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SPECIALIZATIONS AND A LOCAL HOMEOMORPHISM THEOREM FOR REAL RIEMANN SURFACES OF RINGS

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Let $\phi : k \to A$ and $f : A \to R$ be ring morphisms, R a real ring. We prove that if $f : A \to R$ is étale, then the corresponding mapping between real Riemann surfaces $\mathrm{S}_r(f) : \mathrm{S}_r(R/k) \to \mathrm{S}_r(A/k)$ is a local homeomorphism. Several preparatory results are proved, as well. The most relevant among these are: (1) a Chevalley's theorem for real Riemann surfaces on the preservation of constructibility via $\mathrm{S}_r(f)$, and (2) an analysis of the closure operator on real Riemann surfaces. Constructible sets are dealt with by means of a suitable first-order language.

1. Introduction.

Let k be a real ring. In this paper we study a sufficient condition for two real k-algebras A and R to have homeomorphic real Riemann surfaces. More precisely, here we show that if $f : A \to R$ is an étale morphism, then the corresponding mapping between real Riemann surfaces $\mathrm{S}_r(f) : \mathrm{S}_r(R/k) \to \mathrm{S}_r(A/k)$ is a local homeomorphism (Theorem 9). In order to prove this theorem, we need several previous preparatory results, some of which are interesting on their own. Namely,

(a) the functorial character of S_r (Theorem 4),

(b) a Chevalley's theorem for real Riemann surfaces (Theorem 6), which guarantees that if f is finitely presented (in particular, if f is étale) then the image by $\mathrm{S}_r(f)$ of any constructible subset of $\mathrm{S}_r(R/k)$ is a constructible subset of $\mathrm{S}_r(A/k)$,

(c) a good knowledge of the closure operator on real Riemann surfaces (Theorem 1) and of the constructible subsets of real Riemann surfaces in terms of the first-order language of ordered valued fields,

(d) a result relating the notions of constructible, Tychonoff-closed, Tychonoff-clopen, closed and stability of a subset under specialization in real Riemann surfaces (Proposition 8) and, finally,

(e) the known result that if f is étale then $\mathrm{Spec}_r(f) : \mathrm{Spec}_r(R) \to \mathrm{Spec}_r(A)$ is a local homeomorphism. This theorem is due to M. Coste and M.-F. Roy.

Two facts should be pointed out here. First, the étality of f is essential in our proof of Theorem 9 at two stages: (1) when we invoke the Coste-Roy theorem and (2) because the étality of f provides a homeomorphism between the fibers $\pi_R^{-1}(\beta)$ and $\pi_A^{-1}(\alpha)$ of corresponding points β and $\alpha = \mathrm{Spec}_r(f)(\beta)$ (Proposition 5), where π_C is the natural projection from $\mathrm{S}_r(C/k)$ onto $\mathrm{Spec}_r(C)$, for $C = A$ or R. Second, our proof of Chevalley's theorem for real Riemann surfaces uses a result from model theory that allows elimination of quantifiers in the theory of real closed fields endowed with a non-trivial compatible valuation divisibility relation.

2. Notations and Background.

(RR) All rings appearing in this paper are commutative and have an identity element. We call them rings, for short. Ring homomorphisms that preserve the identity are simply called morphisms. When a base ring k is given, morphisms are assumed to preserve the k-algebra structure. By an ordering on a field K we mean a total order relation on K. A ring A is said to be *real* if A has a prime ideal p such that the quotient field of the integral domain A/p can be ordered. Such a prime ideal is called *real*, as well.

(FOL) Given a first-order language $\mathcal{L}$, variables are denoted by $v_1, v_2, v_3, \ldots$ and greek letters ϕ, ψ, θ, $\eta, \ldots$ denote formulas of $\mathcal{L}$. An expression $\phi(v_1, \ldots, v_n)$ means that ϕ is formula whose free variables are a subset of $v_1, \ldots, v_n$. For any family of elements $t_1, \ldots, t_n$ in an $\mathcal{L}$-system R, the expression $\phi[t_1, \ldots, t_n]$ denotes the element in R obtained as a result of the substitution in $\phi(v_1, \ldots, v_n)$ of t_1 by $v_1, \ldots$ and t_n by v_n.

(CVR) Let K be an ordered field and k a subfield of K. If B is a convex valuation ring of K, then *the set of ideals of B is totally ordered by inclusion. Every ideal of B is convex. Any convex valuation ring B' in K containing B satisfies $B' = B_p$, for some prime ideal p of B.* Moreover, *the set of convex valuation rings of K/k (i.e., valuation rings of K which contain k) is totally ordered by inclusion.* These facts follow from various classical results on valuation rings, realizing that convexity is preserved through the proofs; see [**A**], [**E**], [**K**], [**Ri**] or [**Z-S**] for general valuation theory and [**B-C-R**] or [**La**] for results on convex valuation rings.

(RS) Nowadays, the following definition is well-known to specialists in real algebra. Let A be a ring. The *real spectrum of A*, denoted $\mathrm{Spec}_r(A)$, is the collection of pairs $(p_\beta, \leq_\beta)$, where p_β is a real prime ideal of A and $\leq_\beta$ is an ordering on the quotient field of A/p_β. We write β instead of $(p_\beta, \leq_\beta)$. Clearly, $\mathrm{Spec}_r(A)$ is non-empty if and only if the ring A is real. For example, if $A = K$ is a field then $p_\beta = (0)$, for all $\beta \in \mathrm{Spec}_r(K)$, and so any $\beta \in \operatorname{Spec}_r(K)$ is simply an ordering on K. Moreover, if K is real closed, then $\operatorname{Spec}_r(K)$ consists of just one point.

If β belongs to $\operatorname{Spec}_r(A)$, then let $A[\beta]$, $A(\beta)$, $\kappa(\beta)$, $a(\beta)$, $A'[\beta]$ and $|a(\beta)|_\beta$ respectively denote the ring A/p_β endowed with the restriction of the ordering $\leq_\beta$, the quotient field of $A[\beta]$ endowed with $\leq_\beta$, the real closure of $A(\beta)$ (it is unique, up to order-preserving $A(\beta)$-isomorphism), the image of $a \in A$ in $A[\beta]$ by the canonical epimorphism $A \twoheadrightarrow A[\beta]$, the image of $A' \subseteq A$ in $A[\beta]$ by the same epimorphism and $\max\{a(\beta), -a(\beta)\}$. Expressions of the form $a(\beta) \geq_\beta 0$, $|a(\beta)|_\beta$ and the like are simplified to $a(\beta) \geq 0$, $|a(\beta)|$, respectively.

We consider subsets of $\operatorname{Spec}_r(A)$ of the type $U_x := \{\beta \in \operatorname{Spec}_r(A) : x(\beta) > 0\}$, where x runs through A. Then we take in $\operatorname{Spec}_r(A)$ the minimal topology $\mathcal{U}$ (respectively, minimal boolean algebra $\mathcal{C}$) that contains the sets U_x's. The topology $\mathcal{U}$ is the *usual topology* considered on $\operatorname{Spec}_r(A)$. The topological space $\operatorname{Spec}_r(A)$ has some very interesting properties we proceed to recall. As customary, the elements of $\mathcal{U}$ are called open sets. The elements of $\mathcal{C}$ are given a name too: they are called constructible sets. We further consider the minimal topology $\mathcal{T}$ on $\operatorname{Spec}_r(A)$ that contains the constructible sets. It is regarded only as an auxiliary tool for the study of $\operatorname{Spec}_r(A)$. This topology on $\operatorname{Spec}_r(A)$ is called the *Tychonoff topology* (also called the *constructible topology*). Its elements are called Tychonoff-open sets. A subset $F \subseteq \operatorname{Spec}_r(A)$ is Tychonoff-closed if and only if F is an intersection of constructible sets. Tychonoff-clopen means both Tychonoff-open and Tychonoff-closed, obviously. We have that $F \subseteq \operatorname{Spec}_r(A)$ is Tychonoff-clopen if and only if F is constructible. Moreover, it is not hard to show that $\operatorname{Spec}_r(A)$ endowed with $\mathcal{T}$ is a quasi-compact Hausdorff topological space. Now, comparison of both topologies yields that $\mathcal{U}$ is finer than $\mathcal{T}$. Therefore, $\operatorname{Spec}_r(A)$ *endowed with its usual topology is quasi-compact.*

In order to present the constructible subsets of $\operatorname{Spec}_r(A)$, the framework of first-order logic may be used, as an alternative. Namely, consider the first-order language $\mathcal{L}_r$ of ordered fields. A subset L of $\operatorname{Spec}_r(A)$ is constructible if and only if there exists a quantifier-free formula $\phi(v_1, \dots, v_n)$ in $\mathcal{L}_r$ and elements $t_1, \dots, t_n$ in A such that L equals $\{\beta \in \operatorname{Spec}_r(A) : A(\beta) \models \phi[t_1(\beta), \dots, t_n(\beta)]\}$. Notice that $\phi[t_1(\beta), \dots, t_n(\beta)]$ simply consists of finitely many conditions $t_i(\beta) > 0$, $t_j(\beta) = 0$ and $t_h(\beta) < 0$, joined by conjunctive and or disjunctive symbols.

Back to the general properties of the real spectrum of a ring A, it is not Hausdorff, in general (an exception to this is when A is a field, in which case $\mathcal{T} = \mathcal{U}$). Therefore, one important question is to describe the closure of a point β in $\operatorname{Spec}_r(A)$. It is shown that it looks like a spear, i.e., *the closure of a point is a totally ordered set and has a unique maximal element.* The points in the closure of β are called specializations of β and β is said to be a generization of each of them. Given a subset F of $\mathrm{Spec}_r(A)$, we say that F is stable under specialization (respectively, generization) if the conditions $\beta \in F$ and γ is a specialization (respectively, generization) of β imply $\gamma \in F$. The following statement holds true: *if $Y \subseteq \mathrm{Spec}_r(A)$ is Tychonoff-closed and $F \subseteq Y$, then F is closed in Y if and only if F is Tychonoff-closed and stable under specialization in Y.*

Further properties that deserve to be mentioned here are the *functorial* character of the construction as well as the *naturality* of the mapping $\mathrm{Spec}_r(A) \to \mathrm{Spec}(A)$ given by $\beta \mapsto p_\beta$. The real spectrum of a ring was introduced in 1979 by M. Coste and M.-F. Roy. See [**B-C-R**] or [**B**] for details.

(RRS) Once persuaded that the real spectrum of a ring A is a useful tool and that it deserves to be studied and understood, soon we noticed that some facts taking place at points β of $\mathrm{Spec}_r(A)$ are ultimately explained by means of convex valuation rings of the corresponding residue fields $A(\beta)$. The very long remark 10.3.5 in [**B-C-R**] fully justifies our latter statement. With this and certain applications in mind, we introduced the notion of *real Riemann surface of a ring* in [**Pu**]. Notion and name were inspired on both the real spectrum construction and the so called *Riemann surface of a field extension K/k*, introduced by Zariski, see [**Z-S**] VI §17, (and later named *Zariski-Riemann space of K/k* in [**Li**]). In fact, in [**Pu**] we began by introducing a more general space, called the *Riemann surface of a ring*, for which no reality conditions were required. Other authors have also defined such latter spaces (under the name of *valuation spectrum of a ring*) and studied them; see [**H**], [**H-K**] or [**S**].

Back to our presentation, given a ring morphism $\phi : k \to A$, the *real Riemann surface of A/k* is the set $\mathrm{S}_r(A/k)$ consisting of all pairs (β, B) where $\beta \in \mathrm{Spec}_r(A)$ and B is a convex valuation ring in $A(\beta)$ finite over $\phi(k)[\beta]$ i.e., B contains $\phi(k)[\beta]$. Clearly, $\mathrm{S}_r(A/k)$ is non-empty if and only if A is real, since convex hulls of intermediate rings are convex valuation rings (see the next paragraph). The set $\mathrm{S}_r(A/k)$ is endowed with the minimal topology containing the sets of the type $U_{x,y} := \{(\beta, B) : y(\beta) > 0$ and $xy^{-1}(\beta) \in B\}$, where x, y run through A. The choice of this topology is in full accordance with both the usual topology of the real spectrum and the topology given to the Riemann surface of a field extension K/k by Zariski. In particular, the projection onto the first factor $\pi_A : \mathrm{S}_r(A/k) \to \mathrm{Spec}_r(A)$ is a continuous mapping.

Just as in the real spectrum setting, we may consider the constructible subsets of $\mathrm{S}_r(A/k)$ and then the Tychonoff topology on $\mathrm{S}_r(A/k)$. Then it is proved that $\mathrm{S}_r(A/k)$ is Tychonoff quasi-compact, whence quasi-compact with its usual topology. In addition, π_A *has several continuous sections, the images of which are (non-necessarily disjoint) homeomorphic copies of* $\mathrm{Spec}_r(A)$ *lying inside* $\mathrm{S}_r(A/k)$. This ensures that every phenomenon taking place inside $\mathrm{Spec}_r(A)$ is taking place in $\mathrm{S}_r(A/k)$ too. A first instance of such sections of π_A is the mapping $t_A : \mathrm{Spec}_r(A) \to \mathrm{S}_r(A/k)$ where $t_A(\beta) = (\beta, A(\beta))$. We call it the trivial section since $A(\beta)$ is the trivial (convex) valuation ring of $A(\beta)$. Other instances of these sections are the mappings $\rho_{A'}$, which arise by considering convex hulls of intermediate rings of the ring extension $A/\phi(k)$. More precisely, for any intermediate ring A' and any β in $\mathrm{Spec}_r(A)$, consider $\mathcal{O}_\beta(A') := \{x \in A(\beta) : |x| \leq |a| \text{ for some } a \in A'[\beta]\}$. Clearly, $\mathcal{O}_\beta(A')$ is a convex valuation ring of $A(\beta)$. It is shown that the mapping $\rho_{A'} : \mathrm{Spec}_r(A) \to \mathrm{S}_r(A/k)$ given by $\rho_{A'}(\beta) = (\beta, \mathcal{O}_\beta(A'))$ is a section of π_A having the mentioned properties. The most relevant cases are obviously $A' = \phi(k)$ and $A' = A$. Clearly, $\mathcal{O}_\beta(A)$ is the smallest convex valuation ring in $A(\beta)$ finite over $A[\beta]$. All these facts are proved in [**Pu**]. Moreover, the contractions of the ideals of $\mathcal{O}_\beta(A)$ to $A[\beta]$ are precisely the convex ideals of $A[\beta]$. It follows that if J is a convex ideal in $A[\beta]$, then $J\mathcal{O}_\beta(A) \cap A[\beta] = J$. In particular, *if* p *is a prime convex ideal in* $A[\beta]$, *then* $p\mathcal{O}_\beta(A)$ *is prime.*

Let us now briefly look at the *fibers* of π_A. Given $\beta \in \mathrm{Spec}_r(A)$, the set $\pi_A^{-1}(\beta)$ is simply the collection of pairs (β, B), where B runs through all the convex valuation rings of $A(\beta)$ which are finite over $\phi(k)[\beta]$. As seen in **(CVR)**, such a set is totally ordered by inclusion. It has both a minimal and a maximal element, namely $\mathcal{O}_\beta(\phi(k))$ and $A(\beta)$, respectively. On the other hand, $\mathrm{Spec}_r(\kappa(\beta))$ reduces to a point, say $\overline{\beta}$, and therefore $\mathrm{S}_r(\kappa(\beta)/k)$ equals $\pi_{\kappa(\beta)}^{-1}(\overline{\beta})$. By a well-known theorem, there is no content relation between two extensions B' and B'' of a valuation ring B of $A(\beta)$ to the algebraic extension $\kappa(\beta)$ of $A(\beta)$. Thus, $(\beta, B) \mapsto (\overline{\beta}, \mathcal{O}_{\overline{\beta}}(B))$ *defines a bijection from* $\pi_A^{-1}(\beta)$ *onto* $\mathrm{S}_r(\kappa(\beta)/k)$. *The inverse mapping is given by* $(\overline{\beta}, C) \mapsto (\beta, C \cap A(\beta))$.

Back to the constructible sets, it holds that π_A is Tychonoff-continuous. The latter statement simply means that *if* $L \subseteq \mathrm{Spec}_r(A)$ *is constructible, then* $\pi_A^{-1}(L)$ *is constructible.* This is proved in [**Pu**]. We may make use of first-order logic again, in order to deal with constructible subsets of $\mathrm{S}_r(A/k)$. This time we are to talk about systems (K, B), where K is an ordered field and B is a convex valuation ring of K. For technical reasons (namely, the use of [**Pr**] Theorem 4.20), we should rather talk about systems $(K, |)$, where $|$ is a compatible valuation divisibility relation. Each convex valuation ring B gives rise to a compatible valuation divisibility relation $|_B$, and conversely, as follows: for all $a, b \in K$ we have $a|_B b$ if and only if there exists $c \in B$ such that $b = ac$. In view of such systems $(K, |)$, we consider the following first-order language $\mathcal{L}_v = \{+, \cdot, <, |, 0, 1\}$. If $(K, |)$, $(K_1, |_1)$ are $\mathcal{L}_v$-structures, then $(K_1, |_1)$ is an extension of $(K, |)$ if $K \subseteq K_1$ is an extension of ordered fields and $a|b$ if and only if $a|_1 b$, for all $a, b \in K$. The latter condition is equivalent to saying that $B = K \cap B_1$, where $| = |_B$ and $|_1 = |_{B_1}$. Then a subset L of $\mathrm{S}_r(A/k)$ is constructible if L is of the form $\{(\beta, B) \in \mathrm{S}_r(A/k) : (A(\beta), |_B) \models \phi[t_1(\beta), \dots, t_n(\beta)]\}$, where $t_1, \dots, t_n \in A$ and $\phi(v_1, \dots, v_n)$ is a quantifier-free formula of $\mathcal{L}_v$. The set above will be denoted $L_{\phi; t_1, \dots, t_n}$. Notice that $\phi[t_1(\beta), \dots, t_n(\beta)]$ consists of a finite collection of conditions $t_i(\beta) > 0$, $t_i(\beta)|_B t_j(\beta)$ and their negations, joined by conjunctive and or disjunctive symbols.

Let us remark here that in [**Pu**] we allowed B to equal K, the trivial valuation ring of K; in fact, this was an essential requisite to prove compactness. However, the presence of trivial valuation rings prevents us from directly applying [**Pr**] Theorem 4.20 in our proof of Theorem 6.

(EM) Étale morphisms are defined by mimicking the implicit function theorem, which is clearly false in the algebraic case. The notion comprises both non-ramification and smoothness. Concerning language, when a morphism $f : A \to R$ has been fixed, we loosely say that R is étale over A instead of saying that f is étale or that f makes R étale over A.

The concept is very simple in the field case: *an étale extension of fields is just a finite separable extension.* Two results emphasizing the local character of étality are to be pointed out. First, the *étale property is local on* $\mathrm{Spec}(R)$, i.e., *if* $f : A \to R$ *is a morphism and for every* $q \in \mathrm{Spec}(R)$ *there exists an element* $g \in R \setminus q$ *such that* R_g *is étale over* A, *then* R *is étale over* A. Second, we have the so called *local structure theorem for étale morphisms,* which guarantees that *every such ring of fractions* R_g *is* A*-isomorphic to a standard étale* A_h*-algebra, for some* $h \in A \setminus \mathrm{Spec}(f)(q)$ *and that the satisfaction of all these local étality conditions implies that* f *is étale.* A *standard étale* C*-algebra* is simply a C-algebra of the type $(C[X]/(j))_l$, where X is transcendental over C, j, l are polynomials in $C[X]$, with j monic and the class of the formal derivative j' in $(C[X]/(j))_l$ invertible. It is also important that *étality transfers to the fibers of* f. See [**A-K**], [**I**], [**M**] or [**Ra**] for details.

3. Specializations in Real Riemann Surfaces.

Let $\phi : k \to A$ be a morphism of rings. Recall that the topology of $\mathrm{S}_r(A/k)$ is generated by the sets $U_{x,y}$ where x, y run through A. The very definition of this topology yields that given points (β, B) and (γ, C) in $\mathrm{S}_r(A/k)$, then

(γ, C) is a specialization of (β, B) if and only if

$$(1) \quad \text{for all } x, y \in A, \quad [y(\gamma) > 0 \text{ and } xy^{-1}(\gamma) \in C] \qquad \text{imply } [y(\beta) > 0 \text{ and } xy^{-1}(\beta) \in B].$$

Obviously, for $\beta = \gamma$, condition (1) reduces to $C \subseteq B$. So, when restricted to $\pi_A^{-1}(\beta)$, specialization just means inclusion of the valuation rings. This is in full accordance with [**Z-S**] VI §17.

Now, if $\gamma \neq \beta$, then it follows from (1) that γ is a specialization of β. Conversely, if γ is a specialization of β then $p_\gamma \supseteq p_\beta$ and $A[\gamma]$ is naturally order-isomorphic to $A[\beta]/p_\gamma A[\beta]$. Let $A_{\beta,\gamma}$ be the localization of $A[\beta]$ at $p_\gamma A[\beta]$. Then there exists a natural order-preserving epimorphism $\tau_{\beta,\gamma}$: $A_{\beta,\gamma} \to A(\gamma)$ fitting in the following commuting diagram

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ A[\beta] \arrow[rr, two heads] \arrow[d] & & A[\gamma] \arrow[d] \\ A_{\beta,\gamma} \arrow[rr, two heads, "\tau_{\beta,\gamma}"] & & A(\gamma) \end{tikzcd}$$

where the vertical arrows are canonical. Namely, $\tau_{\beta,\gamma}(xy^{-1}(\beta)) = xy^{-1}(\gamma)$, for $x, y \in A$ with $y(\gamma) \neq 0$. We may regard $\tau_{\beta,\gamma}$ as an evaluation on γ. Evidently, $\tau_{\beta,\beta}$ is the identity on $A(\beta)$. The existence of $\tau_{\beta,\gamma}$ implies that (1) is equivalent to

$$(2) \qquad \gamma \text{ is a specialization of } \beta \text{ and } \tau_{\beta,\gamma}^{-1}(C) \subseteq B$$

and to

$$(3) \qquad \gamma \text{ is a specialization of } \beta \text{ and } \mathcal{O}_\beta(\tau_{\beta,\gamma}^{-1}(C)) \subseteq B,$$

since B is convex, and finally to

$$(4) \qquad \gamma \text{ is a specialization of } \beta \text{ and } (\beta, \mathcal{O}_\beta(\tau_{\beta,\gamma}^{-1}(C))) \qquad \text{is a specialization of } (\beta, B).$$

We have thus characterized the closure of the point (β, B) as follows.

Theorem 1. *Let (β, B) belong to $\mathrm{S}_r(A/k)$. Then, the closure of (β, B) is the set $\{(\gamma, C) \in \mathrm{S}_r(A/k) : \gamma$ is a specialization of β and $(\beta, \mathcal{O}_\beta(\tau_{\beta,\gamma}^{-1}(C)))$ is a specialization of $(\beta, B)\}$.*

Note that "γ is a specialization of β" is a statement about $\mathrm{Spec}_r(A)$. On the other hand, the condition "$(\beta, \mathcal{O}_\beta(\tau_{\beta,\gamma}^{-1}(C)))$ is a specialization of (β, B)"

takes place in the fiber $\pi_A^{-1}(\beta)$. The latter is related with the so called secondary specializations in [**H-K**].

Notation 2. If β and γ belong to $\mathrm{Spec}_r(A)$ and γ is a specialization of β, let $\mathcal{O}_{\beta,\gamma}$ denote the localization of $\mathcal{O}_\beta(A)$ at $p_\gamma \mathcal{O}_\beta(A)$. Clearly, $\mathcal{O}_{\beta,\gamma}$ is a convex valuation ring of $A(\beta)$ and thus $\mathcal{O}_\beta(A_{\beta,\gamma}) \subseteq \mathcal{O}_{\beta,\gamma}$.

Corollary 3. *Let (β, B) belong to $\mathrm{S}_r(A/k)$ and let γ be a specialization of β. If $\mathcal{O}_{\beta,\gamma} \subseteq B$, then $\pi_A^{-1}(\gamma)$ is contained in the closure of (β, B).*

Proof. Let (γ, C) be in $\pi_A^{-1}(\gamma)$. The obvious inclusion $\tau_{\beta,\gamma}^{-1}(C) \subseteq A_{\beta,\gamma}$ implies $\mathcal{O}_\beta(\tau_{\beta,\gamma}^{-1}(C)) \subseteq \mathcal{O}_\beta(A_{\beta,\gamma}) \subseteq \mathcal{O}_{\beta,\gamma}$, by convexity. □

4. The Relationship between the Real Riemann Surface and the Real Spectrum of a Ring.

Let A and R be rings, R real and $f : A \to R$ a morphism. We know that f induces a continuous mapping $\mathrm{Spec}_r(f) : \mathrm{Spec}_r(R) \to \mathrm{Spec}_r(A)$. Namely, given $\beta \in \mathrm{Spec}_r(R)$, then $\mathrm{Spec}_r(f)(\beta)$ is, by definition, the point $(f^{-1}(p_\beta), \leq_\alpha)$ where $\leq_\alpha$ is defined by letting $a \leq_\alpha b$ if and only if $f(a) \leq_\beta f(b)$, for all $a, b \in A$. Moreover, Spec_r is a contravariant functor from the category of rings to the category of topological spaces. Similarly, we have the following:

Theorem 4. *Let $\phi : k \to A$ and $f : A \to R$ be morphisms, R a real ring. Then f induces a continuous mapping $\mathrm{S}_r(f) : \mathrm{S}_r(R/k) \to \mathrm{S}_r(A/k)$. Thus, S_r is a contravariant functor from the category of k-algebras to the category of topological spaces. In addition, π is a natural transformation from S_r to Spec_r.*

Proof. Let (β, B) belong to $\mathrm{S}_r(R/k)$ and consider the point $\alpha = \mathrm{Spec}_r(f)(\beta)$. There exists a unique order-preserving morphism f_β such that the following diagram commutes

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & R \arrow[d] \\ A(\alpha) \arrow[r, "f_\beta"] & R(\beta) \end{tikzcd}$$

where the vertical maps are canonical. The mapping f_β is defined by $f_\beta(x(\alpha)) = f(x)(\beta)$, for all $x \in A$. Moreover, it is routinely checked that $f_\beta^{-1}(B)$ is a convex valuation ring in $A(\alpha)$. Then we let $\mathrm{S}_r(f)(\beta, B) := (\alpha, f_\beta^{-1}(B))$, by definition.

In order to prove continuity of $\mathrm{S}_r(f)$, it is sufficient to show that $\mathrm{S}_r(f)^{-1}(U_{x,y})$ is open in $\mathrm{S}_r(R/k)$, for all $x, y \in A$. Actually, we prove the equality $\mathrm{S}_r(f)^{-1}(U_{x,y}) = U_{f(x),f(y)}$. First, given $x, y \in A$, by definition of α we have $y(\alpha) > 0$ if and only if $f(y)(\beta) > 0$. Moreover, $xy^{-1}(\alpha) \in f_\beta^{-1}(B)$ if and only if $f_\beta(xy^{-1}(\alpha)) = f(x)f(y)^{-1}(\beta) \in B$, for all $x, y \in A$. From this, the desired equality follows.

Now the diagram

$$\begin{array}{ccc} \mathrm{S}_r(R/k) & \xrightarrow{\mathrm{S}_r(f)} & \mathrm{S}_r(A/k) \\ {\scriptstyle \pi_R}\downarrow & & {\scriptstyle \pi_A}\downarrow \\ \mathrm{Spec}_r(R) & \xrightarrow{\mathrm{Spec}_r(f)} & \mathrm{Spec}_r(A) \end{array}$$

is commutative, showing that π is a natural transformation. □

Proposition 5. *Let $\phi : k \to A$ be a morphism, R a real ring and $f : A \to R$ an étale morphism. Given β in $\mathrm{Spec}_r(R)$ and $\alpha = \mathrm{Spec}_r(f)(\beta)$, then $\kappa(\beta)$ is isomorphic to $\kappa(\alpha)$ and $\mathrm{S}_r(f)$ maps $\pi_R^{-1}(\beta)$ homeomorphically onto $\pi_A^{-1}(\alpha)$.*

Proof. Clearly, p_β belongs to $\mathrm{Spec}(f)^{-1}(p_\alpha)$ and $f_\beta : A(\alpha) \to R(\beta)$ is an ordered-preserving morphism of fields. By **[Ra]** p. 33 Propositions 10 and 11, if R is étale over A, then $R(\beta)$ is étale over $A(\alpha)$, which amounts to saying that $R(\beta)$ is a finite separable extension of $A(\alpha)$, via f_β. In particular, $R(\beta)$ and $A(\alpha)$ have isomorphic real closures $\kappa(\beta)$ and $\kappa(\alpha)$; we will identify them. As explained in **(RRS)**, $\pi_R^{-1}(\beta)$ and $\pi_A^{-1}(\alpha)$ are both homeomorphic to $\mathrm{S}_r(\kappa(\beta)/k)$ and the resulting composite homeomorphism maps (β, B) to (α, C) if and only if B and C have the same convex hull in $\kappa(\beta)$. If we further identify $A(\alpha)$ with its image in $R(\beta)$, then B is the convex hull of C in $R(\beta)$. □

5. Chevalley's Theorem for Real Riemann Surfaces.

We give a version of Chevalley's theorem, for real Riemann surfaces, which we will use in our proof of Theorem 9. The ideas in the proof of the following theorem are partially due to R. Huber, who kindly discussed with us about this, back in 1991.

Theorem 6. *Let $\phi : k \to A$ be a morphism, R a real ring and suppose that $f : A \to R$ is a morphism that makes R finitely presented over A. ~~If~~ $L \subseteq \mathrm{S}_r(R/k)$ is constructible, then $\mathrm{S}_r(f)(L)$ is constructible.*

Proof. As remarked in **(RRS)**, L equals $L_{\phi;t_1,\dots,t_n}$, for a certain quantifier-free formula $\phi(v_1, \dots, v_n)$ of $\mathcal{L}_v$ and some $t_1, \dots, t_n \in R$. By hypothesis, R

is the quotient of a polynomial ring $A[X]$ modulo a finitely generated ideal J and f is the composite of the canonical morphisms $\iota : A \to A[X]$ and $\epsilon : A[X] \to R$, where $X = (X_1, \dots, X_m)$, $m \in \mathbb{N}$. We choose a finite family $\{P_i(X) : i = 1, \dots, l\}$ generating J and, for each $s = 1, \dots, n$, we choose a preimage $T_s(X)$ of t_s in $A[X]$. Let $\{E_h \in A : h = 1, \dots, g\}$ be the set of coefficients of the polynomials $P_1, \dots, P_l, T_1, \dots, T_n$. We write x_j, e_h for the class of X_j, E_h modulo J and x for $(x_1, \dots, x_m)$.

If (β, B) belongs to L then

$$\bigwedge_{i=1}^{l} P_i(x)(\beta) = 0 \quad \wedge \quad \phi[t_1(\beta), \dots, t_n(\beta)] \tag{5}$$

is a quantifier-free statement about $(R(\beta), |_B)$. It can be obtained by substitution of $e_1(\beta), \dots, e_g(\beta), x_1(\beta), \dots, x_m(\beta)$ by $u_1, \dots, u_g, w_1, \dots, w_m$ in a quantifier-free formula $\psi(u_1, \dots, u_g, w_1, \dots, w_m)$ of $\mathcal{L}_v$. In other words, (5) coincides with

$$\psi[e_1(\beta), \dots, e_g(\beta), x_1(\beta), \dots, x_m(\beta)].$$

Let $\theta := \exists\, w_1 \dots w_m\ \psi$. Then θ is a formula of $\mathcal{L}_v$ whose free variables are a subset of $u_1, \dots, u_g$. By [**Pr**] Theorem 4.20, the theory of real closed fields with a non-trivial compatible valuation divisibility relation admits elimination of quantifiers. Then there exists a quantifier-free formula η, equivalent to θ, the free variables of which are a subset of $u_1, \dots, u_g$.

In order to show that $\mathrm{S}_r(f)(L)$ is constructible, we will prove that the following equalities hold true:

$$\mathrm{S}_r(f)(L) = L_1 = L_2 = L_3 = L_4 = L_{\eta; E_1, \dots, E_g},$$

where

$L_1 := \{(\alpha, C) \in \mathrm{S}_r(A/k) : (K, |_D) \models \theta[E_1(\alpha), \dots, E_g(\alpha)]$
for some extension $(K, |_D)$ of $(A(\alpha), |_C)\}$,

$L_2 := \{(\alpha, C) \in \mathrm{S}_r(A/k) : (K, |_D) \models \theta[E_1(\alpha), \dots, E_g(\alpha)]$
for some extension $(K, |_D)$ of $(A(\alpha), |_C), K$ real closed, D non-trivial$\}$,

$L_3 := \{(\alpha, C) \in \mathrm{S}_r(A/k) : (K, |_D) \models \eta[E_1(\alpha), \dots, E_g(\alpha)]$
for some extension $(K, |_D)$ of $(A(\alpha), |_C), K$ real closed, D non-trivial$\}$,

$L_4 := \{(\alpha, C) \in \mathrm{S}_r(A/k) : (K, |_D) \models \eta[E_1(\alpha), \dots, E_g(\alpha)]$
for some extension $(K, |_D)$ of $(A(\alpha), |_C)\}$.

Proof of $\mathrm{S}_r(f)(L) \subseteq L_1$: Let (β, B) belong to L and write (α, C) for $\mathrm{S}_r(f)(\beta, B)$. Then,

$$(R(\beta), |_B) \models \theta[e_1(\beta), \dots, e_g(\beta)] \tag{6}$$

holds true, since we can take w_j equal to $x_j(\beta)$, for $j = 1, \dots, m$. Consider the point $(\alpha_1, C_1) := \mathrm{S}_r(\epsilon)(\beta, B)$, where $\epsilon : A[X] \to R$ is the canonical epimorphism. Since S_r is functorial by Theorem 4, then we have a commutative diagram

$$\begin{array}{ccccc} A & \overset{\iota}{\hookrightarrow} & A[X] & \overset{\epsilon}{\twoheadrightarrow} & R \\ \downarrow & & \downarrow & & \downarrow \\ A(\alpha) & \overset{\iota_{\alpha_1}}{\hookrightarrow} & A[X](\alpha_1) & \overset{\epsilon_\beta}{\hookrightarrow} & R(\beta) \end{array}$$

where the vertical maps are canonical and $f_\beta = \epsilon_\beta \iota_{\alpha_1}$. Thus, $(A[X](\alpha_1), |_{C_1})$ is an extension of $(A(\alpha), |_C)$. Since (6) holds, θ equals $\exists\, w_1 \dots w_m\ \psi$, and ψ contains no existential quantifiers then, taking w_j equal to $X_j(\alpha_1)$, we see that

$$(A[X](\alpha_1), |_{C_1}) \models \theta[E_1(\alpha), \dots, E_g(\alpha)].$$

Thus (α, C) lies in L_1, because $(A[X](\alpha_1), |_{C_1})$ is an extension of $(A(\alpha), |_C)$.

Proof of $\mathrm{S}_r(f)(L) \supseteq L_1$: Suppose that $(K, |_D)$ is an extension of $(A(\alpha), |_C)$ and that

$$(K, |_D) \models \theta[E_1(\alpha), \dots, E_g(\alpha)]. \tag{7}$$

Let $W_1, \dots, W_m \in K$ be elements satisfying

$$(K, |_D) \models \psi[E_1(\alpha), \dots, E_g(\alpha), W_1, \dots, W_m]. \tag{8}$$

We extend the canonical morphism $A \to A(\alpha)$ to a morphism $\rho : A[X] \to K$ by mapping X_j to W_j, for $j = 1, \dots, m$. Let γ stand for the fixed order in K. Then $K = K(\gamma)$ and $\mathrm{Spec}_r(\rho)(\gamma)$ is a point in $\mathrm{Spec}_r(A[X])$; let us denote it by α_1. Write $W = (W_1, \dots, W_m)$. Since the conditions $0 = P_i(W) = \rho(P_i(X))$ are implicit in (8), then ρ factors through R, yielding a morphism $\varphi : R \to K$ such that $\varphi\epsilon = \rho$. We consider the point $(\beta, B) := \mathrm{S}_r(\varphi)(\gamma, D)$. We have a commutative diagram

$$\begin{tikzcd} A \arrow[r, hook, "\iota"] \arrow[d] & A[X] \arrow[r, two heads, "\epsilon"] \arrow[d, "\rho"'] & R \arrow[d] \arrow[dl, "\varphi"'] \\ A(\alpha) \arrow[r, hook] & K & R(\beta) \arrow[l, hook', "\varphi_\gamma"'] \end{tikzcd}$$

where $f = \epsilon\iota$ and the unlabelled vertical maps are canonical. It follows from the diagram that $(\alpha, C) = \mathrm{S}_r(f)(\beta, B)$. Moreover, since φ_γ is injective, ϕ is a subformula of ψ and ψ contains no existential quantifiers, then (8) implies

$$(R(\beta), |_B) \models \phi[t_1(\beta), \dots, t_n(\beta)],$$

since in K the element $t_s(\beta)$ equals a certain polynomial expression in the $E_h(\alpha)$'s.

Proof of $L_1 = L_2$: Clearly, $L_1 \supseteq L_2$. On the other hand, let $(K, |_D)$ be an extension of (α, C) such that

$$(K, |_D) \models \theta[E_1(\alpha), \dots, E_g(\alpha)].$$

Let K_1 be the real closure of the simple transcendental extension $K(Y)$ of K, where the field $K(Y)$ is endowed with the ordering that makes Y positive and infinitesimal with respect to K. Then K_1 has a non-trivial convex valuation ring D_1 such that $D = K \cap D_1$. Then

$$(K_1, |_{D_1}) \models \theta[E_1(\alpha), \dots, E_g(\alpha)],$$

since $(K_1, |_{D_1})$ is an extension of $(K, |_D)$ and θ contains no universal quantifiers.

Now, $L_2 = L_3$ holds again by [**Pr**] Theorem 4.20, and the proof of $L_3 = L_4$ is similar to that of $L_1 = L_2$, done above.

Proof of $L_4 \subseteq L_{\eta; E_1, \dots, E_g}$: If $(K, |_D)$ is an extension of $(A(\alpha), |_C)$ satisfying

$$(K, |_D) \models \eta[E_1(\alpha), \dots, E_g(\alpha)],$$

then it follows that

$$(A(\alpha), |_C) \models \eta[E_1(\alpha), \dots, E_g(\alpha)],$$

since η contains no existential quantifiers.

Proof of $L_4 \supseteq L_{\eta; E_1, \dots, E_g}$: If $(K, |_D)$ is any extension of $(A(\alpha), |_C)$ and

$$(A(\alpha), |_C) \models \eta[E_1(\alpha), \dots, E_g(\alpha)]$$

holds true, then

$$(K, |_D) \models \eta[E_1(\alpha), \dots, E_g(\alpha)]$$

follows, since η contains no universal quantifiers. □

Remark 7. Coste and Roy have proved that $\mathrm{Spec}_r(f)(L)$ is constructible, for every constructible subset L of $\mathrm{Spec}_r(R)$, when f is finitely presented, see [**C-R 2**] and [**C-R 1**]. Bearing in mind the existence of continuous sections of π_A and π_R which are homeomorphisms onto the image, as remarked in **(RRS)**, the Coste-Roy theorem follows from our Theorem 6.

6. The Local Homeomorphism Theorem for Real Riemann Surfaces.

The following result has been proved in [**Ro**] and [**A-R**]: let $\phi : k \to A$ and $f : A \to R$ be morphisms, R a real ring; if f is étale then $\mathrm{Spec}_r(f) : \mathrm{Spec}_r(R) \to \mathrm{Spec}_r(A)$ is a local homeomorphism. Using this theorem, here we prove an analogous statement for real Riemann surfaces. In order to do so, first we need a result relating the notions of constructible, Tychonoff-closed, Tychonoff-clopen, and closed, as well as the stabilility under specialization, for a subset of $\mathrm{S}_r(A/k)$.

Proposition 8. *Let $\phi : k \to A$ be a morphism, A a real ring. Then*
(a) *$F \subseteq \mathrm{S}_r(A/k)$ is Tychonoff-closed if and only if F is an intersection of constructible sets,*
(b) *$F \subseteq \mathrm{S}_r(A/k)$ is Tychonoff-clopen if and only if F is constructible,*
(c) *if $Y \subseteq \mathrm{S}_r(A/k)$ is Tychonoff-closed and $F \subseteq Y$, then F is closed in Y if and only if F is Tychonoff-closed and stable under specialization in Y.*

Proof. Analogous to the proof done for the real spectrum. □

Theorem 9. *Let $\phi : k \to A$ and $f : A \to R$ be morphisms, R a real ring. If f is étale, then $\mathrm{S}_r(f) : \mathrm{S}_r(R/k) \to \mathrm{S}_r(A/k)$ is a local homeomorphism.*

Proof. Given (β, B) in $\mathrm{S}_r(R/k)$, consider the point $\alpha = \mathrm{Spec}_r(f)(\beta)$ in $\mathrm{Spec}_r(A)$. Because the étale property is local on $\mathrm{Spec}(R)$, (see [**Ra**] p. 16 Propositions 5 and 6) we may replace R by R_g, for some $g \in R \setminus p_\beta$. Now, by the local structure theorem for étale morphisms, (see [**Ra**] p. 51 Theorem 1) there exists $h \in A \setminus p_\alpha$ such that R is A-isomorphic to a standard étale A_h-algebra, i.e., $R \simeq (A_h[X]/(j))_l$, where X is a transcendental element over A_h, j, l are polynomials in $A_h[X]$, with j monic and the class of the formal derivative j' in $(A_h[X]/(j))_l$ invertible.

Since $\mathrm{Spec}_r(f)$ is a local homeomorphism, there exist an open neighborhood H^β of β in $\mathrm{Spec}_r(R)$ and an open neighborhood H^α of α in $\mathrm{Spec}_r(A)$ such that H^α is homeomorphic to H^β. The proof in [**Ro**] shows further that H^α and H^β can be assumed to be constructible. Since π_R and π_A are both continuous and Tychonoff-continuous, then $G^\beta = \pi_R^{-1}(H^\beta)$ and $G^\alpha = \pi_A^{-1}(H^\alpha)$ are open constructible neighborhoods of (β, B) and $\mathrm{S}_r(f)(\beta, B)$, respectively. We will show that G^β and G^α are homeomorphic.

First notice that, by Proposition 5, $\pi_R^{-1}(\gamma)$ is homeomorphic to $\pi_A^{-1}(\mathrm{Spec}_r(f)(\gamma))$, for every $\gamma \in H^\beta$. Since we may express G^β and G^α as unions of π-fibers

$$G^\beta = \bigcup_{\gamma \in H^\beta} \pi_R^{-1}(\gamma)$$
$$G^\alpha = \bigcup_{\delta \in H^\alpha} \pi_A^{-1}(\delta) = \bigcup_{\gamma \in H^\beta} \pi_A^{-1}(\mathrm{Spec}_r(f)(\gamma)),$$

we conclude that $\mathrm{S}_r(f)|_{G^\beta}$ is a bijection onto G^α. By Theorem 4, $\mathrm{S}_r(f)|_{G^\beta}$ is continuous. It only remains to show that $\mathrm{S}_r(f)|_{G^\beta}$ is a closed mapping. In order to do so, let $F \subseteq G^\beta$ be closed in G^β. We want to show that $\mathrm{S}_r(f)(F)$ is closed in G^α, for which it is enough to see that $\mathrm{S}_r(f)(F)$ is Tychonoff-closed in G^α and stable under specialization in G^α, by Proposition 8 (c).

Proof of $\mathrm{S}_r(f)(F)$ *Tychonoff-closed in* G^α: F is closed in G^β, whence Tychonoff-closed in G^β. Then F equals an intersection $\bigcap_{i\in J} G^\beta \cap F_i$, for some constructible subsets F_i of $\mathrm{S}_r(R/k)$, by Proposition 8 (a). Since $\mathrm{S}_r(f)|_{G^\beta}$ is bijective, then $\mathrm{S}_r(f)(F) = \bigcap_{i\in J} G^\alpha \cap \mathrm{S}_r(f)(F_i)$ and $\mathrm{S}_r(f)(F_i)$ is constructible, by Theorem 6. Thus, $\mathrm{S}_r(f)(F)$ is Tychonoff-closed in G^α, again by Proposition 8 (a).

Proof of $\mathrm{S}_r(f)(F)$ *stable under specialization in* G^α: Suppose that (δ_1, D_1) is a specialization of (δ_2, D_2), with (δ_1, D_1) in G^α and (δ_2, D_2) in $\mathrm{S}_r(f)(F)$. We want to show that (δ_1, D_1) belongs to $\mathrm{S}_r(f)(F)$. Let us denote τ_{δ_2,δ_1} by τ_δ. By expression (2), δ_1 is a specialization of δ_2 and $\tau_\delta^{-1}(D_1) \subseteq D_2$. Taking $\gamma_i = \mathrm{Spec}_r(f)^{-1}(\delta_i)$, for $i = 1, 2$, we have that γ_1 is a specialization of γ_2, since H^α and H^β are homeomorphic. Then we find points (γ_1, C_1) in G^β and (γ_2, C_2) in F such that $(\delta_i, D_i) = \mathrm{S}_r(f)(\gamma_i, C_i)$, for $i = 1, 2$. Let us denote τ_{γ_1,γ_2} by τ_γ.

Clearly, it suffices to show that (γ_1, C_1) belongs to F. By expression (2), this holds if $\tau_\gamma^{-1}(C_1) \subseteq C_2$. By the proof of Proposition 5, we know that after adecuate identifications, $R(\gamma_i)$ is a finite ordered field extension of $A(\delta_i)$ and $C_i = \mathcal{O}_{\gamma_i}(D_i)$, for $i = 1, 2$. The following commutative diagrams ilustrate the situation.

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] & \\
A[\delta_2] \arrow[rr, two heads] \arrow[d] & & A[\delta_1] \arrow[d] \\
A_{\delta_2,\delta_1} \arrow[rr, two heads, "\tau_\delta"] \arrow[d] & & A(\delta_1) \arrow[d] \\
A(\delta_2) \arrow[d] & & R(\gamma_1) \\
R(\gamma_2) & &
\end{tikzcd}
\qquad
\begin{tikzcd}
& R \arrow[dl] \arrow[dr] & \\
R[\gamma_2] \arrow[rr, two heads] \arrow[d] & & R[\gamma_1] \arrow[d] \\
R_{\gamma_2,\gamma_1} \arrow[rr, two heads, "\tau_\gamma"] \arrow[d] & & R(\gamma_1) \\
R(\gamma_2) & &
\end{tikzcd}$$

Let z_2 belong to $\tau_\gamma^{-1}(C_1)$ and consider $z_1 = \tau_\gamma(z_2)$ in C_1. Since $C_1 = \mathcal{O}_{\gamma_1}(D_1)$, there exists d_1 in D_1 such that $|z_1| <_{\gamma_1} |d_1|$. Now, d_1 equals $uv^{-1}(\delta_1)$ for some $u, v \in A$ with $v(\delta_1) \neq 0$. Then $v(\delta_2) \neq 0$ and $d_2 = uv^{-1}(\delta_2)$ belongs to $\tau_\delta^{-1}(D_1) \subseteq D_2 \subseteq C_2$. Since τ_γ is order-preserving, then $|z_2| <_{\gamma_2} |d_2|$, and this finishes the proof. □

Remark 10. Note that Theorem 9 is a strengthening of Roy's theorem mentioned above.

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