (two C^*-algebras A and B appear). It would be simpler if $B = A$ or $p = 1$.

The following lemma looks similar to Lemma 5.2 in [**BD**]. But 3.1 and its proof are inspired by [**BDF**, 7.4].

Lemma 3.1. *For any $\epsilon > 0$ and $\eta > 0$, there is $\delta > 0$ such that for any unital C^*-algebra A and $x \in A$ with $\|x\| \le 1$, if*

$$\mathrm{sp}(h_1) \subset [-b, c], \ \mathrm{sp}(h_2) \subset [-\alpha, \beta],$$

where $h_1 = \mathrm{Re}(x)$ and $h_2 = \mathrm{Im}(x)$, and b, c, α and β are positive, and

$$\|xx^* - x^*x\| < \delta,$$

then there is a projection $q \in M_2(A)$ such that

$$\|q(x \oplus ih_2) - (x \oplus ih_2)q\| < \epsilon$$

and

$$\begin{aligned} \mathrm{sp}(\mathrm{Re}[q(x \oplus ih_2)q]) &\subset [-\eta, c + \eta], \\ \mathrm{sp}(\mathrm{Im}[q(x \oplus ih_2)q]) &\subset [-\alpha, \beta], \\ \mathrm{sp}(\mathrm{Re}[(1-q)(x \oplus ih_2)(1-q)]) &\subset [-b - \eta, \eta] \end{aligned}$$

and

$$\mathrm{sp}(\mathrm{Im}[(1-q)(x \oplus ih_2)(1-q)]) \subset [-\alpha, \beta].$$

Furthermore, if x is normal and we denote by P_1 the spectral projection of $x \oplus ih_2$ (in A^{**}) corresponding to the subset

$$R_1 = \{\xi : \mathrm{Re}(\xi) \in [-\eta, c + \eta]\}$$

and by P_2 the spectral projection of $x \oplus ih_2$ (in A^{**}) corresponding to the subset

$$R_2 = \{\xi : \mathrm{Re}(\xi) \in [-b-\eta, \eta]\},$$

then $P_1 \geq q$ and $P_2 \geq (1-q)$.

Proof. Suppose that

$$\|xx^* - x^*x\| < \delta_1$$

where δ_1 is a positive number to be determined. Let δ_2 be another positive number. Set

$$f(t) = \begin{cases} 1 & \text{if } \delta_2 \leq t < \infty \\ 0 & \text{if } t \leq -\delta_2 \\ \text{linear} & \text{if } -\delta_2 < t \leq \delta_2 \end{cases}.$$

If $\delta_3 > 0$ is given, we can choose δ_1 small enough such that

$$\|f(h_1)h_2 - h_2 f(h_1)\| < \delta_3, \quad \text{and} \quad \|f(h_1)x - xf(h_1)\| < \delta_3.$$

We denote $a = f(h_1)$ and

$$q = \begin{pmatrix} a & (a(1-a))^{1/2} \\ (a(1-a))^{1/2} & 1-a \end{pmatrix}.$$

We have

$$\begin{aligned} &q(x \oplus ih_2) - (x \oplus ih_2)q \\ &= \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \end{aligned}$$

where

$$x_{11} = ax - xa, x_{12} = (a(1-a))^{1/2} ih_2 - ih_2 (a(1-a))^{1/2} - h_1(a(1-a))^{1/2},$$
$$x_{21} = (a(1-a))^{1/2} ih_2 - ih_2(a(1-a))^{1/2} + (a(1-a))^{1/2} h_1$$

and

$$x_{22} = (1-a)ih_2 - ih_2(1-a).$$

If δ_2 and δ_3 are small enough, by a direct computation, we obtain

$$\|q(x \oplus ih_2) - (x \oplus ih_2)q\| < \epsilon.$$

Let

$$\omega = \begin{pmatrix} 0 & ah_1(a(1-a))^{1/2} \\ ah_1(a(1-a))^{1/2} & (a(1-a))^{1/2} h_1 (a(1-a))^{1/2} \end{pmatrix}.$$

It is easy to see that

$$\left\|ah_1(a(1-a))^{1/2}\right\| < \delta_2 \quad \text{and} \quad \left\|(a(1-a))^{1/2}h_1(a(1-a))^{1/2}\right\| < \delta_2.$$

Therefore $\|\omega\| < 2\delta_2$. Notice that

$$\mathrm{Re}(q(x \oplus ih_2)q) = \begin{pmatrix} ah_1a & ah_1(a(1-a))^{1/2} \\ ah_1(a(1-a))^{1/2} & (a(1-a))^{1/2}h_1(a(1-a))^{1/2} \end{pmatrix}.$$

So

$$\mathrm{Re}(q(x \oplus ih_2)q) - (ah_1a \oplus 0) = \omega.$$

Since $ah_1 = h_1a$ and $h_1 \le c$, we have

$$\mathrm{Re}(q(x \oplus ih_2)q) \le c + 2\delta_2.$$

On the other hand, if $r < -\delta_2$, then

$$\|(r - ah_1a \oplus 0)^{-1}\| < 1/|r + \delta_2|$$

and

$$\begin{aligned} &\|1 - [r - (ah_1a \oplus 0)]^{-1}[r - \mathrm{Re}(q(x \oplus ih_2)q)]\| \\ &\le \left\|[r - ah_1a \oplus 0]^{-1}\omega\right\| < (1/|r + \delta_2|) \cdot 2\delta_2. \end{aligned}$$

For $\eta > 0$, if δ_2 is small enough, whenever

$$r < -\eta(< -\delta_2/2),$$

$r - \mathrm{Re}(q(x \oplus ih_2)q)$ is invertible. Therefore

$$\mathrm{sp}(\mathrm{Re}[q(x \oplus ih_2)q]) \subset [-\eta, c + \eta].$$

Similarly,

$$\mathrm{sp}(\mathrm{Re}[(1-q)(x \oplus ih_2)(1-q)]) \subset [-b - \eta, \eta].$$

Moreover, since $\mathrm{Im}[q(x \oplus ih_2)q] = q(h_2 \oplus h_2)q$,

$$\mathrm{sp}(\mathrm{Im}[q(x \oplus ih_2)q]) \subset [-\alpha, \beta].$$

Similarly,

$$\mathrm{sp}(\mathrm{Im}[(1-q)(x \oplus ih_2)(1-q)]) \subset [-\alpha, \beta].$$

Finally, if x is normal, and if $\delta_2 < \eta$, one sees easily that

$$P_1 \geq q \quad \text{and} \quad P_2 \geq (1-q).$$

□

In Lemma 3.1, we also have

$$\|x \oplus ih_2 - [q(x \oplus ih_2)q + (1-q)(x \oplus ih_2)(1-q)]\| < \epsilon/2.$$

Lemma 3.1 allows us to "cut" the spectrum of x into two vertical strips. But in order to do that, we have to add to x a normal element with spectrum contained in a vertical line. In the following corollary, by repeating this, we can "cut" the spectrum of x into k vertical strips. But for each cut, we have to add a normal element with spectrum contained in a vertical line. In 3.2, we only do this to a normal element.

Corollary 3.2. *Let*

$$X = \{\lambda : |\operatorname{Re}\lambda| \leq 1, |\operatorname{Im}\lambda| \leq b\},$$

where $b > 0$. For any $\epsilon > 0$ and $\eta > 0$, there is $\delta > 0$, for any unital C^*-algebra B and a normal element $x \in B$ with

$$\|\operatorname{Re}(x)\| \leq 1 \quad \text{and} \quad \|\operatorname{Im}(x)\| \leq b,$$

if

$$\|px - xp\| < \delta,$$

and if

$$-1 = t_0 < t_1 < t_2 < \ldots < t_k = 1,$$

then there are mutually orthogonal normal elements $y_1, y_2, \ldots, y_{k-1} \in M_{2^{k-1}-1}(pBp)$, mutually orthogonal projections $p_1, p_2, \ldots, p_k \in M_{2^{k-1}}(pBp)$ and normal elements $z_1, z_2, \ldots, z_k \in M_{2^{k-1}}(B)$ with $\operatorname{sp}(z_j) \subset X$ satisfying

$$\begin{aligned} \operatorname{sp}(y_j) &\subset \{t_j + ia : |a| \leq b\}, j = 1, 2, \ldots, k-1, \\ \operatorname{sp}(p_j z_j p_j) &\subset X_\eta \cap R_j, \end{aligned}$$

such that

$$\|p_j z_j - z_j p_j\| < \epsilon$$

and

$$\|x \oplus y_1 \oplus \cdots \oplus y_{k-1} - p_1 z_1 p_1 \oplus p_2 z_2 p_2 \oplus \cdots \oplus p_k z_k p_k\| < \epsilon,$$

where

$$X_\eta = \{\lambda : \mathrm{dist}(\lambda, X) < \eta\}$$

and

$$R_j = \{\lambda : t_{j-1} - \eta/2 \le \mathrm{Re}\,\lambda \le t_j + \eta/2, |\,\mathrm{Im}\,\lambda| \le b + \eta/2\}.$$

Furthermore,

(1) *if we denote by P_j the spectral projection of z_j (in B^{**}) corresponding to $X_\eta \cap R_j$, then $P_j \ge p_j$.*

(2) *if A is a C^*-subalgebra of B, $p \in A$, $pf(x)$, $f(x)p \in A$ for any $f \in C(\mathrm{sp}(x))$, then we can have $y_i \in M_{2^{k-1}-1}(pAp)$ and $p_j g(z_j)$, $g(z_j)p \in M_{2^{k-1}}(pAp)$ for any $g \in C(\mathrm{sp}(z_j))$.*

Proof. The proof is a repeated application of Lemma 3.1. Notice that, for any element $z \in A$ and any projection $q \in A$, $\mathrm{Im}(qzq) = q(\mathrm{Im}\,z)q$. We also notice that it follows from [**Lin6**, Lemma 4] that if δ is small enough then $\mathrm{sp}(q(x \oplus ih_2)q) \subset X_\eta$ and $\mathrm{sp}((1-q)(x \oplus ih_2)(1-q)) \subset X_\eta$. The normal element z_j is a direct sum of x with a normal element with spectrum contained in finitely many vertical line segments. □

Now we will "cut" the spectrum of x vertically as well as horizontally. So the spectrum of x is cut into small pieces. Note that for each cut, we have to add a normal element with spectrum contained in a line segment. Note also that 3.2 is not used in the proof of 3.3.

Lemma 3.3. *Let*

$$X = \{z : |\,\mathrm{Re}\,z| \le 1/2, |\,\mathrm{Im}\,z| \le 1/2\}.$$

For any $\epsilon > 0$, there exist $\delta > 0$ and an integer k, for any unital C^*-algebra A and an element $x \in A$, if

$$\|\,\mathrm{Re}(x)\| \le 1/2, \ \ \|\,\mathrm{Im}(x)\| \le 1/2$$

and

$$\|xx^* - x^*x\| < \delta,$$

there are mutually orthogonal normal elements $y_1, y_2, ..., y_m \in M_k(A)$ with $\mathrm{sp}(y_i)$ contained in a straight line segment in X and a normal element $z \in M_{k+1}(A)$ with finite spectrum $\mathrm{sp}(z) \subset X$ such that

$$\|x \oplus y_1 \oplus y_2 \oplus \cdots \oplus y_m - z\| < \epsilon.$$

Proof. We will apply Lemma 3.1 repeatedly. Let ϵ_1 and η_1 be two positive numbers. For $\epsilon = \epsilon_1$ and $\eta = \eta_1$, let $\delta = \delta_1 > 0$ be the number in Lemma 3.1. We will use the notation in Lemma 3.1. Set

$$z_1 = i[q(x \oplus ih_2)q - 1/4], \quad \text{and} \quad z_2 = i[(1-q)(x \oplus ih_2)(1-q) + 1/4].$$

We have

$$\|z_j z_j^* - z_j^* z_j\| < 2\epsilon_1 + \delta_1,$$
$$\|\operatorname{Re} z_j\| \leq 1/2 \quad \text{and} \quad \|\operatorname{Im} z_j\| \leq 1/4 + \eta_1,$$

$j = 1,2$. Let ϵ_2 and η_2 be positive numbers. If ϵ_1, η_1 and δ_1 are small enough, by applying Lemma 3.1 again, there are projections q_1, q_2, p_1, p_2 such that $q_1 \leq q$, $q_2 \leq (1-q)$, $p_1 = q - q_1$ and $p_2 = (1-q) - q_2$, and

$$\begin{aligned}
\|q_j(z_j \oplus i \operatorname{Im} z_j) - (z_j \oplus i \operatorname{Im} z_j) q_j\| &< \epsilon_2, \\
\operatorname{sp}[\operatorname{Re}(q_j(z_j \oplus i \operatorname{Im} z_j) q_j)] &\subset [-\eta_2, 1/2 + \eta_2], \\
\operatorname{sp}[\operatorname{Re}(p_j(z_j \oplus i \operatorname{Im} z_j) p_j)] &\subset [-1/2 - \eta_2, \eta_2], \\
\operatorname{sp}[\operatorname{Im}(q_j(z_j \oplus i \operatorname{Im} z_j) q_j)] &\subset [-1/4 - \eta_2, 1/4 + \eta_2], \\
\operatorname{sp}(\operatorname{Im}(p_j(z_j \oplus i \operatorname{Im} z_j) p_j)] &\subset [-1/4 - \eta_2, 1/4 + \eta_2],
\end{aligned}$$

$j = 1,2$. Denote $x_j = (1/i) q_j (z_j \oplus i \operatorname{Im} z_j) q_j + (-1)^{1+j} 1/4$, $j = 1,2$ and $x_{2+j} = (1/i) p_j (z_j \oplus i \operatorname{Im} z_j) p_j + (-1)^{1+j} 1/4$, $j = 1,2$. Then

$$\left\| x \oplus i h_2 \oplus i \operatorname{Im}[(1/i) z_1 + 1/4] \oplus i \operatorname{Im}[(1/i) z_2 - 1/4] - \sum_{k=1}^{4} x_k \right\| < 2\epsilon_1 + 2\epsilon_2.$$

Notice that there are complex number $\alpha_j = (1/2)(1/2)^{1/2} e^{i(j\pi/4)}$ such that

$$\|x_j - \alpha_j q_j\| < 2(1/4 + \eta_2)$$

and

$$\|x_{2+j} - \alpha_{2+j} p_j\| < 2(1/4 + \eta_2).$$

By continuing to apply 3.1, one sees that for any $\epsilon > 0$, there is an integer k and $\delta > 0$, if

$$\|x^* x - x x^*\| < \delta,$$

there are mutually orthogonal normal elements $y_1, y_2, \ldots, y_m \in M_k(A)$ with $\operatorname{sp}(y_i)$ contained in a straight line segment in X and mutually orthogonal projections $e_1, e_2, \ldots, e_{k+1} \in M_{k+1}(A)$ and complex numbers $\lambda_1, \lambda_2, \ldots, \lambda_{k+1} \in X$ such that

$$\left\| x \oplus y_1 \oplus y_2 \oplus \ldots \oplus y_m - \sum_{j=1}^{k+1} \lambda_j e_j \right\| < \epsilon.$$

□

Corollary 3.4. *Let*

$$X = \{\lambda : |\operatorname{Re} \lambda| \leq 1/2, |\operatorname{Im} \lambda| \leq 1/2\}.$$

For any $\epsilon > 0$, there exist $\delta > 0$ and an integer k, for any unital C^*-algebra A of real rank zero and an element $x \in A$ with

$$\|\operatorname{Re} x\| \leq 1/2 \text{ and } \|\operatorname{Im} x\| \leq 1/2,$$

if

$$\|x^*x - xx^*\| < \delta,$$

there are normal element $y \in M_k(A)$ and normal element $z \in M_{k+1}(A)$ with finite spectra $\operatorname{sp}(y), \operatorname{sp}(z) \subset X$ and

$$\|x \oplus y - z\| < \epsilon.$$

Proof. Since A has real rank zero and a normal element with spectrum contained in a line segment has the form $\alpha a + \beta$, where a is a selfadjoint element in A and α, β are complex numbers, the y_is in Lemma 3.3 are approximated (in norm) by normal elements in $M_k(A)$ with finite spectra. □

Lemma 3.5. *Let X be a compact subset of the plane which is homeomorphic to the unit disk. For any $\epsilon > 0$, there exist $\delta > 0$ and an integer k, if x is a normal element with $\operatorname{sp}(x) \subset X$ in a C^*-algebra B and if p is a nonzero projection in a C^*-subalgebra A of B with real rank zero, such that $pf(x)$, $f(x)p \in A$ for any $f \in C(\operatorname{sp}(x))$,*

$$\|px - xp\| < \delta \text{ and } \operatorname{sp}(pxp) \subset X,$$

then there are normal elements $y \in M_k(pAp)$ and $z \in M_{k+1}(pAp)$ with finite spectra $\operatorname{sp}(y), \operatorname{sp}(z) \subset X$ such that

$$\|x \oplus y - z\| < \epsilon.$$

Proof. Set

$$S = \{\lambda : |\operatorname{Re} \lambda| < 1/2, |\operatorname{Im} \lambda| < 1/2\}.$$

For any $\epsilon > 0$, let X_ϵ be a region such that

$$X \subset X_\epsilon, \;\; X_\epsilon \subset \{\xi : \operatorname{dist}(\xi, X) < \epsilon/2\}$$

and there is a conformal mapping f from X_ϵ onto S. There is $\eta > 0$ such that

$$f(X) \subset \{\xi : |\operatorname{Re}(\xi)| < 1/2 - \eta, \; |\operatorname{Im}(\xi) < 1/2 - \eta\}.$$

Choose $\delta > 0$ such that

$$\|pf(x)p - f(pxp)\| < \eta/4$$

(see 2.8). Since $\mathrm{Re}(pf(x)p) = p\,\mathrm{Re}(f(x))p$, we may also assume that

$$\|\mathrm{Re}(f(pxp)) - p\,\mathrm{Re}(f(x))p\| < \eta/4.$$

Since x is normal, we have

$$\|\mathrm{Re}(f(pxp))\| \le \|p\,\mathrm{Re}(f(x))p\| + \eta/4 \le \|\mathrm{Re}(f(x))\| + \eta \le \|\mathrm{Re}(f)\|_X + \eta.$$

Similarly, we have

$$\|\mathrm{Im}(f(pxp))\| \le \|\mathrm{Im}(f)\|_X + \eta/4.$$

By 3.4, for any $\sigma > 0$, if δ is small enough, there exist an integer k (which does not depend on A, B or x but does depend on X and ϵ), and normal elements $y_1 \in M_k(pAp)$ and $z_1 \in M_{k+1}(pAp)$ with finite spectra $\mathrm{sp}(y_1)$, $\mathrm{sp}(z_1) \subset S$ such that

$$\|f(pxp) \oplus y_1 - z_1\| < \sigma.$$

By 2.8, if σ is small enough,

$$\left\|pxp \oplus f^{-1}(y_1) - f^{-1}(z_1)\right\| < \epsilon/2.$$

Since $\mathrm{sp}(f^{-1}(y_1))$ $\mathrm{sp}(f^{-1}(z_1)) \in X_\epsilon$, by changing the spectrum of $f^{-1}(y_1)$ and $f^{-1}(z_1)$ slightly (within $\epsilon/2$), there are normal elements y and z as required. □

The following lemma is inspired by [**Ph2**].

Lemma 3.6. *For any $\epsilon > 0$ there exist $\delta > 0$ and $d > 0$, for any unital C^*-algebra A and $x \in A$ with the polar decomposition $x = uh$ such that $0 < a \le h \le 1$ (so u is a unitary), if*

$$\|uh - hu\| < \delta,$$

then exists $y \in M_2(A)$ with

$$\mathrm{sp}(y) \subset \{re^{i\theta} : a \le r \le 1,\ -\pi + d/2 < \theta < \pi - d/2\}$$

such that

$$\|x \oplus x^* - y\| < \epsilon.$$

Proof. Let

$$u(\alpha) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} u^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

If $|\alpha| < \pi/2$, by [**Ph2**, 5], $-1 \notin \mathrm{sp}(u(\alpha))$. Let θ be the largest number in $[0, \pi]$ such that $e^{i\theta} \in \mathrm{sp}(u(\alpha))$ and let $d = \pi - \theta$. It is easy to see that

$$\|u(\alpha)(h \oplus h) - x \oplus x^*\| < \epsilon,$$

if α is close to $\pi/2$.
One can verify that

$$\|u(\alpha)(h \oplus h) - (h \oplus h)u(\alpha)\| < 2\delta.$$

Let $\lambda = re^{i\beta}$, where $\beta \notin [-\pi + d/2, \pi - d/2]$, then

$$\begin{aligned}
&[\lambda - u(\alpha)(h \oplus h)]^*[\lambda - u(\alpha)(h \oplus h)] \\
&= r^2 + (h \oplus h)^2 - re^{i\beta}(h \oplus h)u(\alpha)^* - re^{-i\beta}u(\alpha)h(\oplus h) \\
&> r^2 + (h \oplus h)^2 - r(h \oplus h)^{1/2}(e^{i\beta}u(\alpha)^* + e^{-i\beta}u(\alpha))(h \oplus h)^{1/2} - 2r\delta_1 \\
&\geq r^2 + (h \oplus h)^2 - 2r(h \oplus h)\cos(d/2) - 2r\delta_1, \\
&\geq 2r(h \oplus h)[1 - \cos(d/2)] + [r - (h \oplus h)]^2 - 2r\delta_1 \\
&\geq 2ra[1 - \cos(d/2)] - 2r\delta_1,
\end{aligned}$$

where $\delta_1 = \|(h \oplus h)^{1/2}u(\alpha) - u(\alpha)(h \oplus h)^{1/2}\|$. Therefore if δ is small enough,

$$[\lambda - u(\alpha)^*(h \oplus h)][\lambda - u(\alpha)(h \oplus h)]$$

is invertible. Similarly,

$$[\lambda - u(\alpha)(h \oplus h)][\lambda - u(\alpha)(h \oplus h)]^*$$

is invertible. Hence $\lambda \notin \mathrm{sp}((\alpha)(h \oplus h))$. It is clear that if $|\lambda| > 1$, then $\lambda \notin \mathrm{sp}(u(\alpha)(h \oplus h))$. Note $u(\alpha)$ is a unitary. Since $h \oplus h \geq a$, if $|\lambda| < a$, $\lambda \notin \mathrm{sp}(u(\alpha)(h \oplus h))$. □

Corollary 3.7. *Let x be a normal element in a (unital) C^*-algebra B with polar decomposition $x = uh$ such that $0 < a \leq h \leq b$ for some positive numbers a and b (so u is a unitary). For any $\epsilon > 0$, there exist $d > 0$ and a normal element $y \in M_2(B)$ with*

$$\mathrm{sp}(y) \subset \{re^{i\theta} : a \leq r \leq b, -\pi + d/2 \leq \theta \leq \pi - d/2\}$$

such that

$$\|x \oplus x^* - y\| < \epsilon.$$

Furthermore, for any $\eta > 0$, there exists $\delta > 0$ such that if there is a nonzero projection $p \in A$ with

$$\|px - xp\| < \delta, \ \ pf(x), f(x)p \in A,$$

where A is a C^*-subalgebra of B containing the identity of B, then

$$\|(p\oplus p)y - y(p\oplus p)\| < \eta \quad \text{and} \quad (p\oplus p)g(y),\ g(y)(p\oplus p) \in M_2(A)$$

for any $g \in C(\mathrm{sp}(y))$.

Proof. Note that in Lemma 3.6, if x is normal, $u(\alpha)$ commutes with $h\oplus h$ for every α and every $u(\alpha)$ is untary. So the element $y = u(\alpha)(h\oplus h)$ is normal. Note that $h = (x^*x)^{1/2}$ and $u = xh^{-1}$. So $h = f_1(x), u = f_2(x)$ for some $f_1, f_2 \in C(\mathrm{sp}(x))$. Hence, $pu, up, ph, hp \in A$. A direct computation shows that $(p\oplus p)f(y), f(y)(p\oplus p) \in M_2(A)$ for any $f \in C(\mathrm{sp}(y))$. Furthermore, for any η, since x is normal, there is $\delta > 0$ such that if

$$\|px - xp\| < \delta,$$

then

$$\|pu - up\| < \eta/2 \quad \text{and} \quad \|ph - hp\| < \eta/2.$$

So

$$\|(p\oplus p)u(\alpha) - u(\alpha)(p\oplus p)\| < \eta/2$$

and

$$\|(p\oplus p)(h\oplus h) - (h\oplus h)(p\oplus p)\| < \eta/2.$$

We may take

$$y = u(\alpha)(h\oplus h)$$

for some α. □

Lemma 3.8. *Let*

$$X = \{re^{i\theta} : 0 < a \le r \le 1, -\pi \le \theta \le \pi\}.$$

For any $\epsilon > 0$, there exist $\delta > 0$ and an integer k, for any unital C^*-algebra B, any C^*-subalgebra A of B with real rank zero containing the identity of B, and a normal element $x \in B$ with the polar decomposition $x = uh$ such that $0 < a \le h \le \|x\|$ (so u is a unitary), if $p \in A$ is a projection such that

$$\|px - xp\| < \delta,\ \ \mathrm{sp}(pxp) \subset X \ \ \text{and} \ \ pf(x)\, f(x)p \in A$$

for any $f \in C(\mathrm{sp}(x))$, then there are normal elements $y \in M_k(pAp)$ and $z \in M_{k+2}(pAp)$ with finite spectra contained in X such that

$$\|pxp \oplus px^*p \oplus y - z\| < \epsilon.$$

Proof. For any $\epsilon > 0$ and $\eta > 0$, by 3.7, if δ is small enough, there exist $d > 0$ and a normal element $y_1 \in M_2(B)$ with

$$\mathrm{sp}(y_1) \subset \{re^{i\theta} : a \le r \le b, -\pi + d/2 \le \theta \le \pi + d/2\}$$

such that

$$\|x \oplus x^* - y_1\| < \epsilon/3, \ \ \|(p \oplus p)y_1 - y_1(p \oplus p)\| < \eta$$

and $p \oplus pf(y_1)$, $f(y_1)(p \oplus p) \in M_2(A)$.

Set $q = p \oplus p$ and set

$$Y = \{re^{i\theta} : a - \epsilon/3 \le r \le b + \epsilon/3 \text{ and } -\pi + d/4 \le \theta \le \pi - d/4\}.$$

Then, if η is small enough (so δ has to be small enough),

$$\mathrm{sp}(qy_1q) \subset Y.$$

Notice that Y is homeomorphic to the unit disk. By Lemma 3.7 and Lemma 3.5, if η is small enough, there are normal elements $y' \in M_k(qAq)$ and $z' \in M_{k+1}(qAq)$ with finite spectra $\mathrm{sp}(y), \mathrm{sp}(z) \subset Y$ such that

$$\|qy_1q \oplus y' - z'\| < \epsilon/3$$

for some positive integer k. Therefore

$$\|pxp \oplus px^*p \oplus y' - z'\| < 2\epsilon/3.$$

By changing the spectrum of y' and z' slightly (within $\epsilon/3$), we have

$$\|pxp \oplus px^*p \oplus y - z\| < \epsilon,$$

where $\mathrm{sp}(y), \mathrm{sp}(z) \subset X$, $y \in M_k(pAp)$ and $z \in M_{k+1}(pAp)$. □

The idea of using a path of unitaries in the following proof is taken from [**Ph2**].

Lemma 3.9. *For any $\epsilon > 0$, there exist $\delta > 0$ and an integer k, for any unital C^*-algebra B, any C^*-subalgebra A of B with real rank zero containing the identity of B and a normal element $x \in B$ with the polar decomposition (in B) $x = uh$, where $0 < a \le h \le 1$ (a and b are positive numbers) and u is a unitary, and if p is a projection in A such that*

$$\|px - xp\| < \delta, \ pxp \in \mathrm{Inv}_0(pAp) \ \textit{and} \ pf(x), \ f(x)p \in A$$

for any $f \in C(\mathrm{sp}(x))$, *and*

$$\mathrm{sp}(pxp) \subset X = \{re^{i\theta} : a \le r \le 1, -\pi \le \theta \le \pi\},$$

then there are normal elements $y \in M_k(pAp)$ *and* $z \in M_{k+1}(pAp)$ *with finite spectra contained in* X *such that*

$$\|pxp \oplus y - z\| < \epsilon.$$

Proof. Let ϵ be a positive number. For $\epsilon/16$, let δ and k be the numbers in Lemma 3.6. Notice that δ and k does not depend on x or the C^*-algebra A. We may assume that $\delta < \epsilon/8$. Set

$$Y = \{z : a - \epsilon/4 \le |z| \le 1 + \epsilon/4\}.$$

Let $x = uh$ be the polar decomposition. If

$$\|px - xp\| < \delta_0 < \delta,$$

then

$$\|pu - up\| \quad \text{and} \quad \|ph - hp\|$$

are small. We assume that

$$\|pup|pup|^{-1} - pup\| < \epsilon/4,$$
$$\|pu[(1-t)h + t] - u[(1-t)h + t]p\| < \delta.$$

Furthermore, by Lemma 3 in [**Lin6**], we may assume that $\mathrm{sp}(px_tp) \subset Y$ (notice that $(1-t)a + t \le (1-t)h + t$), where $0 \le t \le 1, x_t = u[(1-t)h + t]$. Notice that δ_0 can be chosen such that it does not depend on x or the C^*-algebra A (but depends on ϵ, a and b). Notice also that, if δ_0 is small enough, $v = pup|pup|^{-1} \in U_0(pAp)$. Suppose that $\{v(t),\ 0 \le t \le 1\}$ is a path of unitaries in pAp such that $v(0) = v$ and $v(1) = 1$. Set

$$x(t) = \begin{cases} u[(1-t)h + t] & \text{if } 0 \le t \le 1 \\ v(t-1) & \text{if } 1 \le t \le 2 \end{cases}.$$

So $x(0) = x$ and $x(2) = 1$. For any $\epsilon > 0$, let $x_0 = x$ and $x_i = x(t_i)$, $i = 1, 2, ..., m+1$ such that $0 < t_i \le 1$, $i = 1, 2, ..., n$, $1 < t_i \le 2$, $i = n+1, n+2, ..., m$, $t_{m+1} = 1$,

$$\|x_i - x_{i-1}\| < \epsilon/4, \ i = 1, 2, ..., n \quad \text{and} \quad i = n+2, ..., m+1$$

and

$$\|px_np - px_{n+1}p\| < \epsilon/4.$$

From $pf(u), f(u)p, pg(h), g(h)p \in A$ for any $f \in \mathrm{sp}(\mathrm{sp}(u))$ and $g \in C(\mathrm{sp}(h))$, it is easy to see that $pf(x_i), f(x_i)p \in A$ for any $f \in C(\mathrm{sp}(x_i))$. Notice that from [**Lin5**], $\mathrm{cer}(A) \le 1+\epsilon$. Therefore, the parth $\{v(t)\}$ can be chosen such that the integer m depends on ϵ only. Set

$$y_1 = \sum \oplus_{i=1}^m px_ip \oplus \sum \oplus_{i=1}^m px_i^*p.$$

By 3.8, there are normal elements $y_2 \in M_{km}(pAp)$ and $z_1 \in M_{(k+1)m}(pAp)$ with finite spectra contained in Y such that

$$\|y_1 \oplus y_2 - z_1\| < \epsilon/4.$$

Let

$$y_0 = pxp \oplus y_1 \oplus p, \quad \text{and} \quad y_0' = \sum \oplus_{i=0}^m px_ip \oplus \sum \oplus_{i=0}^m px_i^*p.$$

Then

$$\|y_0 - y_0'\| < \epsilon/4.$$

By 3.8 again, there are normal elements $y_3 \in M_{k(m+1)}(pAp)$ and $z_2 \in M_{(k+1)(m+1)}(pAp)$ with finite spectra contained in Y such that

$$\|y_0' \oplus y_3 - z_2\| < \epsilon/4.$$

Then it is easy to see that there is a unitary $U \in M_{2+(k+1)(m+2)}(pAp)$ such that

$$\|y \oplus p \oplus z_1 \oplus y_3 - U^*(y_2 \oplus z_2)U\| < 3\epsilon/4.$$

By changing the spectrum of y_2, y_3, z_1 and z_2 slightly (within $\epsilon/4$), we may assume that $\mathrm{sp}(y_2), \mathrm{sp}(y_3), \mathrm{sp}(z_1), \mathrm{sp}(z_2) \subset X$. □

3.10. For the convenience, we will consider a special compact subset of the plane which is a square with finitely many holes. Let

$$X' = \{z : 0 \le \mathrm{Re}\, z \le 1, |\mathrm{Im}\, z| \le b\} \setminus \cup_{j=1}^k \{z : |(2j-1)/2k - z| < r\},$$

$0 < r < \min(b, 1/4k)$, and $X = \{z - 1/2 : z \in X'\}$. So X is a square with k holes. Let $d = [b^2 + (1/2k)^2]^{1/2} < 3/4(1/k - 2r)$ and

$$Y' = \cup_{j=1}^k \{z : r/2 \le |(2j-1)/2k - z| \le 4/3d\}.$$

Define $Y = \{z - 1/2 : z \in Y'\}$. Y is a union of finitely many annuli but holes are disjoint. There is a retraction $f : Y \to X$.

Lemma 3.11. *Let Ω be a compact subset of the plane which is homeomorphic to the subset X described in* 3.10. *For any $\epsilon > 0$, there exist $\delta > 0$ and* *an integer L with the property that, for any normal element x in a unital C^*-algebra B, if* $\mathrm{sp}(x) \subset \Omega$ *and p is a projection in a C^*-subalgebra A of B with real rank zero containing the identity of B such that*

$$\|px - xp\| < \delta, \ \ pf(x),\ f(x)p \in A$$

for any $f \in C(\mathrm{sp}(x))$, and

$$\lambda p - pxp \in \mathrm{Inv}_0(pAp)$$

for $\lambda \notin X$, then there are normal elements $y \in M_L(pAp)$ and $z \in M_{L+1}(pAp)$ with finite spectrum $\mathrm{sp}(y), \mathrm{sp}(z) \subset \Omega$ *such that*

$$\|pxp \oplus y - z\| < \epsilon.$$

Proof. We first claim that it suffices to show the case that $\Omega = X$.

There is a homeomorphism $f : \Omega \to X$. For any $\sigma > 0$, if δ is small enough,

$$\|pf(x) - f(x)p\| < \sigma.$$

Suppose that 3.11 holds for $\Omega = X$. Then, for any $\eta > 0$, if σ is small enough, we have y and z as in the conclusion of the lemma but with inequality

$$\|pf(x)p \oplus y - z\| < \eta.$$

There exists a continuous function $\tilde{f} : D \to \mathbf{C}$ such that D is a disk with the center at the origin containing X and $\tilde{f}|_X = f^{-1}$. By 2.11, there is a polynomial P (or rather a linear combination of $z^n(z^*)^m$) of z and z^*, such that

$$\left\|p\tilde{f}(f(x))p - P(pf(x)p)\right\| < \epsilon/4,$$

if σ is small enough. We may further assume that

$$\left\|\tilde{f} - P\right\|_D < \epsilon/4.$$

Hence

$$\left\|\tilde{f}(y) - P(y)\right\| < \epsilon/4 \quad \text{and} \quad \left\|\tilde{f}(z) - P(z)\right\| < \epsilon/4.$$

Note also that $\tilde{f}(f(x)) = x$. Thus, by 2.10, if σ is small enough, we have

$$\|P(pxp) \oplus P(y) - P(z)\| < \epsilon/4.$$

Therefore

$$\begin{aligned} &\left\|pxp \oplus \tilde{f}(y) - \tilde{f}(z)\right\| \\ &\le \|P(pxp) \oplus P(y) - P(z)\| + \epsilon/4 + \epsilon/4 + \epsilon/4 < \epsilon. \end{aligned}$$

Note $\mathrm{sp}\left(\tilde{f}(y)\right)$, $\mathrm{sp}\left(\tilde{f}(z)\right) \subset \Omega$ and $f, \tilde{f}$ and P depend only on Ω and X. This completes the proof of the claim.

For the rest of the proof, we assume that X and Y are as in 3.10.

Set
$$t_i = -1/2 + i/k, i = 0, 1, 2, ..., k.$$

For any $r/4 > \eta > 0$, applying 3.2 (if $\|px - xp\| < \delta$), we obtain normal elements $y_1, y_2, ..., y_{k-1} \in M_{2^{k-1}-1}(pAp)$ with

$$\mathrm{sp}(y_j) \subset \{t_j + is : -b \le s \le b\},$$

mutually orthognal projections $p_1, p_2, ..., p_k \in M_{2^{k-1}}(pAp)$ and normal elements $x_1, x_2, ..., x_k \in M_{2^{k-1}}(B)$ such that $p_j f(x_j), f(x_j)p_j \in M_{2^{k-1}}(A)$ for any $f \in C(\mathrm{sp}(x_j))$,

$$\mathrm{sp}(x_j) \subset X, \quad \|p_j x_j - x_j p_j\| < \eta/4$$

$$\mathrm{sp}(p_j x_j p_j) \subset X_\eta \cap R_j, \mathrm{sp}(\mathrm{Re}(p_j x_j p_j)) \subset \{\lambda : t_{j-1} - \eta/4 \le \mathrm{Re}\,\lambda \le t_j + \eta/4\}$$

and
$$\|pxp \oplus y_1 \oplus \cdots \oplus y_{k-1} - p_1 x_1 p_1 \oplus \cdots \oplus p_k x_k p_k\| < \eta/16,$$

where
$$X_\eta = \{\lambda : \mathrm{dist}(\lambda, X) < \eta/4\}$$

and

$$R_j = \{\lambda : t_{j-1} - \eta/4 \le \mathrm{Re}\,\lambda \le t_j + \eta/4, -b - \eta/4 \le \mathrm{Im}\,\lambda \le b + \eta/4\}.$$

Since A has real rank zero, we may assume that each y_j has finite spectrum. Furthermore, by Lemma 3 in [**Lin6**], we also have

$$\|(\lambda p_j - p_j x_j p_j)^{-1}\| < 1/\mathrm{dist}(\lambda, X_\eta).$$

If $\delta < \eta/16$, from Lemma 3 in [**Lin6**] and 2.7,

$$\lambda q - p_1 x_1 p_1 \oplus p_2 x_2 p_2 \oplus \cdots \oplus p_k x_k p_x \in \mathrm{Inv}_0(M_{2^{k-1}}(pAp)),$$

where $\lambda \notin X_\eta$ and q is the identity of $M_{2^{k-1}}(pAp)$. Let

$$z_j = p_j x_j p_j - [(2j-1)/2k - 1/2]p_j.$$

Then, if η is small enough, $\|z_j\| \le (4/3)d$. Set

$$Y_j = \{\lambda : r/2 \le |\lambda - [(2j-1)/2k - 1/2]| \le (4/3)d\}.$$

By 2.6, if $i \neq j$ and

$$\lambda \in \{\lambda : |\lambda - (2j-1)/2k - 1/2| < (4/3)d\},$$

then

$$\lambda p_i - p_i x_i p_i \in \mathrm{Inv}_0(p_i M_{2^{k-1}}(pAp)p_i).$$

Since

$$\lambda q - p_1 x_1 p_1 \oplus \cdots \oplus p_k x_k p_x \in \mathrm{Inv}_0(M_{2^{k-1}}(pAp)),$$

by 2.3,

$$\lambda p_j - p_j x_j p_j \in \mathrm{Inv}_0(p_j A p_j),$$

if $\lambda \notin Y_j$. Set

$$Y_0 = \{\lambda : r/2 \leq |\lambda| \leq (4/3)d\}.$$

Then

$$\lambda p_j - z_j \in \mathrm{Inv}_0(M_{2^{k-1}}(pAp)),$$

if $\lambda \notin Y_0$. Denote by P_j the spectral projection of $x_j' = x_j - [(2j-1)/2k - 1/2]$ (in B^{**}) corresponding to the subset Y_0. By 3.2, we have $P_j \geq p_j$. So $P_j x_j'$ is a normal element in B^{**} with $\mathrm{sp}(P_j x_j') \subset Y_0$ and $z_j = p_j P_j x_j' p_j$. Furthermore, $p_j f(P_j x_j') = p_j P_j f(x_j') = p_j f(x_j') \in M_{2^{k-1}}(A), f(x_j')p_j \in M_{2^{k-1}}(A)$ for all continuous functions $f \in C(Y_0)$. We are now ready to apply Lemma 3.9. If δ is small enough, there are normal elements $y_j' \in M_L(p_j M_{2^{k-1}}(A)p_j)$ and normal elements $y_j'' \in M_{L+1}(p_j M_{2^{k-1}}(A)p_j)$ with finite spectra contained in Y_j such that

$$\|p_j x_j p_j \oplus y_j - y_j'\| < \eta/4,$$

for some integer L ($p_j x_j p_j = z_j + [(2j-1)/2k - 1/2]p_j$). This implies that

$$\left\| pxp \oplus \sum \oplus_{j=1}^{k-1} y_j \oplus \sum \oplus_{j=1}^{k-1} y_j' - \sum \oplus_{j=1}^{k} y_j'' \right\| < \epsilon/2.$$

Finally, set

$$y = \sum \oplus_{j=1}^{k-1} y_j \oplus \sum \oplus_{j=1}^{k-1} y_j'$$

and

$$z = \sum \oplus_{j=1}^{k} y_j''.$$

Clearly, $\mathrm{sp}(y), \mathrm{sp}(z), X \subset Y$. There is also an integer L (which depends only on ϵ) such that $y \in M_L(pAp)$ and $z \in M_{L+1}(pAp)$. Moreover,

$$\|pxp \oplus y - z\| < \eta.$$

There is a retraction $R : Y \to X$. Let $R' : D \to \mathbf{C}$ be a continuous function such that D is a disk with center at the origin containing Y and $R'|_Y = R$ There is a polynomial P of t, t^* (or rather a linear combination of $t^n(t^*)^m$, see 2.10) such that

$$\|P - R'\|_D < \epsilon/4.$$

By 2.10, if η is small enough,

$$\|P(pxp) \oplus P(y) - P(z)\| < \epsilon/4.$$

By 2.11, if δ is small enough,

$$\|P(pxp) - pR'(x)p\| < \epsilon/4.$$

Note $R'(x) = x$. Therefore we have

$$\|pxp \oplus R'(y) - R'(z)\| < \epsilon.$$

Since $R'|_Y = R$, $\mathrm{sp}(R'(y))$, $\mathrm{sp}(R'(z)) \subset X$. □

Lemma 3.12. *Let Ω be a compact subset of the plane. For any $\epsilon > 0$, there exist $\delta > 0$ and an integer L such that for any (unital) C^*-algebra A of real rank zero and a normal element x in a C^*-algebra $B \supset A$ with $\mathrm{sp}(x) \subset \Omega$, if $p \in A$ is a projection and if $pf(x)$, $f(x)p \in A$ for any $f \in C(\mathrm{sp}(x))$,*

$$\|px - xp\| < \delta \ \textit{and}\ \lambda p - pxp \in \mathrm{Inv}_0(pAp)$$

for $\lambda \notin \Omega$, then there are normal elements $y \in M_L(pAp)$ and $z \in M_{L+1}(pAp)$ with finite spectrum $\mathrm{sp}(y), \mathrm{sp}(z) \subset \Omega$ such that

$$\|pxp \oplus y - z\| < \epsilon.$$

Proof. For any $\epsilon > 0$, there are finitely many closed balls $B_1, B_2, ..., B_n$ with centers in Ω and diameters less than $\epsilon/2$ such that

$$\Omega \subset \cup_{i=1}^n B_i.$$

We may write that $\cup_{i=1}^n B_i = \cup_{j=1}^m X_j$, where each X_j is a connected component of $\cup_{i=1}^n B_i$. So we may write $x = \sum_{j=1}^m x_j$, where each x_k is normal and $\mathrm{sp}(x_j)$ (in a corner of B) is a subset of X_j. Since each X_j is a union of some $B_i's$, X_j is homeomorphic to the subset X described in 3.10. We may assume that

$$\mathrm{dist}(X_j, X_{j'}) \geq \delta/2, \ \ j \neq j',$$

by taking smaller δ. Let $p_1, p_2, ..., p_m$ be m mutually orthogonal projections such that

$$\sum_{j=1}^{m} p_j = 1 \quad \text{and} \quad p_j x_j = x_j p_j = x_j, j = 1, 2, ..., m.$$

For any $\eta > 0$, by taking a small δ, we may assume that

$$\|pp_j - p_j p\| < \eta.$$

Note there is $g_j \in C(\mathrm{sp}(x))$ such that $p_j = g_j(x)$. Thus $p_j p p_j \in A$. If η is small enough, there is $f_j \in C(\mathrm{sp}(p_j p p_j))$ such that $q_j = f_j(p_j p p_j)$ is a projection in A (cf. [**Eff**, A8]). Furthermore, $q_j p_j p p_j = p_j p p_j q_j$ is invertible in $q_j A q_j$. Since $p_j p p_j f(x), f(x) p_j p p_j \in A$ for any $f \in C(\mathrm{sp}(x))$, $q_j f(x)$, $f(x) q_j \in A$. Let $q = \sum_{j=1}^{m} q_j$. Then $qf(x)$, $f(x)q \in A$.

For any $1 > \sigma > 0$, if η is small enough,

$$\|q - p\| < \sigma.$$

There is a unitary $w \in A$ such that

$$\|w - 1\| < 2\sigma \quad \text{and} \quad w^* q w = p.$$

We claim that it suffices to show the case that $q = p$.

Suppose that we have

$$\|qxq \oplus y - z\| < \sigma,$$

where $y \in M_L(qAq)$ and $z \in M_{L+1}(qAq)$. Let $W_1 = \mathrm{diag}(w, w, \cdots, w)$ (there are L copies of w) and $W_2 = \mathrm{diag}(w, w \cdots, w, w)$ (there are $L + 1$ copies of w). We have

$$\|w^*(qxq)w \oplus W_1^* y W_1 - W_2^* z W_2\| < \sigma \quad \text{and} \quad \|w^* qxqw - pxp\| < 2\sigma.$$

Thus

$$\|pxp \oplus W_1^* y W_1 - W_2^* z W_2\| < 3\sigma.$$

Note that $W_1^* y W_1 \in M_L(pAp)$, $W_2^* z W_2 \in M_{L+1}(pAp)$, and $\mathrm{sp}(W_1^* y W_1) = \mathrm{sp}(y)$, $\mathrm{sp}(W_2^* z W_2) = \mathrm{sp}(z)$. This proves the claim.

So, without loss of generality, we may assume that $p = \sum_{j=1}^{m} q_j$ with $q_j \le p_j$. Next we show that if $\mathrm{dist}(\lambda, \Omega) \ge 2\delta$,

$$\lambda q_j - q_j x q_j \in \mathrm{Inv}_0(q_j A q_j).$$

If λ is in the unbounded component of $\mathbf{C} \setminus \mathrm{sp}(x_j)$, then $\lambda q_j - q_j x q_j \in \mathrm{Inv}_0(p_j A p_j)$. Set

$$X_j' = \{\xi : \mathrm{dist}(\xi, X_j) < \delta/2\},\ j = 1, 2, ...m.$$

If λ is in the bounded component of $\mathbf{C} \setminus X_j'$, then λ is in the unbounded component of $\mathbf{C} \setminus \mathrm{sp}(x_k)$ for $k \neq j$. So $\lambda p_k - p_k x p_k \in \mathrm{Inv}_0(p_k A p_k)$ for $k \neq j$. It follows from 2.3 that $\lambda p_j - p_j x p_j \in \mathrm{Inv}_0(p_j A p_j)$. We note that X_j' is homeomorphic to the subset X in 3.10. We then apply 3.11 to each x_j. □

Theorem 3.13. *Let A be a C^*-algebra of real rank zero and let x be a normal element in A. If $\Gamma(x) = 0$, then for any $\epsilon > 0$, there is an integer k, and there are normal elements $y \in M_k(A)$ and $z \in M_{k+1}(A)$ with finite spectra contained in* $\mathrm{sp}(x)$ *such that*

$$\|x \oplus y - z\| < \epsilon.$$

(The integer k depends only on ϵ and $\mathrm{sp}(x)$*.) Moreover, the converse is also true.*

Proof. When A is unital, this follows from 3.12 immediately. Now we assume that A is not unital. Then $0 \in \mathrm{sp}(x)$. As in the proof of Theorem A in [**Lin6**], for any $\epsilon > 0$, there is a projection $e \in A$ and a normal element $x' \in eAe$ such that

$$\|x' - x\| < \epsilon/3.$$

Set

$$X_\epsilon = \{\lambda : \mathrm{dist}(\lambda, X) < \epsilon/3\}.$$

It follows from 2.7 that $\lambda e - x' \in \mathrm{Inv}_0(eAe)$, if $\lambda \in X_\epsilon$. By applying the theorem for the unital case, we obtain normal elements $x' \in M_k(eAe)$ and $z \in M_{k+1}(eAe)$ with finite spectrum contained in X_ϵ such that

$$\|x' \oplus y - z\| < \epsilon/3.$$

Therefore

$$\|x \oplus y - z\| < 2\epsilon/3.$$

By changing spectra of y and z slightly (within $\epsilon/3$), we may assume that $\mathrm{sp}(y),\ \mathrm{sp}(z) \in X$.

Now for the converse, suppose that there are normal elements $y \in M_k(A)$ and $z \in M_{k+1}(A)$ with finite spectra contained in $\mathrm{sp}(x)$. such that

$$\|x \oplus y - z\| < \epsilon.$$

For any fixed $\lambda \in \mathbf{C} \setminus \mathrm{sp}(x)$, set

$$d = \mathrm{dist}(\lambda, \mathrm{sp}(x)).$$

Then

$$\|(\lambda - x \oplus y)^{-1}\| < 1/d.$$

If $\epsilon < 1/d$, by 2.7,

$$[\lambda - x \oplus y] = [\lambda - z] \in \mathrm{Inv}(M_{k+1}(A))/\mathrm{Inv}_0(M_{k+1}(A)).$$

Since $\lambda - z$ has finite spectrum, $\lambda - z \in \mathrm{Inv}_0(M_{k+1}(A))$. This implies that

$$\lambda - x \oplus y \in \mathrm{Inv}_0(M_{k+1}(A)).$$

Since $\lambda - y$ has finite spectrum, $\lambda - y \in \mathrm{Inv}_0(M_k(A))$. It follows from 2.3 that $\lambda - x \in \mathrm{Inv}_0(A)$. So by 2.4, $\Gamma(x) = 0$. □

4. C^*-Algebras with (FN) and Weak (FN).

We start with the following definition.

Definition 4.1. (cf [**Bl1**]). We say a C^*-algebra A has the property (FN) if every normal element can be approximated by normal elements with finite spectra.

It is clear that all W^*-algebras and AW^*-algebras have the property (FN). Some examples of separable C^*-algebras which have the property (FN) can be found in [**Ph1**]. It is also clear that every commutative AF-algebra have the property (FN). But it is not known that all AF-algebras have the property (FN) (see [**Lin6**]). From 3.13 we see that if $U(A)/U_0(A) \neq 0$, then A can not have (FN), even though A has real rank zero. This, of course, is also clear from [**Ph1**], since not every C^*-algebra with real rank zero has (FU). As we stated in Section 1, related to question **Q**, we give the following definition:

Definition 4.2. A simple C^*-algebra A is said to have the property weak (FN) if every normal element $x \in A$ can be approximated by normal elements with finite spectra in A if and only if $\Gamma(x) = 0$.

This terminology is certainly borrowed from N. C. Phillips's weak (FU) ([**Ph1**]). We are grateful to Terry Loring who pointed to us that the definition of weak (FN) for general C^*-algebras will be more complicated. We will discuss this issue elsewhere.

Recall that a simple C^*-algebra is said to be purely infinite if every projection in the algebra is infinite (one may use other definitions in [**LZ**] which do not mention projections).

Lemma 4.3. ([**Lin6**, Lemma 1]). *Let A be a unital C^*-algebra and $x \in A$ be a normal element. Then for any $\epsilon > 0$, if*
(1) $\lambda_1, \lambda_2, ..., \lambda_n \in \mathrm{sp}(x)$ *and* $|\lambda_i - \lambda_j| \geq \epsilon, i \neq j$;
(2) S_k *is an open subset of* $\{\lambda : |\lambda - \lambda_k| < \epsilon\}$ *containing* λ_k;
(3) q_k *is the spectral projection of x in A^{**} corresponding to the open set* S_k;
(4) p_k *is a projection in* $\mathrm{Her}(q_k)$;
(5) $y = (1 - \sum_{i=1}^n p_i)x(1 - \sum_{i=1}^n p_i)$, *then*

$$\left\| x - \left(y + \sum_{k=1}^n \lambda_k p_k \right) \right\| < 2\epsilon$$

and

$$\left\| \left(1 - \sum_{i=1}^n p_i\right) x - x \left(1 - \sum_{i=1}^n p_i\right) \right\| < 2\epsilon.$$

Theorem 4.4. *Let A be a purely infinite simple C^*-algebra and let x be a normal element in A. Then x can be approximated by normal elements with finite spectra if and only if $\Gamma(x) = 0$.*

In other words, every purely infinite simple C^*-algebra has the property weak (FN).

Proof. By [**Zh2**], A has real rank zero. So, every hereditary C^*-subalgebra has real rank zero. It follows from 4.3 that for any $\eta > 0$ and $\delta > 0$, there are mutually orthogonal projections $p_1, p_2, ..., p_n \in A$ and complex numbers $\lambda_1, \lambda_2, ..., \lambda_n \in \mathrm{sp}(x)$ such that

$$\left\| \sum_{i=1}^n \lambda_i p_i + (1-p)x(1-p) - x \right\| < \eta/8,$$

where $p = \sum_{i=1}^n p_i$,

$$\|px - xp\| < \delta$$

and for any $\lambda \in \mathrm{sp}(x)$, there is i such that

$$\mathrm{dis}(\lambda, \lambda_i) < \eta/4.$$

Set $y = (1-p)x(1-p)$. Combine 2.4 with 2.6 and 2.7, if δ is small enough, $\mathrm{sp}(y) \subset X_\eta = \{\lambda : \mathrm{dist}(\lambda, \mathrm{sp}(x)) < \eta/4\}$ and

$$\lambda(1-p) - y \in \mathrm{Inv}_0((1-p)A(1-p))$$

if $\lambda \notin X_\eta$. By 3.12, if δ is small enough, there are normal elements $y_1 \in M_k((1-p)A(1-p))$ and $y_2 \in M_{k+1}((1-p)A(1-p))$ with finite spectra contained in X_η such that

$$\|y \oplus y_1 - y_2\| < \eta/4$$

for some integer k. By changing the spectrum of y_1 and the spectrum of y_2 slightly, we may assume that

$$\|y \oplus y_1 - y_2\| < \eta/2$$

and $\mathrm{sp}(y_1)\ \mathrm{sp}(y_2) \subset \mathrm{sp}(x)$.

Without loss of generality, we may further assume that $y_1 = \sum_{i=1}^n \lambda_i q_i$, where q_i are mutually orthogonal projections in $(1-p)A(1-p)$ such that $\sum_{i=1}^n q_i =$ the identity of $M_k((1-p)A(1-p))$. Since A is purely infinite simple C^*-algebra, there is a partial isometry

$$v \in (p \oplus (1-p) \oplus ... \oplus (1-p))M_{k+1}(A)(p \oplus (1-p) \oplus ... \oplus (1-p))$$

(there are k copies of $1-p$) such that $v^* q_i v \le p_i$, $i = 1, 2, ..., n$,

$$v^*v = \sum_{i=1}^n v^* q_i v \quad \text{and} \quad vv^* = (1-p) \oplus (1-p) \oplus ... \oplus (1-p)$$

(there are k copies of $1-p$).
Set $u = (1-p) \oplus v$. Notice that $uu^* = 1 \oplus (1-p) \oplus \cdots \oplus (1-p)$ (there are k copies of $(1-p)$). We have

$$u^*(y \oplus y_1)u = \sum_{i=1}^n \lambda_i p_i' \oplus y$$

and $u^* y_2 u$ is normal and has finite spectrum, where $p_i' = v^* q_i v$. So

$$\left\| \sum_{i=1}^n \lambda_i p_i \oplus y - \sum_{i=1}^n \lambda_i (p_i - p_i') - u^* y_2 u \right\| < \epsilon/2.$$

Therefore

$$\left\| x - \sum_{i=1}^n \lambda_i (p_i - p_i') - u^* y_2 u \right\| < \epsilon.$$

□

Corollary 4.5. *The Cuntz algebra O_n has (FN), i.e. every normal elements in O_n can be approximated by normal elements in O_n with finite spectra.*

Proof. It is known ([**Cu**]) that O_n is purely infinite simple and $K_1(O_n) = 0$. □

One crucial result in BDF-theory is that the Calkin algebra $B(H)/K$ has weak (FN). The following is a generalization of this fact. The question when corona algebras are simple is discussed in [**Lin1**] and [**Lin7**].

Corollary 4.6. *Let A be a σ-unital simple C^*-algebra of real rank zero. If the corona algebra $M(A)/A$ is simple, then $M(A)/A$ has weak (FN).*

Proof. It follows from [**Zh6**, 1.3] that $M(A)/A$ is a purely infinite simple C^*-algebra. So the corollary follows immediately from 4.4. □

Definition and Remark 4.7. Let A be a separable simple unital C^*-algebra with real rank zero, stable rank one and with weakly unperforated $K_0(A)$ (see [**Bl3**, 6.7.1]). Then $G = K_0(A)/K_0(A)_{\rm tor}$ is a simple ordered group. By [**Zh5**, 1.3] and [**EHS**], G is a simple dimension group. Fix a nonzero projection $e \in A$. Let

$$\Delta = \{\tau \in S : \tau(e) = 1\},$$

where S is the set of positive homomorphisms from G into $\mathbf{R}$ (see [**Eff**], Chapter 4). With weak*-topology, Δ is a compact convex set. Let $p \in A$ be another nonzero projection and let

$$\Delta' = \{\tau \in S : \tau(p) = 1\}.$$

If Δ has countably many extreme points, then Δ' has also countably many extreme points (see [**G2**, 6.17]). It is known that Δ is a Choquet simplex. By [**Al**, I.4.9], every point in Δ is a barycenter of a measure concentrated on its extreme points. So, if Δ has countably many extreme points $\{\tau_n\}$, then for any $\tau \in \Delta$, there is a sequence of nonnegative numbers $\{a_n\}$ such that $\sum_{n=1}^{\infty} a_n = 1$ and

$$\tau = \sum_{n=1}^{\infty} a_n \tau_n.$$

Furthermore, every $\tau \in \Delta$ defines a quasitrace on A (see [**Bl3**, 6.9.1]). We will say that $K_0(A)$ has *countable rank* if Δ has countably many extreme points.

The map:

$$\theta : G \to \mathrm{Aff}(\Delta)$$

determines the order on G in the sense that

$$G_+ = \{g \in G : \theta(g) >> 0\} \cup \{0\},$$

where $\mathrm{Aff}(\Delta)$ is the set of all (real) affine continuous functions on Δ. As in [**Eff**, 4], if $G \neq \mathbf{Z}$, $\theta(G)$ is order isomorphic to a dense subgroup of $\mathrm{Aff}(\Delta)$, provided with the relative strict order. Moreover, $g < f$ in $K_0(A)$ if and only if $\phi(g) < \phi(f)$, where ϕ is the composition map:

$$K_0(A) \to K_0(A)/K_0(A)_{\mathrm{tor}} \to \theta(G).$$

By [**Bl**, 6.9.3], $[p] < [q]$ in $K_0(A)$ if and only if $\tau(p) < \tau(q)$ for all quasitraces in Δ. Furthermore, for any two projections $p, q \in A$, $\tau(p) < \tau(q)$ for all quasitraces in Δ if and only if there is a partial isometry $v \in A$ such that $v^*v = p, vv^* \leq q$.

Lemma 4.8. (cf. [**Lin6**, Lemma 2]). *Let A be a separable simple unital C^*-algebra of real rank zero, stable rank one and with weakly unperforated $K_0(A)$ of countable rank and let x be a normal element in A. For any $\epsilon > 0$ and integer $K > 0$, there are open subsets $O_1, O_2, ..., O_n$ of $\mathrm{sp}(x)$ such that*

$$O_i \cap O_j = \emptyset,\ [\cup_{i=1}^n O_i]^- = \mathrm{sp}(x),$$

$\lambda_i \in O_i$, and there are projections $p_i \in \mathrm{Her}(q_i)$, where q_i are spectral projections of x in A^{**} corresponding to the open subsets O_i, such that

$$\left\| x - \left(y + \sum_{i=1}^n \lambda_i p_i\right)\right\| < \epsilon,$$

where $y = (1 - \sum_{i=1}^n p_i)x(1 - \sum_{i=1}^n p_i)$,

$$\left\|\left(1 - \sum_{i=1}^n p_i\right) x - x\left(1 - \sum_{i=1}^n p_i\right)\right\| < \epsilon,$$

$$[p_k] > K\left[1 - \sum_{i=1}^n p_i\right]$$

and for any $\lambda \in \mathrm{sp}(x)$, there is λ_i such that

$$\mathrm{dis}(\lambda, \lambda_i) < \epsilon.$$

Proof. We set $\Delta = \{\tau \in S : \tau(1) = 1\}$ (see 4.7).

Without loss of generality, we may assume that $\|x\| \leq 1$. Denote by B the C^*-subalgebra generated by x and 1. Then $B \cong C(X)$, where $X = \mathrm{sp}(x)$. Let $\tau \in \Delta$. So τ is a quasitrace defined on A (see 4.7). Since B is commutative, the restriction of τ on B is linear. Hence the restriction of τ on B gives a state on B. By the Riesz representation theorem, the state defines a normalized Borel measure μ_τ on X. Let D denote the unit disk. For any open subset $O \subset D$, let q_o be the spectral projection of x in A^{**} corresponding to the open subset $O \cap X$. The projection q_o is an open projection in A^{**}. Suppose that $h \in B(\cong C(X))$ such that $1 \geq h > 0$ for all $t \in O \cap X$ and $h(t) = 0$ for all $t \in X \cap O$. Then $\{h^{1/n}\}$ forms an approximate identity for $\mathrm{Her}(q_o)$. It is clear that

$$\mu_\tau(O \cap X) = \lim_{n\to\infty} \tau(h^{1/n}).$$

Let $\{e_n^o\}$ be an approximate identity for $\mathrm{Her}(q_o)$ consisting of projections. Then

$$\mu_\tau(O \cap X) = \sup\{\tau(e_n^o)\}.$$

In fact, we have (τ is a quasitrace on A)

$$\tau(h^{1/k}e_n^o) = \tau(h^{1/2k}e_n^o h^{1/2k}) \leq \tau(h^{1/k}) \leq \mu_\tau(O \cap X)$$

and

$$\tau(h^{1/k}e_n^o) = \tau(e_n^o h^{1/k} e_n^o) \leq \tau(e_n^o)$$

for all k and n. Since $h^{1/k}e_n^o \to e_n^o$, if $k \to \infty$, and $h^{1/k}e_n^o \to h^{1/k}$, if $n \to \infty$, from above equalities and inequalities, we conclude that

$$\mu_\tau(O \cap X) = \sup\{\tau(e_n^o)\}.$$

Let Δ_0 denote the countable subset $\{\tau_n\}$. For the simplicity, we use the notation μ_i for the measure μ_{τ_i}.

For any $\epsilon > 0$, there is a finite subsets $\{\zeta_1, \zeta_2, ..., \zeta_m\}$ of D such that for any $\zeta \in D$, there is an integer i such that

$$|\zeta_i - \zeta| < \epsilon/32$$

and for any i, there is $j \neq i$ such that

$$|\zeta_i - \zeta_j| < \epsilon/16.$$

For each i set

$$D_i = \{\zeta : \epsilon/32 \leq |\zeta_i - \zeta| \leq \epsilon/16\}.$$

Fix i, for each $\epsilon/32 \leq r \leq \epsilon/16$, set

$$S_r = \{\zeta : |\zeta - \zeta_i| = r\}.$$

Since $\mu_k(D_i) \leq 1$ and $S_r \cap S_{r'} = \emptyset$, if $r \neq r'$, there are only countably many r in $(\epsilon/32, \epsilon/16)$ such that

$$\mu_k(S_r) > 0.$$

Since the union of countably many countable sets is still countable, we conclude that for each i, there is $r_i \in (\epsilon/32, \epsilon/16)$ such that

$$\mu_k(S_{r_i}) = 0$$

for all k.

Now $D \setminus \cup S_{r_i}$ is a disjoint union of finitely many open sets $O_1, O_2, ..., O_N$ such that the diameter of each O_i is $< \epsilon/4$ and

$$\mu_k(\cup S_{r_i}) = 0$$

for all k.

Let $\{e_n^{(i)}\}$ be an approximate identity for B_{O_i}, where B_{O_i} is the hereditary C^*-subalgebra corresponding to the spectral projection q_{O_i} of x in A^{**} corresponding to the open subset O_i. (Notice that B_{O_i} has real rank zero, whence such an approximate identity exists (see [**BP**])). Then

$$\tau_j(e_n^{(i)}) \nearrow \mu_j(O_i)$$

$j = 1, 2, ...$ and $i = 1, 2, ..., N$. Since $\mu_j\left(D \setminus \cup_{i=1}^N O_i\right) = 0$,

$$\tau_j\left(\sum_{i=1}^N e_n^{(i)}\right) \nearrow 1,$$

as $n \to \infty$, $j = 1, .2, ...,$. Since every $\tau \in \Delta$ has the form

$$\tau = \sum_{j=1}^{\infty} \alpha_j \tau_j,$$

where $\alpha_j \geq 0$ and $\sum_{j=1}^{\infty} \alpha_j = 1$, we conclude that

$$\tau\left(\sum_{i=1}^N e_n^{(i)}\right) \nearrow 1$$

for all $\tau \in \Delta$, as $n \to \infty$. Since Δ is compact, by Dini's theorem, the continuous functions $\sum_{i=1}^N e_n^{(i)}(\tau)$ defined on Δ converges to the constant function 1 uniformly on Δ, as $n \to \infty$. Hence we have projections $p_i \in B_{O_i}$ such that

$$\tau(p_i) > K\tau\left(1 - \sum_{i=1}^{N} p_i\right)$$

for all i and $\tau \in \Delta$. So, by 4.7, we obtain

$$[p_k] > K\left[1 - \sum_{i=1}^{n} p_i\right].$$

The rest of the proof follows from Lemma 4.3. □

Theorem 4.9. *Let A be a separable simple C^*-algebra with real rank zero and stable rank one. If $K_0(A)$ is weakly unperforated and of countable rank, then A has weak (FN). In other words, a normal element $x \in A$ can be approximated by normal elements in A with finite spectra if and only if $\Gamma(x) = 0$.*

Proof. Suppose that $x \in A$ is a normal element with $\Gamma(x) = 0$. Let ϵ be a positive number. For $\epsilon/2$, let δ and k be as in 3.12. We may assume that $\delta < \epsilon/4$. By applying 4.8 , there are complex numbers $\lambda_1, \lambda_2, ..., \lambda_n \in \mathrm{sp}(x)$ and mutually orthogonal projections $p_1, p_2, ..., p_n \in A$ such that

$$\left\| x - \sum_{i=1}^{n} \lambda_i p_i - y \right\| < \epsilon/2,$$

where $p = \sum_{i=1}^{n} p_i$ and $y = (1-p)x(1-p)$,

$$\|(1-p)x - x(1-p)\| < \delta,$$

$$[p_i] > k[1-p], \ i = 1, 2, ..., n$$

and for any $\lambda \in \mathrm{sp}(x)$, there is i such that

$$\mathrm{dis}(\lambda, \lambda_i) < \epsilon/2.$$

As in the proof of 4.4, if δ is small enough, by 3.12, there are normal elements

$$y_1 \in M_k((1-p)A(1-p)) \quad \text{and} \quad z \in M_{k+1}((1-p)A(1-p))$$

with finite spectra contained in $\mathrm{sp}(x)$ such that

$$\|y \oplus y_1 - z\| < \epsilon/2.$$

Without loss of generality, we may assume that $y_1 = \sum_{i=1}^n \lambda_i q_i$, where q_i are mutually orthogonal projections in $M_k((1-p)A(1-p))$ such that $\sum_{i=1}^n q_i = (1-p) \oplus ... \oplus (1-p)$, where the right side has k copies of $(1-p)$. Since

$$[p_i] > k[1-p], \ i = 1, 2, ..., n$$

there is a partial isometry

$$v \in (p \oplus (1-p) \oplus ... \oplus (1-p))M_k(A)(p \oplus (1-p) \oplus ... \oplus (1-p))$$

such that

$$v^* q_i v \leq p_i, \ i = 1, 2, ..., n,$$

$$v^* v = \sum_{i=1}^n p_i', \quad \text{and} \quad vv^* = (1-p) \oplus ... \oplus (1-p)$$

(there are k copies of $(1-p)$), where $p_i' = v^* q_i v$. Set

$$u = (1-p) \oplus v.$$

Notice that

$$u^*(y \oplus y_1)u = \sum_{i=1}^n \lambda_i p_i' \oplus y$$

and $u^* z u$ is still normal and has finite spectrum. Now we have

$$\left\| \sum_{i=1}^n \lambda_i p_i \oplus y - \sum_{i=1}^n \lambda_i (p_i - p_i') - u^* z u \right\| < \epsilon/2.$$

Therefore

$$\left\| x - \sum_{i=1}^n \lambda_i (p_i - p_i') - u^* z u \right\| < \epsilon.$$

□

Corollary 4.10. *All irrational rotation algebras A_θ have the property weak (FN).*

Proof. It has recently been shown that all such C^*-algebras have real rank zero (see [**BKR**]). It is known that A_θ is simple and $K_0(A_\theta) \cong \mathbf{Z} + \mathbf{Z}\theta$ (see [**Rf**]). By [**Pt**], A_θ has stable rank one. So 4.10 follows from 4.9. □

Now we consider the class of C^*-algebras of real rank zero which are inductive limits of finite direct sums of circle algebras classified recently by G. A. Elliott ([**Ell2**]).

Corollary 4.11. *Let A be a C^*-algebra of real rank zero which is an inductive limit of finite direct sums of matrix algebras over $C(S^1)$. If A is simple and $K_0(A)$ has countable rank, then A has weak (FN).*

Proof. All such algebras have stable rank one and (weakly) unperforated $K_0(A)$. □

Corollary 4.12. *The Bunce-Deddens algebra has weak (FN).*

Corollary 4.13. ([**Lin6**]). *Let A be a simple AF-algebra. If $K_0(A)$ is of countable rank, then A has (FN). In particular, every matroid algebra has (FN).*

Proof. All AF-algebras have real rank zero, stable rank one and unperforated $K_0(A)$. Since $K_1(A) = 0$ for all AF-algebras, $\Gamma(x) = 0$ for all normal elements. □

Corollary 4.14. *Let $A = \lim_{\to}(A_n, \phi_n)$ be the C^*-algebra inductive limit of C^*-algebras A_n of the form $C(X, M_{k(n)})$, where ϕ_n are unital homomorphisms and X is a finite CW complex. If A is simple and of real rank zero, then A has weak (FN).*

Proof. As in the proof of [**GL**, 3.3], A has stable rank one and $K_0(A)$ is weakly unperforated and of finite rank. So this corollary follows from 4.9. □

5. Normal Elements with One Dimensional Spectra.

Even though we can not give an affirmative answer to question **Q** for general C^*-algebras with real rank zero at present, we would like to show that for normal elements with dim sp$(x) \leq 1$, the answer to **Q** is affirmative.

Lemma 5.1. *For any $\epsilon > 0$ and integer n, there is $\delta > 0$ such that for any (unital) C^*-algebra A of real rank zero, selfadjoint elements $x_1, x_2, ..., x_n \in A$ and a nonzero projection $p \in A$, if*

$$\|px_i - x_ip\| < \delta \quad \text{and} \quad x_ix_j = 0,$$

$i = 1, 2, ..., n$ and $i \neq j$, then there are selfadjoint elements $y_1, y_2, ..., y_n \in pAp$ satisfying

$$\|y_i - px_ip\| < \epsilon \quad \text{and} \quad y_iy_j = 0,$$

$i = 1, 2..., n$ *and* $i \neq j$.

Proof. We first assume that $x_i \geq 0$. Set $z_i = px_ip$. So $z_i \geq 0$. Suppose that

$$\|px_i - x_ip\| < \delta,$$

where δ is a positive number to be determined. We have

$$\left\| z_1 \left(\sum_{i=2}^{n} z_i \right) \right\| < 2n\delta.$$

Notice that pAp has real rank zero (see [**BP**, 2.8]). It follows from [**BP**, 2.6] that there is a nonzero projection $q_1 \in pAp$ such that

$$\|(p - q_1)z_1\| < (2n\delta)^{1/2} \quad \text{and} \quad \left\| q_1 \left(\sum_{i=2}^{n} z_i \right) \right\| < (2n\delta)^{1/2}.$$

Therefore $\|q_1 z_i\| < (2n\delta)^{1/2}$, $i = 2, 3, ..., n$. Set $y_1 = q_1 z_1 q_1$. Then

$$\|y_1 - z_1\| \leq \|(p - q_1)z_1q_1\| + \|z_1(p - q_1)\| < 2(2n\delta)^{1/2}.$$

We also have

$$\|(p - q_1)z_i(p - q_1) - z_i\| < 2(2n\delta)^{1/2},$$

$i = 2, 3, ..., n$. Notice that $(p - q_1)z_i(p - q_1) \in (p - q_1)A(p - q_1)$. We can then work in $(p - q_1)A(p - q_1)$. So, by induction (on n), the lemma follows for the case that $x_i \geq 0$. For general selfadjoint elements x_i, we notice that $x_i = (x_i)_+ - (x_i)_-$ and $(x_i)_+(x_i)_- = 0$. □

Lemma 5.2. *Let X be a contractible compact subset of the plane which is homeomorphic to a union of finitely many (compact) straight line segments. For any $\epsilon > 0$, there exits $\eta > 0$, for any (unital) C^*-algebra A of real rank zero and a normal element $x \in A$, if p is a projection in A and*

$$\|px - xp\| < \eta,$$

then there is a normal element $y \in pAp$ with finite spectrum $\mathrm{sp}(y) \subset X$ *and*

$$\|pxp - y\| < 2\epsilon.$$

Proof. We may assume that $X = \cup_{i=1}^{n} L_i$, where each L_i is a compact line segment and different L_i lies in a different line. If $n = 1$, we may assume that $X \subset \mathbf{R}$. Therefore we may assume that x is self adjoint. If δ is small enough,

$$\mathrm{sp}(pxp) \subset (X_{\epsilon/2} \cap \mathbf{R}) \cup \{0\},$$

where

$$X_{\epsilon/2} = \{\xi : \mathrm{dist}(\xi, X) < \epsilon/2\}.$$

Since A has real rank zero, there is a selfadjoint element $z \in pAp$ with finite spectrum $\mathrm{sp}(z) \subset (X_{\epsilon/2} \cap \mathbf{R}) \cup \{0\}$ such that

$$\|x - y\| < \epsilon/2.$$

Then we can replace z by a selfadjoint element $y \in pAp$ with finite spectrum $\mathrm{sp}(y) \subset X$ such that

$$\|x - y\| < \epsilon.$$

Now we assume that 5.2 holds for $n \leq k$.

We first show that for any $\sigma > 0$, there are mutually orthogonal normal elements $x_i \in A$ with $\mathrm{sp}(x_i) \subset L_i$ (in the respective corner of A) such that

$$\left\| x - \sum_{i=1}^{n} x_i \right\| < \sigma.$$

We will prove this by induction on n. Assume that the assertion is true for $n \leq k$. We will show it is true for $n = k+1$.

Since X is contractible, there is at least one L_{i_1} such that $L_{i_1} \cap (\cup_{i \neq i_1} L_i)$ has only one point, say ξ_1. We may also assume that $i_1 = 1$. Set $L_1^o = L_1 \setminus \{\xi_1\}$. Let p_1 be the spectral projection of x (in A^{**}) corresponding to L_1^o and p_1' be the spectral projection of x (in A^{**}) corresponding to $(L_1^o)'$, where

$$(L_1^o)' = \{\xi \in L_1^o : \mathrm{dist}(\xi, \xi_1) \leq \sigma/64\}.$$

By Brown's interpolation lemma ([**Bn2**]), there is a projection $q \in A$ such that

$$p_1' \leq q \leq p_1.$$

Note that

$$\|p_1 x - (p_1' x + \xi_1(p_1 - p_1'))\| < \sigma/64.$$

It is then easy to see that

$$\|qx - xq\| < \sigma/32.$$

Let $r : X \to X \setminus L_1^o$ be a retraction. We have

$$\|(1-q)r(x) - r(x)(1-q)\| < \sigma/16$$

and

$$\|(1-q)x(1-q) - (1-q)r(x)(1-q)\| < \sigma/8.$$

By our first inductive assumption, there is a normal elemet $y_1 \in qAq$ with $\mathrm{sp}(y_1) \subset (\cup_{i=2}^{n} L_i)$ such that

$$\|(1-q)r(x)(1-q) - y_1\| < \sigma/8.$$

Then, by our second inductive assumption, there are mutually orthogonal normal elements $x_2, x_3, ..., x_{k+1}$ in $(1-q)A(1-q)$ with $\mathrm{sp}(x_i) \subset L_i$ (in the respective corner), $i = 2, 3, ..., k+1$ such that

$$\left\|(1-q)r(x)(1-q) - \sum_{i=2}^{k+1} x_i\right\| < \sigma/4.$$

So

$$\left\|(1-q)x(1-q) - \sum_{i=2}^{k+1} x_i\right\| < 3\sigma/8.$$

We can write $p_1 x = \alpha h + \beta p_1$, where α and β are complex numbers and h is a self adjoint element in A^{**}. Since $qxq \in A$, $qhq \in A$. So qxq is normal. If σ is small enough, we may assume that $\mathrm{sp}(qxq) \subset L_1$. Set $x_1 = qxq$. Then

$$\left\|x - \sum_{i=1}^{k+1} x_i\right\| < \sigma.$$

This proves the assertion.

Since A has real rank zero, there are mutually orthogonal projections $e_1, e_2, ..., e_{k+1}$, selfadjoint elements $h_i \in e_i A e_i, i = 1, 2, ..., k+1$ and scalars $\alpha_1, \alpha_2, ..., \alpha_{k+1}$ and $\beta_1, \beta_2, ..., \beta_{k+1}$ such that $x_i = \alpha_i h_i + \beta_i$. If $\sigma < \delta/2$ and $\eta < \delta/2$, by Lemma 5.1, there are mutually orthogonal normal elements $y_1, y_2, ..., y_n$ with spectrum $\mathrm{sp}(y_i) \subset L_i$ (in the respective corner of pAp) such that

$$\left\|p\left(\sum_{i=1}^{n} x_i\right)p - \sum_{i=1}^{n} y_i\right\| < \epsilon.$$

Since each $e_i A e_i$ has real rank zero, there is a normal element $y \in A$ with finite spectrum $\mathrm{sp}(y) \subset X$ such that

$$\left\|\sum_{i=1}^{n} y_i - y\right\| < \epsilon/2.$$

Therefore, if σ is small enough, we have

$$\|pxp - y\| < 2\epsilon.$$

□

Lemma 5.3. (cf. [**Lin5**, Lemma 3]). *Let X be a compact subset of the plane with* $\dim X \le 1$. *Suppose that F and Ω are two proper closed subset of X such that $\Omega \cap F = \emptyset$ and the closure of $X \setminus \Omega$ is a compact subset described in* 5.2. *Then for any $\epsilon > 0$, there is $\delta > 0$ satisfying the following: for any unital C^*-algebra A with real rank zero, a nonzero projection $p \in A$ and a normal element $x \in pAp$ with* $\mathrm{sp}(x) = X$, *if there exist normal elements $y \in (1-p)A(1-p)$ with* $\mathrm{sp}(y) \subset F$ *(as an element in $(1-p)A(1-p)$) and $z \in A$ with finite spectrum such that*

$$\|x \oplus y - z\| < \delta,$$

then there is a normal element $z' \in pAp$ with finite spectrum satisfying

$$\|x - z'\| < \epsilon.$$

Proof. Let f and g be two continuous functions defined on X such that $0 \le f \le 1, f(\zeta) = 0$, if $\zeta \in F$, $f(\zeta) = 1$ if $\zeta \in \Omega$; and $0 \le g \le 1, g(\zeta) = 1$, if $\zeta \in F$, $g(\zeta) = 0$, if $\zeta \in \Omega$ and $fg = 0$. For any $\delta > 0$, by 2.8 (1), there is $0 < \eta \le \delta$ such that for any two normal elements x_1, $x_2 \in A$ with spectra contained in X if $\|x_1 - x_2\| < \eta$, then

$$\|f(x_1) - f(x_2)\| < \delta$$

and

$$\|g(x_1) - g(x_2)\| < \delta.$$

Now we suppose that

$$\|x \oplus y - z\| < \eta.$$

We assume that

$$z = \sum_{i=1}^{n} \lambda_i p_i,$$

where $\{p_i\}_{i=1}^n$ is a set of mutually orthogonal projections in A and $\lambda_i \in X$, $i = 1, 2, ..., n$. Set $z_1 = \sum_{\lambda_i \in \Omega} \lambda_i p_i$, $z_2 = \sum_{\lambda_i \notin \Omega} \lambda_i p_i$ and $r = \sum_{\lambda_i \in \Omega} p_i$. Since r commutes with z and $(1-p)$ commutes with $x \oplus y$, we have

$$f(z)r = rf(z) = r \text{ and } g(x \oplus y)(1-p) = (1-p)g(x \oplus y) = (1-p).$$

Therefore

$$\begin{aligned} \|r(1-p)\| &= \|rf(z)g(x \oplus y)(1-p)\| \\ &\le \|rf(x \oplus y)g(x \oplus y)(1-p)\| \\ &\quad + \|r(f(z) - f(x \oplus y))(1-p)\| < \delta. \end{aligned}$$

Consequently,

$$\|r - prp\| < 2\delta.$$

If $\delta < 1/4$, by [**Eff**, A8], there is a projection $r' \leq p$ such that

$$\|r' - r\| < 2\delta$$

and there is a unitary $u \in A$ such that

$$\|u - 1\| < 4\delta, \quad \text{and} \quad u^*ru = r'.$$

Thus

$$\|u^*z_1u - z_1\| < 8\delta.$$

Then

$$\begin{aligned}\|pz_2 - z_2p\| &\leq \|pz_1 - z_1p\| + \|pz - zp\| \\ &\leq \|prz_1 - z_1rp\| + 2\delta < 2(\delta + \eta).\end{aligned}$$

Since

$$\|u^*z_2u - z_2\| < 8\delta,$$

we obtain that

$$\|p(u^*z_2u) - (u^*z_2u)p\| \leq 2(\delta + \eta) + 16\delta = 18\delta + 2\eta.$$

Put $p' = p - r'$. Since

$$r'u^*z_2u = u^*z_2ur' = 0,$$

then

$$\|p'(u^*z_2u) - (u^*z_2u)p'\| < 32\delta + 2\eta.$$

We notice that $\mathrm{sp}(u^*z_2u)$ is a subset of the closure of $X \setminus \Omega$. It follows from 5.2 that, if both η and δ are small enough, there is a normal element $y \in p'Ap'$ with finite spectrum $\mathrm{sp}(y) \subset \overline{(X \setminus \Omega)}$ such that

$$\|p'(u^*z_2u)p' - y\| < \epsilon/2.$$

Therefore, if δ and η are small enough,

$$\|x - u^*z_1u - y\| < \epsilon.$$

Notice that $u^*z_1u = \sum_{\lambda_i \Omega} \lambda_i(u^*p_iu)$ and $p' = p - u^*ru$. □

Theorem 5.4. *Let A be a C^*-algebra of real rank zero and let x be a normal element in A with* $\dim \mathrm{sp}(x) \leq 1$. *Then x can be approximated by normal elements in A with finite spectra if and only if $\Gamma(x) = 0$.*

Proof. As in 3.13, we may assume that A has unit. If $\dim \mathrm{sp}(x) = 0$, then $\mathrm{sp}(x)$ is totally disconnected. The conclusion of 5.4 is trivial in this case.

Now we assume that dim sp$(x) = 1$. It follows from [**F**, Satz 1; p. 229] that $C(\mathrm{sp}(x)) = \lim_{n\to\infty}(C(X_n, \phi_{n,n+1}))$, where X_n are one dimensional polyhedra. For any $\eta > 0$, there exists n and $y \in \phi_{n,\infty}(C(X_n))$, where $\phi_{n,\infty}$ is the map from $C(X_n)$ into $C(X)$, such that

$$\|x - \phi_{n,\infty}(y)\| < \eta.$$

Note that $\phi_{n,\infty}(C(X_n)) \cong C(Y_n)$, where Y_n is a compact subset of X_n. Without loss of generality, we may assume that X_n is a union of finitely many line segments in the plane. It is also clear that it suffices to show that every element in $\phi_{n,\infty}(C(X_n))$ can be approximated by normal elements with finite spectrum. Thus, we reduce the general case to the case that sp(x) is a union of finitely many line segments. By 3.13, for any $\delta > 0$, there are normal elements $y \in M_k(A)$ and $z \in M_{k+1}(A)$ with finite spectra contained in sp(x) such that

$$\|x \oplus y - z\| < \delta.$$

Since y has finite spectrum, we may write $y = y_1 \oplus y_2$ such that $\mathrm{sp}(y_1) \cap \mathrm{sp}(y_2) = \emptyset$ and $\mathrm{sp}(y_2) \subset F$, where both F and the closure of $S \setminus F$ satisfy the description of the contractible compact subset in 5.2. By applying 5.3, we have a normal element z' with finite spectrum contained in sp(x) such that

$$\|x \oplus y_1 - z'\| < \delta.$$

By applying 5.3 again, we finally obtain a normal element $z'' \in A$ with finite spectrum contained in sp(x) such that

$$\|x - z''\| < \epsilon.$$

□

Corollary 5.4. *Every normal element x in an AF-algebra with* dim sp$(x) \leq$ *1 can be approximated by normal elements with finite spectra.*

6. Applications.

In [**Ell2**], George A. Elliott shows that C^*-algebras of real rank zero which are inductive limits of the form

$$\sum \oplus_{i=1}^{n} C(X_i) \otimes M_{k(i)},$$

where X_i is homeomorphic to the unit circle or unit interval, can be completely determined by their K_*-groups. Conversely, if G is a countable unperforated graded ordered group with Riesz decomposition property, then there exists a C^*-algebra A of real rank zero which is an inductive limit of the above form such that $(K_0(A), K_1(A))$ is isomorphic to G (see [**Ell2**] for his definition of unperforated graded ordered group). These remarkable results also provide the way to construct such C^*-algebras with given K_*-groups. In particular, A is an AF-algebra if and only $K_1(A) = 0$. Recently, it was shown that all irrational rotation C^*-algebras are in fact in this class of C^*-algebras of real rank zero (see [**EE**]). As indicated in Section 1, Elliott's algebras of real rank zero may well include all C^*-algebras of real rank zero which are inductive limits of the form

$$\sum \oplus_{i=1}^{k(n)} C(X_n^i) \otimes M_{m(n)},$$

where each X_n^i is a compact subset of the plane. We show in this section that at least it is true in some special cases.

Theorem 6.1. *Let $A = \lim_{\to}(A_n, \phi_n)$ be a simple C^*-algebra of real rank zero, where each A_n has the form*

$$A_n = \sum \oplus_{i=1}^{k} C(X_n^i) \otimes M_{m_i},$$

where X_n^i is a compact subset of the plane. Suppose that $K_0(A)$ has countable rank. Then A is an AF-algebra if and only if $K_1(A) = 0$.

Proof. Since every AF-algebra A has trivial $K_1(A)$, we need only to show the "if" part. From [**DNNP**], A has stable rank one. It follows from 4.9 that A has (FN).

We now assume that A is unital. For any $\epsilon > 0$, and $x_1, x_2, ..., x_m \in A$ there are an integer N and $y_1, y_2, ..., y_m \in \phi_\infty(A_N)$ such that

$$\|x_i - y_i\| < \epsilon/2.$$

We will show that there are a finite dimensional C^*-subalgebra $B \subset A$ and $z_1, z_2, ..., z_m \in B$ such that

$$\|y_i - z_i\| < \epsilon/2.$$

To save notation, (without loss of generality), we may assume that

$$A_N = C(X) \otimes M_k (\cong C(X, M_k)),$$

where X is a compact subset of the plane. Since $\phi_\infty(A_n)$ is isomorphic to a C^*-algebras with the form $C(Y, M_k)$, where Y is a compact subset of X, we may simply assume that $A_N = \phi_\infty(A_N)$.

Let $\{e_{ij}\}$ be a matrix unit for M_k. Set $\epsilon_{ij} = 1 \otimes e_{ij}$. We view $\epsilon_{ij} \in A$. Notice that $\epsilon_{11} A_N e_{11} \cong C(X)$. Let y be a normal element in $\epsilon_{11} A_N \epsilon_{11}$ with $\mathrm{sp}(y) = X$ such that y is a generator for $\epsilon_{11} A_n \epsilon_{11} \cong C(X)$. It is easy to see that it is sufficient to show that for any $\eta > 0$, there is a normal element $z \in \epsilon_{11} A \epsilon_{11}$ with finite spectrum such that

$$\|y - z\| < \eta.$$

Notice that $\epsilon_{11} A \epsilon_{11}$ is simple C^*-algebra with real rank zero and, by [**Bn1**], $A \cong \epsilon_{11} A \epsilon_{11} \otimes \mathcal{K}$. So $K_0(\epsilon_{11} A \epsilon_{11}) = K_0(A)$ and $K_1(\epsilon_{11} A \epsilon_{11}) = K_1(A) = 0$. By 4.9, such z exists. This proves the case that A is unital.

If A is not unital, then A has an approximate identity $\{e_n\}$ consisting of projections (see [**BP**]). Fixed n. There are integers $n(1), n(2), ..., n(k), ...$, such that $n(k) < n(k+1)$ and, for each k, there is a positive element $a_{n(k)} \in A_{n(k)}$ such that

$$\|a_{n(k)} - e_n\| < 1/k.$$

By [**Eff**, A], there is a projection $q_{n(k)} \in A_{n(k)}$ such that

$$\|q_{n(k)} - e_n\| < 2/k$$

and there is a unitary $u \in \tilde{A}$ such that

$$\|u - 1\| < 4/k \quad \text{and} \quad u^* e_n u = q_{n(k)}.$$

Clearly, $q_{n(k)} A_{n(k)} q_{n(k)}$ is isomorphic to a direct sum of finitely many C^*-algebra each of which is of the form

$$C(X) \otimes M_m$$

for some compact subset of the plane X and positive integer m. This implies that $e_n A e_n$ is a unital simple C*-algebra satisfying the conditions of the theorem. From what we have proved for the unital case, we know that $e_n A e_n$ is an AF-algebra. Consequently, A is an AF-algebra. □

Corollary 6.2. *Let $A = \lim_{\to}(A_n, \phi_n)$ be a simple C^*-algebra. Suppose that each A_n is isomorphic $C(X_n) \otimes M_{m(n)}$ for some integer $m(n)$, where X_n is a contractive compact subset of the plane. Then A is an AF-algebra if and only if A has real rank zero (or equivalently, the projections of A separate the traces* (see [**BDR**])).

Proof. It follows from 4.14 that A has week (FN). Since each X_n is contractive, $K_1(A_n) = 0$. Consequently, $K_1(A) = 0$. Then Theorem 6.1 applies. □

The following is due to H. Su. We present here as a simple application of Theorem 5.3.

Corollary 6.3. ([Su]). *Let A be a C^*-algebra of real rank zero. Suppose that A is an inductive limit of finite direct sums of matrix algebras over one-dimensional spaces, then A is an AF-algebra if and only if $K_1(A) = 0$.*

Proof. By 5.4, every normal element x with $\mathrm{sp}(x) \leq 1$ can be approximated by normal elements with finite spectra, if $\Gamma(x) = 0$. The proof is the same as that of 6.1. □

Remark 6.4. Even if $K_1(A) \neq 0$, both algebras in 6.1 and 6.3 are inductive limits of finite direct sums of matrix algebras over circles. This can be done by applying 4.9 and 5.4 more carefully. We will discuss these elsewhere.

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