PACIFIC JOURNAL OF MATHEMATICS
Vol. 173, No. 2, 1996

H^p-ESTIMATES OF HOLOMORPHIC DIVISION FORMULAS

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We prove that an explicit formula, due to Berndtsson, for representation of solutions of holomorphic division problems in a strictly pseudoconvex domain admit H^p-estimates and provides a solution to the following problem: Given bounded holomorphic functions $G_1, ..., G_m$ such that $\sum |G_j|^2 \geq \delta^2$, and $\phi \in H^p$, find $u_j \in H^p$ such that $\sum G_j u_j = \phi$. The estimates are based on careful estimates of Hefer functions and a $T1$-theorem for Carleson measures, due to Christ and Journé.

1. Introduction.

Let $G_1, G_2, ..., G_m$ be holomorphic functions in some pseudoconvex domain D in $\mathbb{C}^n$ without common zeros. For any holomorphic ϕ one can then find holomorphic $u_1, ..., u_m$ such that

$$\sum G_j u_j = \phi. \tag{1.1}$$

Formulas for explicit solutions of such division problems were introduced in [**B1**]. These formulas have been used by several authors to obtain estimates in various situations and norms, see e.g. [**B2**] and the references given there. Our main purpose in this note is to show that appropriate such formulas admit H^p estimates in strictly pseudoconvex domains. More precisely we have

Theorem 1.1. *Let D be a strictly pseudoconvex domain with C^3-boundary and let $G_1, ..., G_m \in H^\infty$ be given such that*

$$\sum |G_j|^2 \geq \delta^2 \tag{1.2}$$

for some $\delta > 0$. Then there are explicit integral operators $T_1, ..., T_m$ which are bounded on H^p, $1 \leq p < \infty$, take H^∞ into $H^\infty \cdot \mathrm{BMO}$ and satisfy

$$\sum G_j T_j \phi = \phi.$$

A function f is in $H^\infty \cdot \mathrm{BMO}$ if it is holomorphic and its boundary values can be written as a finite sum $\sum a_j b_j$ where $a_j \in H^\infty$ and $b_j \in \mathrm{BMO}$.

Thus, in strictly pseudoconvex domains, we have solved, for $1 \leq p \leq *$, what we call the H^p-corona problem:
Given $G_1, ..., G_m \in H^\infty$ such that (1.2) holds, find to each $\phi \in H^p$ solutions u_j of (1.1) in H^p, if $p < \infty$. If $p = *$ it means that, for each $\phi \in H^\infty$, one has to find a solution in $H^\infty \cdot \mathrm{BMO}$.

Of course, the true corona problem is to find bounded solutions u_j when ϕ is bounded.

For $0 < p \leq 2$ this problem was solved by L^2-methods in quite general pseudoconvex domains with C^2-boundary in [**An1**], [**An2**], for an arbitrary (even infinite) m. The proof was based on a modification of a technique due to Skoda, see [**Sk**], where L^2-estimates of solutions of division problems are obtained.

Another way to deal with division problems is to use the Koszul complex to reduce it to (systems of) $\bar\partial$-equations. When $m = 2$ (or $n = 1$) this is particularily simple as one just ends up with a finite number of equations $\bar\partial u = f$ where f is a (0,1)-form. In this method, as well as in the method exploited in this paper, one starts with a smooth solution γ_j to $\sum G_j \gamma_j = 1$. On can make such a choice so that $\bar\partial \gamma_j$ are Carleson measures, and then one can apply any popular weighted solution formula for $\bar\partial$ and get a solution of the H^p-corona problem in a strictly pseudoconvex domain for $1 \leq p \leq *$, see e.g. Varopoulos [**V1**]. However, the simplest choice $\gamma_j = \bar G_j / |G|^2$ requires a Wolff-type estimate of the corresponding $\bar\partial$-problem. This approach was carried out for the ball in [**Am**] and in [**AnC2**] in the general strictly pseudoconvex case.

When the number of generators exceeds two, the situation is more complicated. In this case the Koszul complex provides a scheme for solving the division problem by iteratively solving equations $\bar\partial u = f$ for various $(0, q)$-forms q. In §7 we indicate how one can use integral formulas to solve the $\bar\partial$-equations and obtain H^p-estimates for the division problem.

However, our main purpose is to prove Theorem 1.1, i.e. solve the H^p-problem with Berndtsson's explicit formulas. Also this method is considerably simpler when $m = 2$. The main ingredients in the proof are careful estimates of certain Hefer functions of a bounded function, see §4, and the $T1$-theorem for Carleson mesures from [**ChJ**].

The plan of this paper is the following. First we recall some preliminary facts about integral formulas and harmonic analysis in strictly pseudoconvex domains (§2) and then in §3 we give the proof of our main result, relying on a some propositions which are proved in the succeeding paragraphs, §4-§7. Finally in §8 we briefly discuss the Koszul complex approach.

2. Preliminaries.

We start by recalling some facts about harmonic analysis and integral representation in a strictly pseudoconvex domain $D = \{\rho < 0\}$ with C^3 boundary (although C^2 is enough at several instances) where ρ is strictly plurisubharmonic in a neighborhood of $\bar{D}$ and $d\rho \neq 0$ on ∂D. For more details see e.g. [**CoW**], [**AnC1**] and the references given there.

A vector v at $p \in \partial D$ is complex tangential if v is a tangent vector, i.e. $d\rho|_p v = 0$, and $d^c\rho|_p v = 0$. Here d^c is the real operator $i(\bar{\partial} - \partial)$. A K-basis (K =Koranyi) at $p \in \partial D$ is a basis of neighborhoods $B_t(p) \subset \partial D$, $t > 0$, at p such that $B_t(p)$ has length $\sim \sqrt{t}$ in all complex tangential directions and $\sim t$ in the last one. Then clearly $|B_t(p)| \sim t^n$. Sometimes we consider neighborhoods $Q_t(p) \subset \bar{D}$ which have also extension $\sim t$ into D, so that $|Q_t(p)| \sim t^{n+1}$. Any two K-bases $B_t(p)$ and $B'_t(p)$ are equivalent, i.e. $B_{ct} \subset B'_t \subset B_{t/c}$, $t > 0$, for some constant $c > 0$. For instance, it $x_2, ..., x_{2n}$ are local coordintes at $p \in \partial D$ such that $x(p) = 0$ and $dx_2|_p$ and $d^c\rho|_p$ are colinear, then $B_t(p) = \{x; |x_2| + \sum x_j^2 < t\}$ is a K-basis at p.

If now $B_t(p)$ is any continuous choice of a K-basis at each $p \in \partial D$ one can put $\sigma(p, z) = \inf\{t;\ z \in B_t(p)\}$ and $d(z, w) = \frac{1}{2}(\sigma(z, w) + \sigma(w, z))$. Then

$$d(z, w) + d(w, \zeta) \leq Cd(z, \zeta).$$

Since also

$$|B_{2t}(p)| \leq C|B_t(p)|,$$

∂D is a homogeneous space, so a lot of tools of harmonic analysis are available. By replacing $B_t(p)$ by $Q_t(p)$, $d(\zeta, z)$ extends to $D \times \partial D$.

For $p > 0$ we put

$$H^p = \left\{ f \in \mathcal{O}(D);\ \sup_{\epsilon > 0} \int_{\partial D_\epsilon} |f|^p d\sigma < \infty \right\},$$

where $D_\epsilon = \{\rho < -\epsilon\}$ and $d\sigma$ is (some) surface measure. It is well-known that any $f \in H^p$ has admissible (i.e. "non-tangential" with respect to the balls $B_t(p)$) boundary values f^* a.e. $[d\sigma]$ and that f is the Poisson integral (or the Bergman-Poisson dito) of f^* if $p \geq 1$.

An $f \in L^1_{\text{loc}}(\partial D)$ is in BMO if

$$\sup_{t>0, p\in\partial D} \frac{1}{|B_t(p)|} \int_{B_t(p)} |u - u_{B_t(p)}| d\sigma = \|u\|_* < \infty,$$

where $u_{B_t(p)}$ is the mean value of u over $B_t(p)$. We also put BMOA $=$ BMO $\cap \mathcal{O}(D)$.

Since ∂D is a homogeneous space there is also an atomic $\mathcal{H}^1$-space on ∂D whose dual is BMO.

A measure μ in D is a Carleson measure if

$$|\mu(Q_t(p))| \leq Ct^n, \quad t \in \partial D, \quad t > 0,$$

and for such measures the Carleson-Hörmander inequality holds;

$$\int_D |g|^p d\mu \leq C_p \|g\|_{H^p}^p, \quad g \in H^p, \quad p > 0.$$

If $f \in H^p$, $p < \infty$, then

$$\int_D (-\rho|\partial f|^2 + |\partial\rho \wedge \partial f|^2)|f|^{p-2} \lesssim \|f\|_{H^p}^p, \tag{2.1}$$

and if $f \in$ BMOA, then

$$-\rho|\partial f|^2 + |\partial\rho \wedge \partial f|^2 \tag{2.2}$$

is a Carleson measure with Carleson norm bounded by $\|f\|_*^2$.

For smooth functions f and g we put

$$(f, g) = \int_{\partial D} f\bar{g}d\sigma. \tag{2.3}$$

Let $H_0^1 = \{f \in H^1;\ f(0) = 0\}$ (0 is any point in D). Via the pairing (2.3), BMOA is the dual space of H_0^1.

If $v(\zeta, z) : \overline{D \times D} \to \mathbb{C}^n$ satisfies

$$2\text{Re}v \geq -\rho(\zeta) - \rho(z) + \delta|\zeta - z|^2$$

and

$$d_\zeta\bar{v}|_{\zeta=z} = -d_z v|_{\zeta=z} = -\partial\rho(\zeta).$$

then $Q_t(p) = \{\zeta \in D;\ |v(p, \zeta)| < t\}$ is a K-basis at p (take local coordinates $x_1 = -\rho$, $x_2 = \text{Im}v$ and $x_3, \ldots, x_{2n}$ arbitrary). In particular, $|v(\zeta, z)|$ is compatible with $d(\zeta, z)$, and we have the well-known estimates

$$\int_{d(w,z)<d} \frac{d\sigma(w)}{|v(w,z)|^{n-\alpha}} \lesssim d^\alpha, \tag{2.4}$$

$$\int_{d(w,z)>d} \frac{d\sigma(w)}{|v(w,z)|^{n+\alpha}} \lesssim d^{-\alpha}, \tag{2.5}$$

$$\int_{\partial D} \frac{d\sigma(w)}{|v(w,z)|^{n+\alpha}} \lesssim \left(\frac{1}{-\rho(z)}\right)^\alpha, \tag{2.6}$$

and

$$\int_D \frac{(-\rho(w))^{\beta-1} d\lambda(w)}{|v(w,z)|^{n+\alpha+\beta}} \lesssim \left(\frac{1}{-\rho(z)}\right)^{\alpha}, \tag{2.7}$$

if α and β are positive. Furthermore, see [**AnC1**, Lemma 5.2], if $\alpha, \beta < n$

$$\int_{\partial D} \frac{d\sigma(w)}{d(w,z)^{\alpha} d(w,\zeta)^{\beta}} \lesssim \frac{1}{d(z,\zeta)^{\alpha+\beta-n}} \tag{2.8}$$

if $\alpha+\beta > n$ and the integral is bounded if $\alpha+\beta < n$.

We also need the simple estimate

$$|v(w,z) - v(w,\zeta)| \lesssim \sqrt{d(\zeta,z)(d(w,z)+d(w,\zeta))}, \tag{2.9}$$

see [**AnC1**, formula (6.1)].

One can choose such a v that is holomorphic in z for fixed $\zeta \in \bar{D}$, and for the rest of this paper v denote such a choice. Then we have the representation formula

$$Hu(z) = c \int_{\partial D} \frac{q \wedge (\bar{\partial} q)^{n-1} u}{v(\zeta,z)^n}, \quad z \in D, \tag{2.10}$$

where $q = \sum Q_j d\zeta_j$, $\sum Q_j(z_j - \zeta_j) = v(\zeta,z)$ and $q(\zeta,z)$ is holomorphic in z. Moreover, $q(\zeta,z) = \partial\rho(\zeta) + \mathcal{O}(|\zeta - z|)$ and $\bar{\partial} q = \partial\bar{\partial}\rho + \mathcal{O}(|\zeta - z|)$. Clearly, Hu is holomorphic in D if $u \in L^1(\partial D)$ and, by the Cauchy-Fantappie formula, $Hu = u$ if u is (the boundary values of) a holomorphic function. In fact, Hu has admissible boundary values a.e. if $u \in L^p(\partial D)$, $p > 1$, and this operator maps $L^p(\partial D)$ into H^p, BMO into BMOA, and $\mathcal{H}^1$ boundedly into $L^1(\partial D)$.

When $z \in \partial D$, we let $Hu(z)$ denote the boundary values of Hu. This operator is closely related to a singular integral operator on ∂D. Let

$$H_\delta u(z) = c \int_{d(\zeta,z) > \delta} \frac{q \wedge (\bar{\partial} q)^{n-1} u}{v(\zeta,z)^n}, \quad z \in \partial D,$$

and

$$H_{pv} u = \lim_{\delta \to 0} H_\delta u.$$

Then,

$$Hu = \frac{1}{2} u + H_{pv} u.$$

In particular, if G is holomorphic, then $H_{pv} G = \frac{1}{2} G$. Furthermore, if $H^* u = \sup_{\delta > 0} |H_\delta u|$ and M is the Hardy-Littlewood maximal operator on ∂D (w.r.t. the balls $B_t(p)$), then by Cotlar's inequality, see e.g. [**J**],

$$H^* u \lesssim Mu + MH_{pv} u.$$

Thus

$$\left|\int_{d(\zeta,z)>\delta} \frac{q\wedge(\bar\partial q)^{n-1}G}{v(\zeta,z)^n}\right| \le C\|G\|_{H^\infty}, \tag{2.11}$$

with C independent of δ.

Finally we also recall the weighted representation formulas

$$B_\alpha u(z) = c_\alpha \int_D \frac{[(-\rho)^\alpha \bar\partial q + n(-\rho)^{\alpha-1}\bar\partial\rho\wedge q]}{v(\zeta,z)^{n+\alpha}}(\bar\partial q)^{n-1}u(\zeta), \quad z\in D$$

for holomorphic functions u and $\alpha>0$. In fact, if $\alpha\to 0$ one obtains (2.10).

3. The division formulas.

Suppose that $G_1,\dots,G_m$ are given holomorphic functions without common zeros, so that $|G|^2=\sum|G_j|^2>0$, and let $H_j^\ell(\zeta,z)$ be Hefer functions to G_j. This means that they are holomorphic solutions of

$$\sum_{\ell=1}^n H_j^\ell(\zeta,z)(\zeta_\ell - z_\ell) = G_j(\zeta)-G_j(z).$$

We also define the $(1,0)$-forms,

$$h_j = \sum_{\ell=1}^n H_j^\ell d\zeta_\ell, \tag{3.1}$$

and

$$g_i = \partial\frac{G_i}{|G|^2}. \tag{3.2}$$

If ρ is a strictly plurisubharmonic C^3 defining function for D and $v(\zeta,z)$ and the $(1,0)$-form $q=\sum Q_\ell d\zeta_\ell$ are as in §2, then (for $\alpha>0$) we can define the operators, cf. [**B1**],

$$T_j\phi(z) = \int_{\zeta\in D}\frac{\bar G_j}{|G|^2}\sum_{k=0}^n c_{k,\alpha}\Big(\frac{G(z)\cdot\bar G}{|G|^2}\Big)^{n-k}(-\rho)^{\alpha-1} \tag{3.3}$$
$$\frac{[-\rho\bar\partial q+(n-k)\bar\partial\rho\wedge q]\,(\bar\partial q)^{n-k-1}\sum_{|K|=k} h_{K_1}\wedge\dots\wedge h_{K_k}\wedge\bar g_{K_1}\wedge\dots\wedge\bar g_{K_k}}{v(\zeta,z)^{n+\alpha-k}}\phi.$$

Here

$$G(z)\cdot\bar G = \sum_{j=1}^m G_j(z)\bar G_j(\zeta).$$

If ϕ is holomorphic and the integrals converge in some reasonable way then, see [**B1**],

$$\sum_{j=1}^{m} G_j T_j \phi = \phi,$$

so $T\phi$ provides a solution to the division problem. Hence Theorem 1.1 is a direct consequence of the next theorem.

Theorem 3.1. *Let D be a strictly pseudoconvex domain with C^3-boundary and $G_1, ..., G_m$ bounded holomorphic functions in D such that*

$$\sum_{1}^{m} |G_j|^2 \geq \delta^2, \tag{3.4}$$

for some $\delta > 0$. Then for $\alpha > \min(n, m-1)/2$ each operator T_j defined above can be written as a sum $\sum G^\ell T^\ell$, where each G^ℓ is a product of some of the functions G_j and the operators T^ℓ take H^p into $L^p(\partial D)$, $1 \leq p < \infty$, and H^∞ into BMO.

Remark 3.1. Since

$$0 = \bar\partial 1 = \bar\partial \sum_{j=1}^{m} G_j \frac{\bar G_j}{|G|^2} = \sum_{j=1}^{m} G_j \bar g_j,$$

$g_1, ..., g_m$ are linearly dependent and hence the terms for $k \geq m$ vanish in the sum (3.3). In particular, if $m = 2$ then only terms with one factor h_j occur; this simplifies the argument in §6.

Remark 3.2. The formula (3.3) provides a solution to the division problem if $\bar G/|G|^2$ is replaced by any smooth solution γ_j to $\sum G_j \gamma_j = 1$. In particular, one can choose γ_j such that such that $\bar g_j = \bar\partial \gamma_j$ are Carleson measures (see [**Ca**] for $n = 1$ and [**V2**] for the multidimensional case). Then still Theorem 3.1 holds and, as for the Kozsul complex method cf. §7, this choice considerably simplifies the estimation of the, though less explicit, solutions. However, even in this case the estimates of the Hefer functions in Proposition 3.2. are required.

We now state some results (mainly) about the Hefer functions, Propositions 3.2 and 3.3, from which we can conclude the proof of Theorem 3.1, whereas the proofs of these propositions are left to later paragraphs.

Proposition 3.2. *If $G_1, ..., G_m$ are bounded holomorphic functions and the forms g_j are defined by (3.2), then*

$$|g_1 \wedge ... \wedge g_k| \lesssim \frac{|\partial G|}{(-\rho)^{\frac{k-1}{2}}} \lesssim \frac{1}{(-\rho)^{1+\frac{k-1}{2}}}, \tag{3.5}$$

and

$$|\partial\rho \wedge g_1 \wedge \ldots \wedge g_k| \lesssim \frac{|\partial\rho \wedge \partial G|}{(-\rho)^{\frac{k-1}{2}}} \lesssim \frac{1}{(-\rho)^{\frac{k}{2}}}, \tag{3.6}$$

where $|G| = \sqrt{\sum_1^m |G_j|^2}$. *Moreover, there are "good" Hefer functions* H_j^ℓ *corresponding to the* G_j, *such that for* $z \in \partial D$ *the Hefer forms* h_j *defined by* (3.1) *satisfy*

$$|h_1 \wedge \ldots \wedge h_k| \lesssim \frac{1}{d(\zeta, z)^{1+\frac{k-1}{2}}}, \tag{3.7}$$

and

$$|\partial\rho \wedge h_1 \wedge \ldots \wedge h_k| \lesssim \frac{1}{d(\zeta, z)^{\frac{k}{2}}}. \tag{3.8}$$

The estimates (3.5) and (3.6) are simple and well known whereas the estimates of the Hefer functions are more delicate. We postpone the proof to §4 where also our exact choice of the Hefer functions is described.

Remark 3.3. Using Proposition 3.2 and that

$$|q - \partial\rho| \lesssim |\zeta - z| \lesssim \sqrt{d(\zeta, z)},$$

we get (if α is large enough) the rough estimate

$$|T\phi(z)| \lesssim \int_D \frac{|\phi(\zeta)| d\lambda(\zeta)}{d(\zeta, z)^{n+1}}. \tag{3.9}$$

Unfortunately this integral is infinite if $z \in \partial D$ but if we for some reason can gain just an ϵ in the exponent in the denominator, then the $L^p(\partial D)$-norm of the integral is less than a constant times $\|\phi\|_{H^p}$. In fact, if ϕ is in BMOA or even in H^p for a sufficiently large p, then the integral is bounded. On the other hand, if ϕ is in H^1 then by (2.6), the $L^1(\partial D)$-norm is

$$\lesssim \int_D (-\rho)^{-1+\epsilon} |\phi| \lesssim \|\phi\|_{H^1}.$$

By this observation we may diregard various error term.

We will also need

Proposition 3.3. *If* $\mathcal{L}$ *is a smooth* $(1,0)$*-field, then the forms* g_j *from Proposition* 3.2 *satisfy the following additional estimates,*

$$|\mathcal{L}(\bar{g}_1 \wedge \ldots \wedge \bar{g}_k)| \lesssim \frac{|\partial G|^2}{(-\rho)^{\frac{k-1}{2}}} \tag{3.10}$$

and

$$(3.11)\qquad |\bar\partial\rho\wedge\mathcal{L}(\bar g_1\wedge\ldots\wedge\bar g_k)|\lesssim\frac{|\partial G||\partial\rho\wedge\partial G|}{(-\rho)^{\frac{k-1}{2}}}.$$

To decompose T_j we first put each factor $G^\ell(z)$ of G_j:s occurring from the factors $\left(\frac{G(z)\cdot\bar G}{|G|^2}\right)^{n-k}$ in (3.3) outside the integrals (they will be incorperated in the factors $G^\ell(z)$ in Theorem 3.1), and for simplicity we just denote the remaining factor by $\mathcal{G}$. Thus $\mathcal{G}$ is just a product of a certain number of factors $\bar G_j/|G|^2$.

First consider the terms corresponding to $k\geq 1$. In view of Remark 3.3 the factors $\bar\partial q$ can be replaced by $\bar\partial\partial\rho$ (modulo negligable terms) and to simplify notation we omit them in the sequel. They play no other role then to achieve full bidegree in the $d\zeta$. Any remaining integral from (3.3) is then either

$$T_K\phi(z)=\int_{\zeta\in D}(-\rho)^{\frac{k-1}{2}}\mathcal{G}\bar g_{K_1}\wedge\ldots\wedge\bar g_{K_k}\wedge M_K(\zeta,z)\phi(\zeta),$$

where

$$(3.12)\qquad M_K(\zeta,z)=(-\rho)^{\alpha-\frac{k-1}{2}}\frac{h_{K_1}(\zeta,z)\wedge\ldots\wedge h_{K_k}(\zeta,z)}{v(\zeta,z)^{n+\alpha-k}},$$

or

$$T'_K\phi(z)=\int_{\zeta\in D}(-\rho)^{\frac{k}{2}-1}\mathcal{G}\bar\partial\rho\wedge\bar g_{K_1}\wedge\ldots\wedge\bar g_{K_k}\wedge M'_K(\zeta,z)\phi(\zeta),$$

where

$$(3.13)\qquad M'_K(\zeta,z)=(-\rho)^{\alpha-\frac{k}{2}}\frac{q\wedge h_{K_1}(\zeta,z)\wedge\ldots\wedge h_{K_k}(\zeta,z)}{v(\zeta,z)^{n+\alpha-k}}.$$

In view of (3.7) and (3.8) (and assuming that G_j and ϕ are smooth up to the boundary) $T_K\phi(z)$ and $T'_K\phi(z)$ has meaning even for $z\in\partial D$. However, in order to be able to estimate them we are forced to decompose the kernels $M_K(\zeta,z)$ and $M'_K(\zeta,z)$ further. We say that a locally integrable function b in D is a Carleson function if $\|b\|_C^2=$ the Carleson norm of $-\rho|b|^2+\sup(-\rho|b|)^2$ is finite.

Proposition 3.4. *Any kernel M_K or M'_K can be written as a sum $\sum G^\ell(z)M^\ell(\zeta,z)$, where each $G^\ell(z)$ is a product of some of the G_j and such that the dual operators $\mathcal{M}_\ell^*$ satisfy the following estimates.*

a)

$$\|\mathcal{M}_\ell^*\psi\|_{L^\infty} \lesssim \|\psi\|_{L^\infty(\partial D)}. \tag{3.14}$$

If $d\tau$ is a Carleson measure, then

$$\int_D |\mathcal{M}_\ell^*\psi|^p d\tau \lesssim \|\psi\|^p_{L^p(\partial D)} \tag{3.15}$$

for ψ in $L^p(\partial D)$, $1 < p < \infty$, and

$$\int_D |\mathcal{M}_\ell^*\psi| d\tau \lesssim \|\psi\|_{\mathcal{H}^1}. \tag{3.16}$$

b) *Moreover, if $\mathcal{L}$ is a smooth $(1,0)$-field then*

$$\|\mathcal{M}_\ell^*\psi\|_C \le \|\psi\|_{L^\infty(\partial D)},$$

$$\int_D (-\rho)|\mathcal{L}\mathcal{M}_\ell^*\psi|^2 \lesssim \|\psi\|^2_{L^2(\partial D)}, \tag{3.17}$$

and if b is a Carleson function then

$$\int_D (-\rho)|\mathcal{L}\mathcal{M}_\ell^*\psi||b| \lesssim \|\psi\|_{\mathcal{H}^1}\|b\|_C. \tag{3.18}$$

This proposition gives rise to the decomposition $G^\ell T^\ell$ of each T_K and T'_K and hence of the proposed decomposition of T in Theorem 3.1 and is the key point in the proof.

To estimate the boundary values of each $T^\ell\phi(z)$ we shall use duality, and hence we integrate against some locally integrable function ψ on ∂D. Then by Fubini's theorem,

$$\int_{\partial D} (T^\ell\phi)(z)\psi(z)d\sigma(z) = \int_{\zeta\in D} (-\rho)^{\frac{k-1}{2}}\mathcal{G}\bar{g}_{K_1}\wedge\ldots\wedge\bar{g}_{K_k}\mathcal{M}_\ell^*\psi(\zeta)\phi(\zeta),$$

or

$$\int_{\partial D} (T^\ell\phi)(z)\psi(z)d\sigma(z) = \int_{\zeta\in D} (-\rho)^{\frac{k}{2}-1}\mathcal{G}\bar\partial\rho\wedge\bar{g}_{K_1}\wedge\ldots\wedge\bar{g}_{K_k}\mathcal{M}_\ell^*\psi(\zeta)\phi(\zeta).$$

Now we apply the Wolff trick that allow us to increase the power of $-\rho$ with one unit at the cost of the action of a certain smooth $(1,0)$-vector field $\mathcal{L}$

on the rest of the integrand. In the ball $\mathcal{L}$ can be chosen as $\sum \zeta_j \frac{\partial}{\partial \zeta_j}$. Then, modulo negligable terms e.g.,

$$\int_{\partial D}(T^\ell\phi)(z)\psi(z)d\sigma(z) \sim \int_{\zeta\in D}(-\rho)^{\frac{k+1}{2}}\mathcal{L}[\mathcal{G}\bar{g}_{K_1}\wedge\ldots\wedge\bar{g}_{K_k}]\mathcal{M}_\ell^*\psi(\zeta)\phi(\zeta)$$
$$+\int_{\zeta\in D}(-\rho)^{\frac{k+1}{2}}\mathcal{G}\bar{g}_{K_1}\wedge\ldots\wedge\bar{g}_{K_k}\mathcal{L}(\mathcal{M}_\ell^*\psi)(\zeta)\phi(\zeta)$$
$$+\int_{\zeta\in D}(-\rho)^{\frac{k+1}{2}}\mathcal{G}\bar{g}_{K_1}\wedge\ldots\wedge\bar{g}_{K_k}(\mathcal{M}_\ell^*\psi)(\zeta)\mathcal{L}(\phi)(\zeta).$$

and analogously for the other type of terms.

In view of Propositions 3.2 and 3.3 we get the estimate

$$\left|\int_{\partial D}T^\ell\phi(z)\psi(z)d\sigma(z)\right| \lesssim \int_D(-\rho)|\partial G|^2|\mathcal{M}_\ell^*\psi||\phi| + \int_D(-\rho)|\partial G||\mathcal{L}\mathcal{M}_\ell^*\psi||\phi| + \int_D(-\rho)|\partial G||\mathcal{M}_\ell^*\psi||\partial\phi|, \tag{3.19}$$

or

$$\left|\int_{\partial D}(T^\ell\phi)\psi(z)d\sigma(z)\right| \lesssim \int_D\sqrt{-\rho}|\partial G||\partial\rho\wedge\partial G||\mathcal{M}_\ell^*\psi||\phi| + \int_D\sqrt{-\rho}|\partial\rho\wedge\partial G||\mathcal{L}\mathcal{M}_\ell^*\psi||\phi| + \int_D\sqrt{-\rho}|\partial\rho\wedge\partial G||\mathcal{M}_\ell^*\psi||\partial\phi|. \tag{3.20}$$

We want the estimates

$$||\psi||_{\mathcal{H}^1}||\phi||_{H^\infty}, \tag{3.21}$$
$$||\psi||_{L^q(\partial D)}||\phi||_{H^p} \tag{3.22}$$

and

$$||\psi||_{L^\infty(\partial D)}||\phi||_{H^1} \tag{3.23}$$

for each term in (3.19) and (3.20). These together imply Theorem 3.1.

The first terms in any of (3.19) and (3.20) are handled by Hölder's inequality, (3.14)-(3.16) and the Carleson Hörmander inequality, as $(-\rho)|\partial G|^2$ and $\sqrt{-\rho}|\partial G||\partial\rho\wedge\partial G|$ both are Carleson measures, see §2.

The third term in (3.19) is by (3.15) and (3.16) estimated by $||\psi||_{L^\infty(\partial D)}||\phi||_{H^1}$ and $||\psi||_{\mathcal{H}^1}||\phi||_{H^\infty}$, but the last estimate, cf. §2, can be sharpened to $||\psi||_{\mathcal{H}^1}||\phi||_{\mathrm{BMOA}}$, and hence we can interpolate and get the intermediate estimates (3.22). The third term in (3.20) is handled similarily.

For the second term in (3.19) we notice that

$$|\partial G||\phi| \lesssim |\partial(G\phi)| + |\partial\phi|$$

(and similarily for the second one in (3.20)) so it is enough to get the desired estimates for the term

$$\int_D (-\rho)|\partial\phi||\mathcal{L}\mathcal{M}_\ell^*\psi|,$$

(letting ϕ play the role of both ϕ and $G\phi$) and this is accomplished as for the third term(s), using (3.17) and and that $\mathcal{L}\mathcal{M}_\ell^*\psi$ is a Carleson function if ψ is in L^∞.

Finally we consider the term corresponding to $k = 0$ in (3.3),

$$\int_D \mathcal{G}\,[(-\rho)^\alpha + n(-\rho)^{\alpha-1}\bar\partial\rho\wedge q]\wedge(\bar\partial q)^{n-1}\frac{\phi(\zeta)}{v(\zeta,z)^{n+\alpha}} = \sum G^l T^l\phi$$

where $T^l\phi$ are nothing but the weighted Bergman-type operator B_α acting on the function $\mathcal{G}\phi$, and the necessary estimate follows from

Proposition 3.5. *Let B_α, $\alpha > 0$, and $\mathcal{G}$ be as before. Then*

$$T^l : \phi \mapsto B_\alpha(\mathcal{G}\phi)$$

maps $H^p \to H^p, 1 \le p < \infty$ and $H^\infty \to$ BMOA.

Proof. We may assume that $\mathcal{G}$ and ϕ are smooth up to the boundary. Then $B_\alpha(\mathcal{G}\phi)$ has continuous boundary values defined by the formula

$$B_\alpha(\mathcal{G}\phi)(z) = \int_D B(\zeta,z)(\mathcal{G}(\zeta)\phi(\zeta) - \mathcal{G}(z)\phi(z))d\lambda(\zeta) + \mathcal{G}(z)\phi(z),$$

since $B_\alpha 1 = 1$. It is clearly enough to estimate the integral, and to this end we integrate against some ψ on the boundary and get by Fubini's theorem modulo innocent terms

$$I = \int_D (-\rho)^{\alpha-1}\int_{\partial D}\frac{(\mathcal{G}(\zeta)\phi(\zeta) - \mathcal{G}(z)\phi(z))\psi(z)}{v(\zeta,z)^{n+\alpha}}d\sigma(z).$$

Let

$$P\psi(\zeta) = \int_{\partial D}(-\rho)^\alpha\frac{\psi(z)d\sigma(z)}{v(\zeta,z)^{n+\alpha}}.$$

By the Wolff trick I is comparable to

$$\int_D \mathcal{G}\partial\phi P\psi$$

since $v(\zeta, z)$ is anti-holomorphic in ζ to the first order. It can also occur derivatives of $\mathcal{G}$ but as above these terms can be reduced to the case when the derivatives occur on ϕ.

Now we can apply the Wolff trick again but with the vector field $\bar{\mathcal{L}}$ instead. We then get the terms like

$$\int_D (-\rho)\overline{\partial \mathcal{G}}\partial\phi P\psi + \int_D (-\rho)\mathcal{G}\partial\phi\nabla P\psi$$

and both of these admit the estimates (3.21) to (3.23) by means of Proposition 7.1. This concludes the proof. □

4. The Hefer functions and proof of Propositions 3.2 and 3.3.

In this section we define our Hefer functions and prove Propositions 3.2 and 3.3.

First notice that the Cauchy-Fantappie representation formula (2.10) for a holomorphic function G can be written

$$G(z) = \int_{\partial D} \frac{A(w,z)G(w)d\sigma(w)}{v(w,z)^n},$$

where $A(w, z)$ is of class C^1 (since ρ is assumed to be C^3) and holomorphic in z. Now

$$\begin{aligned} G(z) - G(\zeta) &= \int_{\partial D} \left(\frac{A(w,z)}{v(w,z)^n} - \frac{A(w,\zeta)}{v(w,\zeta)^n} \right) G(w)d\sigma(w) \\ &= \int_{\partial D} \frac{A(w,z) - A(w,\zeta)}{v(w,z)^n} G(w)d\sigma(w) \\ &\quad + \sum_{k=1}^{n} \int_{\partial D} \frac{A(w,\zeta)(v(w,\zeta) - v(w,z))G(w)d\sigma(w)}{v(w,z)^k v(w,\zeta)^{n+1-k}}. \end{aligned}$$

We can write

$$A(w,z) - A(w,\zeta) = \sum_1^n A_j(w,z,\zeta)(z_j - \zeta_j),$$

for some C^1-functions A_j (they are still C^1 since A is holomorphic in z), which in addition are holomorphic in ζ and z. Moreover, as $v(w, z) = Q(w, z) \cdot (z - w)$, we have

(4.1)
$$\begin{aligned} v(w,\zeta) - v(w,z) &= -Q(w,\zeta) \cdot (z - \zeta) + (Q(w,\zeta) - Q(w,z)) \cdot (z - w) \\ &= (-Q(w,\zeta) + \mathcal{O}(|z - w|)) \cdot (z - \zeta) = U^0(z,\zeta,w) \cdot (z - \zeta) \end{aligned}$$

and similarily

(4.2)
$$v(w,\zeta)-v(w,z)=(-Q(w,z)+\mathcal{O}(|\zeta-w|))\cdot(z-\zeta)=U^1(z,\zeta,w)\cdot(z-\zeta).$$

If we now let $a=\sum A_j d\zeta_j$, $u_n=\sum U_j^0 d\zeta_j$ and $u_k=\sum U_j^1 d\zeta_j$ if $k<n$ we can define the Hefer form of the function G as

$$(4.3)\quad h(\zeta,z)=\int_{\partial D}\frac{a(w,\zeta,z)}{v(w,z)^n}G(w)d\sigma(w)$$
$$+\sum_{k=1}^{n}\int_{\partial D}\frac{A(w,\zeta)u_k(w,\zeta,z)G(w)d\sigma(w)}{v(w,z)^k v(w,\zeta)^{n+1-k}}=\sum_{k=0}^{n}h^k(\zeta,z).$$

The reason for having different definitions of u_k when k is n and less than n is merely technical. Then $u_n=-q(z,\zeta)+\mathcal{O}(|z-w|)$ (with the obvious definition of $q(z,\zeta)$) so that the term corresponding to the error $\mathcal{O}$ is not singular.

Proof of Proposition 3.2. The statements concerning the g_j are well known and quite simple. First observe that if G is any bounded holomorphic function in D, then

$$|\partial G|\lesssim\frac{1}{-\rho},\quad |\partial\rho\wedge\partial G|\lesssim\frac{1}{\sqrt{-\rho}}.$$

This follows e.g. from the representation formula (2.10), noting that $\partial_z v(w,z)=-\partial\rho+\mathcal{O}(|w-z|)$. Thus

$$(4.4)\qquad \partial G(z)=a(z)\partial\rho(z)+b(z),$$

where $|a|\lesssim 1/(-\rho)$ and $|b|\lesssim 1/\sqrt{-\rho}$. Next notice that

$$(4.5)\qquad g_j=\sum\omega_{jk}\partial G_k,$$

where ω_{jk} are bounded. The estimates (3.5) and (3.6) now follow from (4.4) and (4.5).

The corresponding estimates for the Hefer forms, (3.7) and (3.8), are more involved. It is enough to prove that

$$h^k(\zeta,z)=\mathcal{O}\left(\frac{1}{d(\zeta,z)}\right)\partial\rho(\zeta)+\mathcal{O}\left(\frac{1}{\sqrt{d(\zeta,z)}}\right),\quad k=0,1,...,n.$$

The term $h^0(\zeta, z)$ is bounded. To see this, first observe that

$$\begin{aligned} a(w,\zeta,z) &= \frac{A(w,z)+\mathcal{O}(|w-z|)}{A(z,z)}(a(z,\zeta,z)+\mathcal{O}(|w-z|)) \\ &= \frac{A(w,z)a(z,\zeta,z)}{A(z,z)}+\mathcal{O}(|w-z|). \end{aligned}$$

Hence

$$h^0(\zeta,z)=\frac{a(z,\zeta,z)}{A(z,z)}G(z)+\mathcal{O}(1)\int_{\partial D}\frac{d\sigma(w)}{d(w,z)^{n-1/2}}=\mathcal{O}(1).$$

The estimate for the terms h^k when $2 \le k \le n-1$ are also simple. By (4.3), $u_k(w,\zeta,z) = -q(\zeta,z)+\mathcal{O}(|\zeta - w|)$, and hence by (2.8),

$$\begin{aligned} &h^k(\zeta,z) \\ &= q(\zeta,z)\mathcal{O}(1)\int_{\partial D}\frac{d\sigma(w)}{d(w,z)^k d(w,\zeta)^{n+1-k}}+\mathcal{O}(1)\int_{\partial D}\frac{d\sigma(w)}{d(w,z)^k d(w,\zeta)^{n+1/2-k}} \\ &= q(\zeta,z)\mathcal{O}\left(\frac{1}{d(z,\zeta)}\right)+\mathcal{O}\left(\frac{1}{\sqrt{d(z,\zeta)}}\right)=\mathcal{O}\left(\frac{1}{d(z,\zeta)}\right)\partial\rho+\mathcal{O}\left(\frac{1}{\sqrt{d(z,\zeta)}}\right). \end{aligned}$$

The terms h^1 and h^n are harder as they involve singular integrals. We first consider h^n. As above it is easy to see that the contribution from the $\mathcal{O}$-term in (4.1) is bounded by $\mathcal{O}\left(\frac{1}{\sqrt{d(\zeta,z)}}\right)$. It remains to consider

$$\begin{aligned} J &= q(z,\zeta)\int_{\partial D}\frac{G(w)A(w,z)d\sigma(w)}{v(w,z)^n v(w,\zeta)} \\ &= q(z,\zeta)\left[\frac{G(z)}{v(z,\zeta)}+\int_{\partial D}\frac{G(w)A(w,z)}{v(w,z)^n}\left(\frac{1}{v(w,\zeta)}-\frac{1}{v(z,\zeta)}\right)d\sigma(w)\right]. \end{aligned}$$

Clearly the first term is $\mathcal{O}(1/d(z,\zeta))$. As the kernel in the second one is integrable, we may assume that $z \in \partial D$. Write

$$\begin{aligned} q(z,\zeta) &\int_{\partial D} \frac{G(w)A(w,z)}{v(w,z)^n}\left(\frac{1}{v(w,\zeta)}-\frac{1}{v(z,\zeta)}\right)d\sigma(w) \\ &= q(z,\zeta)\left[\int_{d(w,z)\le Cd(z,\zeta)} \frac{G(w)A(w,z)}{v(w,z)^n}\left(\frac{1}{v(w,\zeta)}-\frac{1}{v(z,\zeta)}\right)d\sigma(w)\right. \\ &\quad + \int_{d(w,z)>Cd(z,\zeta)} \frac{G(w)A(w,z)d\sigma(w)}{v(w,z)^n v(w,\zeta)} \\ &\quad \left. - \frac{1}{v(z,\zeta)}\int_{d(w,z)>Cd(z,\zeta)} \frac{G(w)A(w,z)d\sigma(w)}{v(w,z)^n}\right] \\ &= q(z,\zeta)[\mathrm{I}+\mathrm{II}+\mathrm{III}]. \end{aligned}$$

By (2.11), III is $\mathcal{O}(1/d(z,\zeta))$. When $d(w,z) > Cd(z,\zeta)$, then $d(w,z) \sim d(w,\zeta)$ and

$$|\mathrm{II}| \lesssim \int_{d(w,z)>Cd(z,\zeta)} \frac{d\sigma(w)}{d(w,z)^{n+1}} \lesssim \frac{1}{d(z,\zeta)}.$$

Finally, since $d(w,\zeta) \lesssim d(z,\zeta)$ in I we have by (2.9) that

$$\left|\frac{1}{v(w,\zeta)}-\frac{1}{v(z,\zeta)}\right| \lesssim \frac{\sqrt{d(w,z)(d(w,\zeta)+d(z,\zeta))}}{d(w,\zeta)d(z,\zeta)} \lesssim \frac{\sqrt{d(w,z)}}{\sqrt{d(z,\zeta)}d(w,\zeta)}.$$

Thus by (2.8)

$$|\mathrm{I}| \lesssim \frac{1}{\sqrt{d(z,\zeta)}}\int_{\partial D} \frac{d\sigma(w)}{d(w,z)^{n-1/2}d(w,\zeta)} \lesssim \frac{1}{d(z,\zeta)}.$$

Summing up we have

$$h^n(\zeta,z) = q(z,\zeta)\mathcal{O}\left(\frac{1}{d(z,\zeta)}\right) = \partial\rho\,\mathcal{O}\left(\frac{1}{d(z,\zeta)}\right) + \mathcal{O}\left(\frac{1}{\sqrt{d(z,\zeta)}}\right).$$

By symmetry we get the same estimate for the term $h^1(\zeta,z)$, at least when $\zeta, z \in \partial D$. But then the estimate follows for $\zeta \in D$ from the maximum principle, since $d(z,\zeta) \sim |v(z,\zeta)|$ and $v(z,\zeta)$ is holomorphic in ζ. □

Proof of Proposition 3.3. Just notice that the ω_{jk} in (4.5) satisfy

$$\mathcal{L}\bar{\omega}_{jk} = \mathcal{O}(\partial G).$$

Then

$$\mathcal{L}\bar{g}_j = \sum(\mathcal{L}\bar{\omega}_{jk})\overline{\partial G_k} = \sum \mathcal{O}(\partial G)\overline{\partial G_k}$$

so the proposition follows from (3.5) and (3.6). □

5. Decomposition of the Hefer forms, proof of Proposition 3.4 a).

To make the arguments as comprehensive as possible, we mostly restrict ourselves to the ball. The general case is handled along the same lines, but with a myriad of various error terms.

We first describe the decomposition of each $M_K(\zeta, z)$ into the sum $\sum G^\ell(z) M^\ell(\zeta, z)$. For this we need a lemma which we prove in a moment.

Lemma 5.1. *Let D be the unit ball. The Hefer form $h = \sum H^j d\zeta_j$ of some bounded function G, as defined in §4, can be decomposed as*

$$h = \frac{G(z) d(\bar{z} \cdot \zeta)}{v(z, \zeta)} + \tilde{h},$$

where $\tilde{h}$ satisfies the same estimates as h, i.e.

$$\left|\tilde{h}\right| \lesssim \frac{1}{d(\zeta, z)}, \quad \left|\partial\rho \wedge \tilde{h}\right| \lesssim \frac{1}{\sqrt{d(\zeta, z)}}, \tag{5.1}$$

but, in addition, also

(5.2)

$$\left|\tilde{h}(\zeta, z) - \tilde{h}(\zeta, z')\right| \lesssim \frac{d(z, z')^{\frac{1}{4}}}{d(\zeta, z)^{1+\frac{1}{4}}}, \quad \left|\partial\rho \wedge \left(\tilde{h}(\zeta, z) - \tilde{h}(\zeta, z')\right)\right| \lesssim \frac{d(z, z')^{\frac{1}{4}}}{d(\zeta, z)^{\frac{1}{2}+\frac{1}{4}}},$$

if $d(z, z') \leq c d(\zeta, z)$. Here d denotes the Koranyi distance, cf. §2.

The lemma says that h is the sum of one term, $\tilde{h}$, which satisfies a certain Hölder condition in the z-variable (intuitively it is differentiable a $1/4$ time), and one term that certainly does not, but which instead is quite simple. In the one-variable case,

$$h(\zeta, z) = \frac{G(z) - G(\zeta)}{z - \zeta}$$

and here of course $\tilde{h}$ is $G(\zeta)/(z - \zeta)$. Notice that $\tilde{h}$ is no longer holomorphic.

We are now prepared to make the decomposition of the kernels $M_K(\zeta, z)$ and $M'_K(\zeta, z)$ stated in Proposition 3.4. Recall that, cf. (3.12),

$$M_K(\zeta, z) = (-\rho)^{\alpha - \frac{k-1}{2}} \frac{h_{K_1}(\zeta, z) \wedge \ldots \wedge h_{K_k}(\zeta, z)}{v(\zeta, z)^{n+\alpha-k}}.$$

If we successively replace h_{K_j} with $\tilde{h}_{K_j}$, we get

(5.3)

$$M_K(\zeta, z) = \sum_j (-\rho)^{\alpha - \frac{k-1}{2}} \frac{\tilde{h}_{K_1}(\zeta, z) \wedge \ldots \wedge G_{K_j}(z) d(\bar{z} \cdot \zeta) \wedge \ldots \wedge \tilde{h}_{K_k}(\zeta, z)}{v(\zeta, z)^{n+\alpha-k} v(z, \zeta)}$$
$$+ (-\rho)^{\alpha - \frac{k-1}{2}} \frac{\tilde{h}_{K_1}(\zeta, z) \wedge \ldots \wedge \tilde{h}_{K_k}(\zeta, z)}{v(\zeta, z)^{n+\alpha-k}},$$

which defines the desired decomposition $M_K(\zeta,z)\psi = \sum G^\ell(z)M^\ell(\zeta,z)$. In the same way we get from (3.13),

(5.4)

$$M'_K(\zeta,z) = \sum_j (-\rho)^{\alpha-\frac{k}{2}} \frac{q\wedge \tilde h_{K_1}(\zeta,z)\wedge ... \wedge G_{K_j}(z)d(\bar z\cdot\zeta)\wedge\ldots\wedge \tilde h_{K_k}(\zeta,z)}{v(\zeta,z)^{n+\alpha-k}v(z,\zeta)}$$
$$+ (-\rho)^{\alpha-\frac{k}{2}} \frac{q\wedge \tilde h_{K_1}(\zeta,z)\wedge\ldots\wedge \tilde h_{K_k}(\zeta,z)}{v(\zeta,z)^{n+\alpha-k}},$$

which defines the decomposition $M'_K(\zeta,z) = \sum G^\ell(z)M^\ell(\zeta,z)$.

Thus in the ball we get exactly $k+1$ terms, involving respectively G_{K_1}, ...,G_{K_k} and 1. However, in the general case, products G^ℓ with several factors may occur.

Proof of Proposition 3.4a). Let $M(\zeta,z)$ denote one of the kernels M^ℓ from (5.3) or (5.4). Using Proposition 3.2 one readily verifies that

$$|M(\zeta,z)| \lesssim \frac{-\rho(\zeta)}{d(\zeta,z)^{n+1}}. \tag{5.5}$$

By (2.6), (3.14) immediately follows and also (3.15) is quite easily verified. To prove (3.16), we must use the additional property

$$|M(\zeta,z) - M(\zeta,z')| \lesssim \frac{-\rho(\zeta)d(z,z')^{\frac14}}{d(\zeta,z)^{n+1+\frac14}}, \tag{5.6}$$

if $d(z,z') \le cd(\zeta,z)$. This follows by iterated use of (5.2).

Now one can get (3.16) either by first applying the kernel to an atom, and then use the atomic decomposition of $\mathcal{H}^1$, or use the duality with BMO. In the latter case one has to show that the dual kernel takes Carleson measures into BMO. Such a calculation is done e.g. in in §6 in [**AnC1**]. One just has to check that essentially only the properties (5.5) and (5.6) are used. □

Proof of Lemma 5.1. By (3.7) and (3.8), (5.1) is obvious. Notice that, in the ball,

$$h(\zeta,z) = c\sum_{k=1}^{n}\int_{\partial D}\frac{d(\bar w\cdot\zeta)G(w)d\sigma(w)}{v(w,z)^k v(w,\zeta)^{n+1-k}}, \tag{5.7}$$

where $v(w,z) = 1-\bar w\cdot z$. We claim that all the terms in this sum, but the one corresponding to $k=n$ satisfy (5.2). The problem with this term is that it already is a singular integral when $z\in\partial D$, and hence it does not admit the estimate (5.2) (unless $G(z)$ is Hölder continuous on ∂D).

To handle this term we first replace $d(\bar{w}\cdot\zeta)$ by $d(\bar{z}\cdot\zeta)$ and then the factor $v(w,\zeta)$ by $v(z,\zeta)$. Then we get the Cauchy integral

$$c\int_{\partial D}\frac{d(\bar{z}\cdot\zeta)G(w)d\sigma(w)}{v(w,z)^n v(z,\zeta)}=\frac{G(z)d(\bar{z}\cdot\zeta)}{v(z,\zeta)}$$

plus the error terms

$$\begin{aligned}&c\int_{\partial D}\frac{d((\bar{z}-\bar{w})\cdot\zeta)G(w)d\sigma(w)}{v(w,z)^n v(w,\zeta)}\\ &+c\int_{\partial D}\frac{d(\bar{z}\cdot\zeta)(v(z,\zeta)-v(w,\zeta))G(w)d\sigma(w)}{v(w,z)^n v(w,\zeta)v(z,\zeta)}.\end{aligned}\tag{5.8}$$

We have to verify that these two terms and the ones in the sum (5.7) for $k<n$ satisfy the estimate (5.2).

The estimate is based on (2.8) but we have to be careful only to apply it when α and β are less than n. This is hardest for the terms in (5.8), and we only consider them.

Let $\tilde{h}_0(\zeta,z)$ be the last term in (5.8). Note that $d(z,z')\leq cd(z,\zeta)$ for c small enough, implies that $d(z,\zeta)\approx d(z',\zeta)$. Also

$$d(\bar{z}\cdot\zeta)=\partial\rho(\zeta)+\mathcal{O}(|\zeta-z|).$$

Let

$$\Delta=\Delta(z,z',\zeta,w)=\frac{d(\bar{z}\cdot\zeta)(v(z,\zeta)-v(w,\zeta))}{v(w,z)^n v(z,\zeta)}-\frac{d(\bar{z}'\cdot\zeta)(v(z',\zeta)-v(w,\zeta))}{v(w,z')^n v(z',\zeta)}.$$

Thus

$$\tilde{h}_0(\zeta,z)-\tilde{h}_0(\zeta,z')=\int_{\partial D}\Delta(z,z',\zeta,w)\frac{G(w)d\sigma(w)}{v(w,\zeta)}.$$

To estimate this integral we split the range of integration into two parts. First we consider $w\in A=\{w;\,d(w,z)\leq Cd(z,z')\}$. In this part of the integral we estimate the two terms in Δ separately. Note that when $w\in A$, we have $d(w,z)\leq cCd(\zeta,z)$ and hence (if cC is small enough), $|v(w,\zeta)|\approx d(w,\zeta)\approx d(z,\zeta)$, and by (2.9), $|v(z,\zeta)-v(w,\zeta)|\leq(d(z,w)d(z,\zeta))^{1/2}$. Thus the contribution from the first term is $d(\bar{z}\cdot\zeta)$ times a factor that is bounded by

$$\int_{d(w,z)\leq cd(z,z')}\frac{d\sigma(w)}{d(w,z)^{n-1/2}d(z,\zeta)^{3/2}}\lesssim\frac{d(z,z')^{1/2}}{d(z,\zeta)^{3/2}}\lesssim\frac{d(z,z')^{1/4}}{d(z,\zeta)^{5/4}}.$$

Since $\partial\rho\wedge d(\bar{z}\cdot\zeta)=\mathcal{O}(d(z,\zeta)^{1/2})$ the estimate follows.

As $A\subset\{w;\,d(w,z')\leq Cd(z,z')\}$, the second term can be estimated in the same way.

When $w \notin A$, we use that Δ is smaller than the individual terms. By successively replacing z by z' in each of the factors we obtain four terms of which

$$\Delta_0 = \frac{d(\bar{z}\cdot\zeta)(v(z,\zeta)-v(w,\zeta))}{v(z,\zeta)}\left(\frac{1}{v(w,z)^n}-\frac{1}{v(w,z')^n}\right)$$

is the hardest. When $w \notin A$, $|v(w,z')| \approx |v(w,z)| \approx d(w,z)$. Thus by (2.9),

$$|v(z,\zeta)-v(w,\zeta)| \lesssim (d(w,z)(d(z,\zeta)+d(w,\zeta)))^{1/2}$$

and

$$\left|\frac{1}{v(w,z)^n}-\frac{1}{v(w,z')^n}\right| \lesssim \frac{(d(z,z')d(w,z))^{1/2}}{d(w,z)^{n+1}}.$$

By (2.8) we obtain that $\int_{\partial D\setminus A}\Delta_0\frac{G(w)d\sigma(w)}{v(w,\zeta)}$ is $d(\bar{z}\cdot\zeta)$ times

$$\int_{d(w,z)>Cd(z,z')}\frac{d(z,z')^{1/2}(d(z,\zeta)+d(w,\zeta))^{1/2}}{d(w,z)^n d(z,\zeta)d(w,\zeta)}d\sigma(w)$$
$$\lesssim \int_{\partial D}\frac{d(z,z')^{1/4}(d(z,\zeta)+d(w,\zeta))^{1/2}}{d(w,z)^{n-1/4}d(z,\zeta)d(w,\zeta)}d\sigma(w) \lesssim \frac{d(z,z')^{1/4}}{d(z,\zeta)^{5/4}}$$

as desired. (Of course $1/4$ can be replaced with any $\epsilon, 0<\epsilon<1/2$.)

For the first term in (5.8) we get the estimate

$$\lesssim \frac{d(z,z')^{1/4}}{d(z,\zeta)^{3/4}},$$

(but without the factor $d(\bar{z}\cdot\zeta)$). This follows by the same argument as above. □

6. The $T1$-theorem, proof of Proposition 3.4 b).

The proof of Proposition 3.4 b) relies on the $T1$-theorem for Carleson measures, due to Christ and Journé, see [**CJ**]. Here we formulate it in the setting of a strictly pseudoconvex domain. The proof is a straight forward modification of the proof in the euclidean case.

Let $T(\zeta,z)$ be a kernel on $D\times\partial D$. We say that T is a CJ-kernel if for some $\epsilon>0$,

$$|T(\zeta,z)| \le C\frac{(-\rho(\zeta))^{\epsilon-1}}{d(\zeta,z)^{n+\epsilon}}, \tag{6.1}$$

and

$$|T(\zeta,z) - T(\zeta,z')| \leq C\left(\frac{d(z,z')}{d(\zeta,z)}\right)^{\epsilon} \frac{(-\rho(\zeta))^{\epsilon-1}}{d(\zeta,z)^{n+\epsilon}}, \tag{6.2}$$

if $d(z,z') \leq cd(\zeta,z)$.

Let $\mathcal{T}$ denote the corresponding operator,

$$\mathcal{T}\psi(\zeta) = \int_{\partial D} T(\zeta,z)\psi(z)d\sigma(z).$$

Theorem 6.1. *Let $D = \{\rho < 0\}$ be a strictly pseudoconvex domain with C^3-boundary, $d\rho \neq 0$ on ∂D and let T be a CJ-kernel. If $\mathcal{T}$ is L^2-bounded, i.e.*

$$\int_D (-\rho)|\mathcal{T}\psi|^2 \lesssim \int_{\partial D} |\psi|^2 d\sigma, \tag{6.1}$$

then $\mathcal{T}$ maps L^∞ into (the space of) Carleson functions. Moreover, if $\mathcal{T}1$ is a Carleson function, then (6.3) *holds, i.e $\mathcal{T}$ is L^2-bounded.*

Of course it is the last statement which is of most interest.

To obtain (3.18) we need the following simple additional result.

Proposition 6.2. *If $\mathcal{T}$ is an L^2-bounded CJ-operator as in Theorem* 6.1, *ψ is in $\mathcal{H}^1$ and b is a Carleson function, then*

$$\int_D (-\rho)|\mathcal{T}\psi||b| \lesssim \|\psi\|_{\mathcal{H}^1}\|b\|_C.$$

The proof of this proposition is standard, and we omit the argument.

We also need a complement to Lemma 5.1.

Lemma 6.3. *With the same notation as in Lemma* 5.1 *we have*

$$\left|\mathcal{L}\tilde{h}\right| \lesssim \frac{1}{(-\rho)|d(\zeta.z)|}, \quad \left|\partial\rho\wedge\mathcal{L}\tilde{h}\right| \lesssim \frac{1}{(-\rho)|d(\zeta.z)|^{\frac{1}{2}}}, \tag{6.4}$$

$$\left|\mathcal{L}\left(\tilde{h}(\zeta,z) - \tilde{h}(\zeta,z')\right)\right| \lesssim \frac{d(z,z')^{\frac{1}{4}}}{(-\rho)d(\zeta,z)^{1+\frac{1}{4}}}, \tag{6.5}$$

and

$$\left|\partial\rho\wedge\mathcal{L}\left(\tilde{h}(\zeta,z) - \tilde{h}(\zeta,z')\right)\right| \lesssim \frac{d(z,z')^{\frac{1}{4}}}{(-\rho)d(\zeta,z)^{\frac{1}{2}+\frac{1}{4}}},\tag{6.6}$$

if $d(z,z') \le cd(\zeta,z)$.

Proof. From Lemma 5.1 it follows that $\tilde{h}(\zeta,z)v(z,\zeta)$ is a bounded holomorphic function i ζ. Hence

$$\mathcal{L}\left(\tilde{h}(z,\zeta)v(z,\zeta)\right) \lesssim \frac{1}{-\rho},$$

and since $\mathcal{L}v(z,\zeta)$ is bounded, we get the first part of (6.4). The second part follows in the same way, noting that $\partial\rho\wedge\mathcal{L}\tilde{h} = \mathcal{O}(|\zeta - z|)\mathcal{L}\tilde{h} + \mathcal{L}(d(\bar{z}\cdot\zeta)\wedge\tilde{h})$ and that $\sqrt{v(z,\zeta)}d(\bar{z}\cdot\zeta)\wedge\tilde{h}(\zeta,z)$ is bounded and holomorphic in ζ. The estimates (6.5) and (6.6) follow in the same way. □

Proof of Proposition 3.4 b). Put $T^l(\zeta,z) = \mathcal{L}M^\ell(\zeta,z)$ (for the definition of M^ℓ see §5, in particular (5.3) and (5.4)). The corresponding operator $\mathcal{T}_\ell$ then satisfies $\mathcal{T}_\ell\psi = \mathcal{L}\mathcal{M}_\ell^*\psi$.

There are two kinds of $\mathcal{T}_\ell = \mathcal{L}\mathcal{M}^\ell$ from (5.3),

(6.7)

$$\mathcal{T}_\ell\psi(\zeta) = \mathcal{L}\int_{\partial D}(-\rho)^{\alpha-\frac{k-1}{2}}\frac{d(\bar{z}\cdot\zeta)\wedge\tilde{h}_2(\zeta,z)\wedge\ldots\wedge\tilde{h}_k(\zeta,z)\psi(z)}{v(\zeta,z)^{n+\alpha-k}v(z,\zeta)}$$

and

(6.8)

$$\mathcal{T}_\ell\psi(\zeta) = \mathcal{L}\int_{\partial D}(-\rho)^{\alpha-\frac{k-1}{2}}\frac{\tilde{h}_1(\zeta,z)\wedge\ldots\wedge\tilde{h}_k(\zeta,z)\psi(z)}{v(\zeta,z)^{n+\alpha-k}}.$$

We first consider the term (6.7) with $k=1$. It is

$$\mathcal{T}_\ell\psi(\zeta) = \mathcal{L}\int_{\partial D}(-\rho)^{\alpha}\frac{\psi(z)d(\bar{z}\cdot\zeta)d\sigma(z)}{v(\zeta,z)^{n+\alpha-1}v(z,\zeta)},$$

which is an instance of a differentiated Poisson type integral, and the desired estimate follows from Proposition 7.1 below.

If $k>1$ in (6.7), Lemmas 5.1 and 6.3 imply that T^ℓ is a CJ-kernel (with $\epsilon = 1/4$). In view of the $T1$-theorem (Theorem 6.1) and Proposition 6.2, it is enough to show that $\mathcal{T}_\ell 1$ is a Carleson function.

First note that $\tilde{h}_2$ can be replaced by h_2 due to the presence of the factor $d(\bar{z}\cdot\zeta)$. If we then use the definition (5.7) (of h_2), and apply Fubini's theorem, we (formally) get

$$\mathcal{L}\int_{w\in\partial D}\sum_{j=1}^{n}\frac{(-\rho)^{\alpha-\frac{k-1}{2}}}{v(w,\zeta)^{n+1-j}}$$

$$\times\int_{z\in\partial D}\frac{d(\bar{z}\cdot\zeta)\wedge d(\bar{w}\cdot\zeta)\wedge\tilde{h}_3(\zeta,z)\wedge\ldots\tilde{h}_k(\zeta,z)}{v(\zeta,z)^{n+\alpha-k}v(z,\zeta)v(w,z)^j}G_2(w)d\sigma(w).$$

This computation is legitimate since the kernel is

$$\mathcal{O}\left(\frac{d(\bar{w}\cdot\zeta)\wedge d(\bar{z}\cdot\zeta)}{|v(w,z)|^n}\right)=\mathcal{O}\left(\frac{1}{|v(w,z)|^{n-1/2}}\right),$$

and hence integrable over $\partial D\times\partial D$.

The resulting integral may be viewed as an operator acting on G_2.

Claim. The corresponding kernel $K(\zeta,w)$ is a CJ-kernel.

Taking this claim for granted, to see that (6.7) is a Carleson function, again by Theorem 6.1 it is enough to do this when G_2 is replaced by 1. But then the integral even vanishes because of

Lemma 6.4. *In the ball the Hefer form* (4.3) *to the function* 1 *is identically zero.*

Proof. If $G\equiv 1$, then the integrand in (5.7) is antiholomorphic and hence the integrals vanish by the mean value property. □

In the general strictly pseudoconvex case the corresponding Hefer form is $\sim\partial\rho\mathcal{O}\left(1/\sqrt{|v|}\right)+\mathcal{O}(1)$ which is also good enough.

To see that $\mathcal{T}_l 1$ from (6.8) is a Carleson function, we notice (cf. (5.3)) that it is the sum of operators of the type (6.7) acting on certain G_j (which hence are Carleson functions by Theorem 6.1), plus the integral

$$\mathcal{L}\int_{\partial D}(-\rho)^{\alpha-\frac{k-1}{2}}\frac{h_1(\zeta,z)\wedge\ldots\wedge h_k(\zeta,z)}{v(\zeta,z)^{n+\alpha-k}},$$

which by the mean value property is

$$\mathcal{L}[(-\rho(\zeta))^{\alpha-\frac{k-1}{2}}h_1(\zeta,0)\wedge\ldots\wedge h_k(\zeta,0)],$$

and this is a Carleson function if $\alpha>k/2$. Hence Proposition 3.4 is proved.

Proof of the claim. The kernel is

$$K(\zeta,w)=\mathcal{L}\sum_{j=1}^{n}\frac{(-\rho)^{\alpha-\frac{k-1}{2}}}{v(w,\zeta)^{n+1-j}}\int_{z\in\partial D}\frac{d(\bar{z}\cdot\zeta)\wedge d(\bar{w}\cdot\zeta)\wedge\tilde{h}_3(\zeta,z)\wedge\ldots\tilde{h}_k(\zeta,z)}{v(\zeta,z)^{n+\alpha-k}v(z,\zeta)v(w,z)^j}.$$

Cancelling a suitable power of $-\rho/|v|$, we obtain by (2.8)

$$|K(\zeta,w)|\lesssim\sum_{j=1}^{n}\frac{(-\rho)^{-1+1/4}}{|v(w,\zeta)|^{n+1-j}}\int_{z\in\partial D}\frac{d\sigma(z)}{d(\zeta,z)^{n+1/4-1/2}d(w,z)^{j-1/2}}$$
$$\sim\frac{(-\rho)^{-1+1/4}}{d(w,\zeta)^{n+1/4}},$$

and hence it satisfies (6.1) with $\epsilon = 1/4$. Moreover, since the exponent of $d(w, z)$ is at most $n - 1/2$ in the integral, it also satisfy (6.2) with $\epsilon = 1/4$, by the same argument as for Lemma 5.1. □

The terms occurring from (5.4) are handled in the same way. This concludes the proof of Proposition 3.4 b).

7. Poisson-type integrals.

Let $\rho(\zeta)$ and $v(\zeta, z)$ be as before and put

$$P\psi(\zeta) = \int_{\partial D} \frac{(-\rho)^{\epsilon} a_k(\zeta, z)\psi(z)}{v(\zeta, z)^{n+1+\frac{k}{2}-j} v(z, \zeta)^j},$$

where $a_k(\zeta, z)$ is C^1 and $\mathcal{O}(|\zeta - z|^k)$. Then we say that $P\psi(\zeta)$ is a Poisson-type integral. For instance, the Poisson-Szegö integral in the ball is of this type. Another example is the dual of the approximate solution kernels for the $\partial\bar{\partial}$-equation from [**AnC1**]. However, the interesting example in this paper is the integrals in the proof of Proposition 3.5 and in (6.7) for $k = 1$.

Our main result is that a Poisson-type integral satisfies the same estimates as the integrals $\mathcal{M}_\ell^* \psi$ in Proposition 3.4. For the reader's convenience we reformulate it.

Proposition 7.1. *Let P be a Poisson-type integral.*
a) *If $d\tau$ is a Carleson measure, then*

$$\int_D |P\psi|^p d\tau \lesssim \|\psi\|_{L^p(\partial D)} \tag{7.1}$$

for ψ in $L^p(\partial D)$, $1 < p < \infty$,

$$\int_D |P\psi| d\tau \lesssim \|\psi\|_{\mathcal{H}^1}$$

and

$$\|P\psi\|_{L^\infty} \lesssim \|\psi\|_{L^\infty(\partial D)}.$$

b) *Moreover, if b is a Carleson function, then*

$$\int_D (-\rho)|\mathcal{L}P\psi||b| \lesssim \|\psi\|_{\mathcal{H}^1} \|b\|_C,$$

$$\int_D (-\rho)|\mathcal{L}P\psi|^2 \lesssim \|\psi\|^2_{L^2(\partial D)},$$

and $-\rho|\mathcal{L}P\psi|^2$ *is a Carleson measure (i.e.* $\mathcal{L}P\psi$ *is a Carleson function) if* $\psi \in L^\infty(\partial D)$.

Proof. The proof of part a) is similar to the corresponding one for M_ℓ^*. It just depends on the estimates

$$|P(\zeta, z)| \lesssim \frac{(-\rho)^\epsilon}{d(\zeta, z)^{n+\epsilon}}$$

and

$$|P(\zeta, z) - P(\zeta, z')| \lesssim \left(\frac{d(z, z')}{d(\zeta, z)}\right)^\epsilon \frac{(-\rho)^\epsilon}{d(\zeta, z)^{n+\epsilon}}, \quad d(z, z') \leq cd(\zeta, z).$$

To prove part b), first notice that

$$|\mathcal{L}P(\zeta, z)| \lesssim \frac{(-\rho)^{-1+\epsilon}}{d(\zeta, z)^{n+\epsilon}}$$

and

$$|\mathcal{L}P(\zeta, z) - \mathcal{L}P(\zeta, z')| \leq C\left(\frac{d(z, z')}{d(\zeta, z)}\right)^\epsilon \frac{(-\rho(\zeta))^{\epsilon-1}}{d(\zeta, z)^{n+\epsilon}},$$

if $d(z, z') \leq cd(\zeta, z)$. In view of the $T1$-theorem, it remains to show that $\mathcal{L}P1$ is a Carleson function.

Choose local coordinates on $\overline{D}$ at some point $p \in \partial D$, $\zeta = (y, x_2, ..., x_{2n})$, such that $y = -\rho$. If $z = (0, t_2, ..., t_{2n})$, then

$$v(\zeta, z) = y + i\alpha \cdot (t - x) + \sum \beta'_{jk}(t_j - x_j)(t_k - x_k),$$

where $\alpha|_{\zeta=z} = d^c\rho|_\zeta$ and $\operatorname{Re}\sum \beta'_{jk}(t_j - x_j)(t_k - x_k) \geq \delta|t - z|^2$, for some $\delta > 0$, cf. §2. Moreover,

$$v(z, \zeta) = y - i\alpha \cdot (t - x) + \sum \beta''_{jk}(t_j - x_j)(t_k - x_k),$$

where β''_{jk} satisfy the same relation as β'_{jk}. To simplify notation, in the sequel we denote any of them by

$$y \pm i\alpha \cdot (t - x) + \sum \beta_{jk}(t_j - x_j)(t_k - x_k).$$

Then for $\zeta = (y, x)$ near p, modulo negligable terms we have

$$(7.2) \quad P1(y, x) = \int_t \frac{y^\epsilon \mathcal{O}((|x - t| + y)^k)\chi(t)dt}{(y \pm i\alpha \cdot (t - x) + \sum \beta_{jk}(t_j - x_j)(t_k - x_k))^{n+\epsilon+\frac{k}{2}}},$$

where $\chi(t)$ is a cut-off function.

To begin with, we assume that $\mathcal{L}$ only involves derivatives of $x_2, ..., x_{2n}$. After a translation in the integral in (7.2), we get

$$P1(y,x) = \int_t \frac{y^\epsilon \mathcal{O}((|t|+y)^k)\chi(t+x)dt}{\left(y \pm i\tilde{\alpha}\cdot t + \sum \tilde{\beta}_{jk}t_jt_k\right)^{n+\epsilon+\frac{k}{2}}},$$

and hence

$$\mathcal{L}P1(y,x) = \int_t \frac{y^\epsilon \mathcal{O}((|t|+y)^k)\mathcal{O}(|t|)dt}{\left(y \pm i\tilde{\alpha}\cdot t + \sum \tilde{\beta}_{jk}t_jt_k\right)^{n+\epsilon+\frac{k}{2}+1}} = \mathcal{O}\left(\frac{1}{\sqrt{y}}\right),$$

and thus it is a Carleson function.

If $\mathcal{L} = \frac{\partial}{\partial y}$, we can fix $x = 0$, and assume that

$$v = y \pm i\alpha t_2 + \sum_{j,k\geq 3} \beta_{jk}t_jt_k.$$

We put $t' = (t_3, ..., t_{2n})$ an we may assume that $\mathcal{O}(|\zeta - z|^k)$ is $\mathcal{O}(|t'|^k)$ since $\mathcal{O}((y+|t_2|)^k)$ gives rise to a less singular integral. Then

$$P1(y,0) = \int_t \frac{y^\epsilon \mathcal{O}(|t'|^k)\chi(t)dt}{(y \pm i\alpha t_2 + \sum_{j,k\geq 3}\beta_{jk}t_jt_k)^{n+\epsilon+\frac{k}{2}}}.$$

We now make a change of variables, by putting $t_2 = ys_2$ and $t_j = \sqrt{y}s_j$ for $j = 3, ..., 2n$. Then we get

$$P1(y,0) = \int_s \frac{\tilde{\mathcal{O}}(|s'|^k)\tilde{\chi}ds}{\left(1 \pm i\tilde{\alpha}s_2 + \sum_{j,k\geq 3}\tilde{\beta}_{jk}s_js_k\right)^{n+\epsilon+\frac{k}{2}}},$$

where $\tilde{\alpha} = \alpha(y, ys_2, \sqrt{y}s_3, ..., \sqrt{y}s_{2n})$, so that

$$\frac{\partial}{\partial y}\tilde{\alpha} = \mathcal{O}(1) + \mathcal{O}(s_2) + \mathcal{O}\left(\frac{|s'|}{\sqrt{y}}\right),$$

and analogously for $\tilde{\beta}_{jk}$, $\tilde{\chi}$ and $\tilde{\mathcal{O}}$. Thus,

$$\frac{\partial}{\partial y}P1(y,0) = \int_s \frac{\tilde{\mathcal{O}}(|s'|^k)\left(\mathcal{O}(1) + \mathcal{O}(s_2) + \mathcal{O}\left(\frac{|s'|}{\sqrt{y}}\right)\right)\tilde{\chi}ds}{\left(1 \pm i\tilde{\alpha}s_2 + \sum_{j,k\geq 3}\tilde{\beta}_{jk}s_js_k\right)^{n+\epsilon+\frac{k}{2}}} + \text{similar}$$

$$+\int_s \frac{\tilde{\mathcal{O}}(|s'|^k)(\mathcal{O}(s_2)+\mathcal{O}(|s'|^2))\left(\mathcal{O}(1)+\mathcal{O}(s_2)+\mathcal{O}\left(\frac{|s'|}{\sqrt{y}}\right)\right)\tilde{\chi}ds}{\left(1\pm i\tilde{\alpha}s_2+\sum_{j,k\geq 3}\tilde{\beta}_{jk}s_js_k\right)^{n+\epsilon+\frac{k}{2}+1}},$$

which after substituting back, yields the estimate

$$\frac{\partial}{\partial y}P1(y,0)\lesssim y^{-1+\epsilon}\int_{|t|\leq C}\frac{dt}{|(y\pm i\alpha t_2+\sum_{j,k\geq 3}\beta_{jk}t_jt_k)|^{n+\epsilon-\frac{1}{2}}}.$$

The latter integral is bounded if $\epsilon<1/2$ and $\sim y^{1/2-\epsilon}$ for $\epsilon>1/2$, and again it follows that $\frac{\partial}{\partial y}P1$ is a Carleson function. □

8. The Koszul complex approach.

We conclude this paper with a brief discussion of the Koszul complex method for solving division problems of our kind. The case with two generators is already studied in [**AnC2**]. It just amounts to solving one single $\bar{\partial}$-equation $\bar{\partial}u=w$ for a certain (0,1)-form w. Even in this case, the solution could be simplified by using Proposition 7.1 above. In this case the equation to be solved is $\bar{\partial}u=w$, where

$$w=\omega\phi=\bar{G}_1\bar{\partial}\psi_2-\bar{G}_2\bar{\partial}\psi_1\phi,$$

where we have used the notation from §3 (so that ψ_j are a smooth solution to $\sum G_j\psi_j=1$). Boundary values of a solution to $\bar{\partial}u=w$ is given by a formula of type

$$Kw(z)=\int_D\frac{(-\rho)^\alpha w\wedge\mathcal{O}(1)+(-\rho)^{\alpha-1}w\wedge\bar{\partial}\rho\wedge\mathcal{O}(|\zeta-z|)}{v(\zeta,z)^{n+\alpha-1}v(z,\zeta)},$$

where the functions $\mathcal{O}$ are C^1. To estimate the solution we integrate against $\psi(z)$, and get

$$\int_{\partial D}Kw(z)\psi(z)d\sigma(z)=\int_D w\wedge Tw+\int_D\frac{1}{\sqrt{-\rho}}w\wedge\bar{\partial}\rho\wedge T'\psi,$$

where

$$T\psi(\zeta)=\int\sigma_{\partial D}\frac{(-\rho)^\alpha\mathcal{O}(1)\psi(z)d\sigma(z)}{v(\zeta,z)^{n+\alpha-1}v(z,\zeta)},$$

and

$$T'\psi(\zeta)=\int_{\partial D}\frac{(-\rho)^{\alpha-1/2}\mathcal{O}(|\zeta-z|)\psi(z)d\sigma(z)}{v(\zeta,z)^{n+\alpha-1}v(z,\zeta)}.$$

If we use the simple choice $\psi_j = \bar{G}_j/|G|^2$ we have to apply the Wolff trick again, and (modulo error terms) we arrive at

$$\int_{\partial D} Kw(z)\psi(z)d\sigma(z) = \int_D (-\rho)\mathcal{L}w \wedge Tw + \int_D (-\rho)w \wedge \mathcal{L}Tw$$
$$+ \int_D \frac{1}{\sqrt{-\rho}} w \wedge \bar\partial\rho \wedge T'\psi + \int_D \sqrt{-\rho}\mathcal{L}w \wedge \bar\partial\rho \wedge T'\psi$$
$$+ \int_D \sqrt{-\rho} w \wedge \bar\partial\rho \wedge \mathcal{L}T'\psi,$$

and arguing as in §3, and using Proposition 7.1 (and Propositions 3.2 and 3.4) one get the estimates (3.21) to (3.23) of $\int_{\partial D} Kw(z)\psi(z)d\sigma(z)$, which solves the H^p-corona problem for $1 \le p \le *$ for two generators.

Now suppose that we have an arbitrary but finite number m of generators. Let K denote (good, apropriately weighted) homotopy operators for $\bar\partial$, cousins to the operator K above; thus mapping $(0, q+1)$-forms into $(0, q)$-forms, such that $\bar\partial K + K\bar\partial =$ identity. Then the Koszul complex, see e.g. [**G**], furnishes a holomorphic solution to the division problem that can be written as a sum of terms of the type

$$(GK)^k \psi_{K_0} \wedge \bar\partial\psi_{K_1} \wedge \dots \wedge \bar\partial\psi_{K_k}\phi, \tag{8.1}$$

where $|K| = k+1$, k ranges from 0 to $\min(n, m-1)$, and

$$(GK)^k = GKGK...GK,$$

where GK means the operator K followed by multiplication with one of the generators G_j.

If now $|\bar\partial\psi_j| + \frac{1}{\sqrt{-\rho}}|\bar\partial\rho \wedge \bar\partial\psi_j|$ are Carleson measures, $|\bar\partial\psi_j| \lesssim 1/(-\rho)$ and $|\bar\partial\rho \wedge \bar\partial\psi_j| \lesssim 1/\sqrt{-\rho}$, then

$$(-\rho)^{\frac{k-1}{2}} |\psi_{K_0} \wedge \bar\partial\psi_{K_1} \wedge \dots \wedge \bar\partial\psi_{K_k}\phi|$$

and

$$(-\rho)^{\frac{k-2}{2}} |\bar\partial\rho \wedge \psi_{K_0} \wedge \bar\partial\psi_{K_1} \wedge \dots \wedge \bar\partial\psi_{K_k}\phi|$$

are Carleson measures, and then a size estimate of the integrals in (8.1) gives a desired estimate for the solution. The necessary estimates for the kernel of $(GK)^k$ follow from Lemma 5.2 in [**AnC1**].

The explicit choice of starting solution $\psi = \bar{G}/|G|^2$ offer additional techniqual problems that we do not pursue here.

References

[Am] E. Amar, *On the corona problem*, J.of Geom. Anal., **1** (1991), 291-305.

[An1] M. Andersson, *The H^2 corona problem and $\bar{\partial}_b$ in weakly pseudoconvex domains*, TAMS, **342** (1994), 241-255.

[An2] ———, *On the H^p-corona problem*, Bull. Sci. Math., **118** (1994), 287-306.

[AnC1] M. Andersson and H. Carlsson, *Formulas for approximate solutions of the $\partial\bar{\partial}$-equation in a strictly pseudoconvex domain*, Revista Mat. Iberoamericana, **11** (1995), 67-101.

[AnC2] ———, *Wolff-type estimates and the H^p-corona problem in strictly pseudoconvex domains*, Ark. Math., (1992), **32** (1994), 255-276.

[B1] B. Berndtsson, *A formula for division and interpolation*, Math. Ann., **263** (1983), 399-418.

[B2] ———, *Weighted integral formulas*, Several complex variables: Proceedings of the Mittag-Leffler Institute, 1987-1988, Math. Notes Princeton Univ Press, **38** (1993).

[Ca] L. Carleson, *Interpolation by bounded analytic functions and the corona theorem*, Ann. of Math., **76** (1962).

[ChJ] M. Christ and J.-L. Journé, *Polynomial growth estimates for multilinear singular integral operators*, Acta Math, **159** (1987), 51-80.

[CoW] R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull.Amer. Math., **83** (1977), 569-645.

[G] J. Garnett, *Bounded analytic functions*, Academic Press, 1981.

[Hö1] L. Hörmander, *Generators for some rings of holomorphic functions*, Bull. Am. Math. Soc., **73** (1967), 943-949.

[J] J.-L. Journé, *Calderón-Zygmund operators, pseudo-differential operators, and the Cauchy integral of Calderón, Lecture Notes in Math.*, Springer-Verlag, **994** (1983).

[Sk] H. Skoda, *Morphismes surjectifs de fibres vectoriels semi-positifs*, Ann. Sci. Ec. Norm. Super., **11** (1978), 577-611.

[V1] N.Th. Varopoulos, BMO *functions and the $\bar{\partial}$-equation*, Pacific J. Math., **71** (1977), 221-273.

[V2] ———, *A remark on* BMO *and bounded harmonic functions*, Pacific J. Math., **74** (1977), 257-259.

Received August 18, 1993 and revised June 15, 1994. The first author was partially supported by the Swedish Natural Research Council.

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