Twisted Alexander invariants and hyperbolic volume

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Abstract: We give a volume formula of hyperbolic knot complements using twisted Alexander invariants.

Key words: Twisted Alexander polynomial; hyperbolic knot; volume.

1. Introduction. The purpose of this note is to give a formula of the hyperbolic volume of a knot complement using twisted Alexander invariants.

A twisted Alexander polynomial was first defined in [3] for knots in the 3-sphere, and Wada ([10]) generalized this work and showed how to define a twisted Alexander polynomial given only a presentation of a group and representations to $\mathbf{Z}$ and $\mathrm{GL}(V)$ where V is a finite dimensional vector space over a field. In [2], Kitano proved that in the case of knot groups the twisted Alexander polynomial can be regarded as a Reidemeiser torsion.

Let M be a compact and oriented 3-manifold whose interior admits a finite volume hyperbolic structure. Porti ([8]) has investigated the Reidemeister torsion of M associated with the adjoint representation $\mathrm{Ad}\circ\mathrm{Hol}_M$ of its holonomy representation $\mathrm{Hol}_M : \pi_1(M) \to \mathrm{PSL}(2, \mathbf{C})$, and then Yamaguchi showed in [13] a relationship between the Porti's Reidemeister torsion and the twisted Alexander invariant explicitly.

Müller's work ([7]) provides the relation between the Ray-Singer torsion and the hyperbolic volume of a compact hyperbolic 3-manifold. By another work ([6]) of Müller on the equivalence between the Reidemeister torsion and the Ray-Singer torsion for unimodular representations, we know the hyperbolic volume of a compact 3-manifold can be expressed using a Reidemeister torsion. After the works, Menal-Ferrer and Porti ([5]) obtained a formula of the volume of a cusped hyperbolic 3-manifold M using 'Higher-dimensional Reidemeister torsion invariants', which are associated with representations $\rho_n : \pi_1(M) \to \mathrm{SL}(n, \mathbf{C})$ corresponding to the holonomy representation $\mathrm{Hol}_M : \pi_1(M) \to \mathrm{PSL}(2, \mathbf{C})$ (see Section 3 for the detail).

In this note, we show that the Yamaguchi's method in [12,13] is applicable to Higher-dimensional Reidemeister torsion invariants, so that we have a formula of the hyperbolic volume of a knot complement using twisted Alexander invariants. Let $\Delta_{K,\rho_n}(t)$ be the twisted Alexander invariant of Wada's notation ([10]). For the integer $k(>1)$, set $\mathcal{A}_{K,2k}(t) := \frac{\Delta_{K,\rho_{2k}}(t)}{\Delta_{K,\rho_2}(t)}$ and $\mathcal{A}_{K,2k+1}(t) := \frac{\Delta_{K,\rho_{2k+1}}(t)}{\Delta_{K,\rho_3}(t)}$.

Theorem 1.1. *Let K be a hyperbolic knot in the 3-sphere. Then*

$$\lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k+1}(1)|}{(2k+1)^2} = \lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k}(1)|}{(2k)^2} = \frac{\mathrm{Vol}(K)}{4\pi}.$$

In the last section, we give some calculations for the figure eight knot. The details, including link case, will be given elsewhere.

2. Reidemeister torsions and twisted Alexander invariants. Following [9] and [13], we review some definitions and conventions in this section.

Let $\mathbf{F}$ be a field and $C_* = (C_*, \partial)$ a chain complex of finite dimensional $\mathbf{F}$-vector spaces:

$$0 \to C_d \xrightarrow{\partial} C_{d-1} \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_0 \to 0.$$

For each i, we denote by $B_i = \mathrm{Im}(C_{i+1} \xrightarrow{\partial} C_i)$, $Z_i = \ker(C_i \xrightarrow{\partial} C_{i-1})$, and the homology is denoted by $H_i = Z_i/B_i$. By the definition of Z_i, B_i and H_i, we obtain the following exact sequence:

$$0 \to Z_i \to C_i \xrightarrow{\partial} B_{i-1} \to 0,$$
$$0 \to B_i \to Z_i \to H_i \to 0.$$

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Let $\widetilde{B}_{i-1}$ be a lift of B_{i-1} to C_i, and $\widetilde{H}_i$ a lift of H_i to Z_i. Then we can decompose C_i as follows:

$$\begin{aligned} C_i &= Z_i \oplus \widetilde{B}_{i-1} \\ &= B_i \oplus \widetilde{H}_i \oplus \widetilde{B}_{i-1}. \end{aligned}$$

Let c^i be a basis for C_i and $\mathbf{c}$ the collection $\{c^i\}_{i\geq 0}$. Similarly, let h^i be a basis for H_i, if nonzero, and $\mathbf{h}$ the collection $\{h^i\}_{i\geq 0}$. We choose b^i a basis of B_i. Let $\widetilde{b}^{i-1}$ be a lift of b^{i-1} to C_i, and $\widetilde{h}^i$ a lift of h^i to Z_i, then we have a new basis $b^i \sqcup \widetilde{b}^{i-1} \sqcup \widetilde{h}^i$ of C_i, where $\sqcup$ means a disjoint union. We denote by $[b^i, \widetilde{b}^{i-1}, \widetilde{h}^i / c^i]$ the determinant of the transformation matrix from the basis c^i to $b^i \sqcup \widetilde{b}^{i-1} \sqcup \widetilde{h}^i$.

Definition 2.1. The torsion of the chain complex C_* with basis $\mathbf{c}$ and $\mathbf{h}$ for H_i is:

$$\mathrm{tor}(C_*, \mathbf{c}, \mathbf{h}) = \prod_{i=0}^{d} [b^i, \widetilde{b}^{i-1}, \widetilde{h}^i / c^i]^{(-1)^{i+1}} \quad \in \mathbf{F}^*/\{\pm 1\}.$$

It is known that $\mathrm{tor}(C_*, \mathbf{c}, \mathbf{h})$ is independent of the choice of b^i and the lifts $\widetilde{b}^{i-1}$ and $\widetilde{h}^i$.

Remark 2.2. In [5], Menal-Ferrer and Porti use $(-1)^i$ instead of $(-1)^{i+1}$ in Definition 2.1. Then the sign of the right-hand side of the equation in Theorem 7.1 in [5] becomes opposite. See Remark 2.2 and Theorem 4.5 in [9].

Let W be a finite CW-complex, and $\rho : \pi_1(W, *) \to \mathrm{SL}(n, \mathbf{F})$ a representation of its fundamental group. Consider the chain complex of vector spaces

$$C_*(W, \rho) := \mathbf{F}^n \otimes_\rho C_*(\widetilde{W}; \mathbf{Z})$$

where $C_*(\widetilde{W}, \mathbf{Z})$ denotes the simplicial complex of the universal covering of W and $\otimes_\rho$ means that one takes the quotient of $\mathbf{F}^n \otimes_{\mathbf{Z}} C_*(\widetilde{W}; \mathbf{Z})$ by $\mathbf{Z}$-module generated by

$$\rho(\gamma)^{-1} v \otimes c - v \otimes \gamma \cdot c.$$

Here, $v \in \mathbf{F}^n$, $\gamma \in \pi_1(W, *)$ and $c \in C_*(\widetilde{W}; \mathbf{Z})$. Namely,

$$v \otimes \gamma \cdot c = \rho(\gamma)^{-1} v \otimes c \quad \forall \gamma \in \pi_1(W, *).$$

The boundary operator is defined by linearity and $\partial(v \otimes c) = (\mathrm{Id} \otimes \partial)(v \otimes c) = v \otimes \partial c$. We denote by $H_*(W, \rho)$ the homology of this complex.

Let $\{v_1, \ldots, v_n\}$ be a basis of $\mathbf{F}^n$ and let $c_1^i, \ldots, c_{k_i}^i$ denote the set of i-dimensional cells of W. We take a lift $\tilde{c}_j^i$ of the cell c_j^i in $\widetilde{W}$. Then, for each i, $\tilde{c}^i = \{\tilde{c}_1^i, \ldots, \tilde{c}_{k_i}^i\}$ is a basis of the $\mathbf{Z}[\pi_1(W)]$-module $C_i(\widetilde{W}; \mathbf{Z})$. Thus we have the following basis of $C_i(W, \rho)$:

$$c^i = \{v_1 \otimes \tilde{c}_1^i, v_2 \otimes \tilde{c}_1^i, \ldots, v_n \otimes \tilde{c}_{k_i}^i\}.$$

Suppose $H_i(W, \rho) \neq 0$, and let h^i be a basis of $H_i(W; \rho)$. We denote by $\mathbf{h}$ the basis $\{h^0, \ldots, h^{\dim W}\}$ of $H_*(W, \rho)$. Then $\mathrm{tor}(C_*(W, \rho), \mathbf{c}, \mathbf{h})$ $(\in \mathbf{F}^*/\{\pm 1\})$ is well defined. Note that it does not depend on the lifts of the cells $\tilde{c}^i$ since $\det \rho = 1$. Further, if the Euler characteristic of W is equal to zero (e.g. the case that W corresponds to a knot exterior), it does not depend on the choice of a basis $\{v_1, \ldots, v_n\}$ (cf. Lemma 2.4.2 in [13]).

Remark 2.3. The Reidemeister torsion is independent of the choice of a base point $*$ of the fundamental group $\pi_1(W, *)$. Furthermore, it is known that the Reidemeister torsion is an invariant under subdivision of the cell decomposition of W with ρ-coefficients up to factor ± 1.

Remark 2.4. Let K be a knot in the 3-sphere S^3 and $M_K = S^3 - \mathrm{int}N(K)$. We denote by $G(K)$ the fundamental group of M_K. From the result of Waldhausen ([11]), the Whitehead group $\mathrm{Wh}(G(K))$ is trivial. In such a case, the Reidemeister torsion does not depend on the choice of its CW-structure. Suppose $H_*(M_K, \rho) = 0$. Then the Reidemeister torsion does not depend on $\mathbf{h} = \emptyset$. In this case we denote by $\mathrm{tor}(M_K, \rho)$ the Reidemeister torsion.

Let α be a surjective homomorphism from $\pi_1(W, *)$ to the multiplicative group $\langle t \rangle$. Instead of a representation $\rho : \pi_1(W, *) \to \mathrm{SL}(n, \mathbf{F})$, consider the twisted representation:

$$\alpha \otimes \rho : \pi_1(W, *) \to \mathrm{GL}(\mathbf{F}(t)),$$

where $\mathbf{F}(t)$ is the field of fraction of the polynomial ring $\mathbf{F}[t]$. By the same method as above, we can define $\mathrm{tor}(C_*(W, \alpha \otimes \rho), \mathbf{1} \otimes \mathbf{c}, \mathbf{h})$ $(\in \mathbf{F}^*(t)/\{\pm t^{n\mathbf{Z}}\})$. As the determinant is not one, there is an independency factor t^{nm}, for some integer m. More precisely, we define:

$$C_*(W, \alpha \otimes \rho) = \mathbf{F}(t) \otimes_{\mathbf{F}} \mathbf{F}^n \otimes_\rho C_*(\widetilde{W}; \mathbf{Z}),$$

where the action is given by $f \otimes v \otimes (\gamma \cdot c) = f \cdot t^{\alpha(\gamma)} \otimes \rho(\gamma)^{-1} v \otimes c$ for $\gamma \in \pi_1(W, *)$. The boundary operator is defined by linearity and $\partial(f \otimes v \otimes c) = f \otimes v \otimes \partial c$.

Kitano ([2]) investigated the relationship between the Reidemeister torsions and the twisted Alexander invariants for knots. Namely, he proved

Theorem 2.5 ([2])**.** *Let K be a knot in the 3-sphere S^3 and $M_K = S^3 - \mathrm{int}N(K)$. Suppose ρ is a* *non-trivial representation such that* $H_*(M_K, \rho) = 0$. *Then,* $H_*(M_K, \alpha \otimes \rho) = 0$ *and* $\mathrm{tor}(M_K, \alpha \otimes \rho) = \Delta_{K,\rho}(t)$, *where* $\Delta_{K,\rho}(t)$ *is the twisted Alexander invariant.*

See also Theorem 2.13 in [9]. The twisted Alexander invariant can be computed using the Fox calculus ([1,2,10]).

3. Representations of the fundamental groups of hyperbolic 3-manifolds. Let M be an oriented, complete, hyperbolic 3-manifold of finite volume. Then M has the holonomy representation: $\mathrm{Hol}_M : \pi_1(M, *) \to \mathrm{Isom}^+\mathbf{H}^3$, where $\mathrm{Isom}^+\mathbf{H}^3$ is the orientation preserving isometry group of hyperbolic 3-space $\mathbf{H}^3$. Using the upper half-space model, $\mathrm{Isom}^+\mathbf{H}^3$ is identified with $\mathrm{PSL}(2, \mathbf{C}) = \mathrm{SL}(2, \mathbf{C})/\{\pm 1\}$. It is known that Hol_M can be lifted to $\mathrm{SL}(2, \mathbf{C})$, and such lifts are in canonical one-to-one correspondence with spin structures on M. Thus, attached to a fixed spin structure η on M, we get a representation:

$$\mathrm{Hol}_{(M,\eta)} : \pi_1((M, \eta), *) \to \mathrm{SL}(2, \mathbf{C}).$$

Let W be a finite CW-complex and ρ a representation of $\pi_1(W, *)$ to $\mathrm{SL}(2, \mathbf{C})$. Then the pair $(\mathbf{C}^2, \rho)$ is an $\mathrm{SL}(2, \mathbf{C})$-representation of $\pi_1(W, *)$ by the standard action $\mathrm{SL}(2, \mathbf{C})$ to $\mathbf{C}^2$. It is known that the pair of the symmetric product $\mathrm{Sym}^{n-1}(\mathbf{C}^2)$ and the induced action by $\mathrm{SL}(2, \mathbf{C})$ gives an n-dimensional irreducible representation of $\mathrm{SL}(2, \mathbf{C})$. More precisely, let V_n be the vector space of homogeneous polynomials on $\mathbf{C}^2$ with degree $n - 1$, that is,

$$V_n = \mathrm{span}_{\mathbf{C}}\langle x^{n-1}, x^{n-2}y, \ldots, xy^{n-2}, y^{n-1}\rangle.$$

Then the symmetric product $\mathrm{Sym}^{n-1}(\mathbf{C}^2)$ can be identified with V_n and the action of $A \in \mathrm{SL}(2, \mathbf{C})$ is expressed as

$$A \cdot p\begin{pmatrix} x \\ y \end{pmatrix} = p\left(A^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

where $p\begin{pmatrix} x \\ y \end{pmatrix}$ is a homogeneous polynomial and the right-hand side is determined by the action of A^{-1} on the column vector as a matrix multiplication. We denote by (V_n, σ_n) the representation given by this action of $\mathrm{SL}(2, \mathbf{C})$ where σ_n means the homomorphism from $\mathrm{SL}(2, \mathbf{C})$ to $\mathrm{GL}(V_n)$. It is known that each representation (V_n, σ_n) turns into an irreducible $\mathrm{SL}(n, \mathbf{C})$-representation of $\mathrm{SL}(2, \mathbf{C})$ and that every irreducible n-dimensional representation of $\mathrm{SL}(2, \mathbf{C})$ is equivalent to (V_n, σ_n). Composing $\mathrm{Hol}_{(M,\eta)}$ with σ_n, we obtain the following representation:

$$\rho_n : \pi_1((M, \eta), *) \to \mathrm{SL}(n, \mathbf{C}).$$

In the following sections, we will discuss Reidemeister torsions associated with this representation ρ_n. Note that there are several computations of the Reidemeister torsions associated with σ_{2k} in [14,15].

4. The results of Menal-Ferrer and Porti. In this note, we focus on a knot complement. We introduce the results of Menal-Ferrer and Porti ([4,5]) in this setting.

Let K be a hyperbolic knot in the 3-sphere S^3, that is, $S^3 - K$ is an oriented, complete, finite-volume hyperbolic manifold with only one cusp. Then, $S^3 - K$ may be regarded as the interior of a compact manifold M_K such that $\partial M_K = T$ where T is homeomorphic to a torus T^2. In what follows, we consider the compact manifold M_K instead of $S^3 - K$.

By Corollary 3.7 in [4], we have that $\dim_{\mathbf{C}} H^i(M_K, \rho_n) = 0$ $(i = 0, 1, 2)$ if n is even, and that $\dim_{\mathbf{C}} H^0(M_K, \rho_n) = 0$, $\dim_{\mathbf{C}} H^1(M_K, \rho_n) = \dim_{\mathbf{C}} H^2(M_K, \rho_n) = 1$ if n is odd. Further, in [5], Menal-Ferrer and Porti proved the following. (Note that Poincaré duality with coefficients in ρ_n holds (Corollary 3.7 in [5]).)

Proposition 4.1 (Proposition 4.6 in [5]). *Suppose that* $H_*(T; \rho_n) \neq 0$. *Let* $G < \pi_1(M_K, *)$ *be some fixed realization of the fundamental group of* T *as a subgroup of* $\pi_1(M_K, *)$. *Choose a non-trivial cycle* $\theta \in H_1(T; \mathbf{Z})$, *and a non-trivial vector* $v \in V_n$ *fixed by* $\rho_n(G)$. *Then the following holds*:
(a) *A basis for* $H_1(M_K, \rho_n)$ *is given by* $i_*([v \otimes \widetilde{\theta}])$.
(b) *A basis for* $H_2(M_K, \rho_n)$ *is given by* $i_*([v \otimes \widetilde{T}])$.
Here, $i : T \hookrightarrow M_K$ *denotes the inclusion.*

Set $h^1 = i_*([v \otimes \widetilde{\theta}])$, $h^2 = i_*([v \otimes \widetilde{T}])$, and $\mathbf{h} = \{h^1, h^2\}$. On the other hand, Menal-Ferrer and Porti (Theorem 0.2 in [4]) proved that $H^*(M_K, \rho_{2k}) = 0$ for $k \geq 1$. Therefore, we may define the following quotients:

$$\mathcal{T}_{2k+1}(M_K, \eta) := \frac{\mathrm{tor}(M_K, \rho_{2k+1}, \mathbf{h})}{\mathrm{tor}(M_K, \rho_3, \mathbf{h})} \quad \in \mathbf{C}^*/\{\pm 1\},$$

$$\mathcal{T}_{2k}(M_K, \eta) := \frac{\mathrm{tor}(M_K, \rho_{2k})}{\mathrm{tor}(M_K, \rho_2)} \quad \in \mathbf{C}^*/\{\pm 1\}.$$

The quantity $\mathcal{T}_{2k+1}$ is independent of the spin structure because of the fact that an odd-dimensional irreducible complex representation of $\mathrm{SL}(2,\mathbf{C})$ factors through $\mathrm{PSL}(2,\mathbf{C})$. Since $S^3 - K$ has only one cusp, then all spin structures on M_K are acyclic (Corollary 3.4 in [5]). This means that the limit of $\mathcal{T}_{2k}$ is also independent of the spin structure (Theorem 7.1 in [5]). Thus it is not necessary to consider a spin structure on M_K in our setting. Hence, the above definition may be simplified to the following form deleting η.

Definition 4.2.

$$\mathcal{T}_{2k+1}(M_K) := \frac{\mathrm{tor}(M_K, \rho_{2k+1}, \mathbf{h})}{\mathrm{tor}(M_K, \rho_3, \mathbf{h})} \quad \in \mathbf{C}^*/\{\pm 1\},$$
$$\mathcal{T}_{2k}(M_K) := \frac{\mathrm{tor}(M_K, \rho_{2k})}{\mathrm{tor}(M_K, \rho_2)} \quad \in \mathbf{C}^*/\{\pm 1\}.$$

Note that it is proved that the quotient is independent of the choices $\mathbf{h}$ (Proposition 4.2 in [5]). Then, we can reduce Theorem 7.1 in [5] to the following statement:

Theorem 4.3 (Theorem 7.1 in [5])**.**

$$\lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k+1}(M_K)|}{(2k+1)^2} = \lim_{k\to\infty} \frac{\log|\mathcal{T}_{2k}(M_K)|}{(2k)^2} = \frac{\mathrm{Vol}(K)}{4\pi}.$$

As in Remark 2.2, the sign of the right-hand side is plus.

5. Proof of Theorem 1.1.

Case 1. Even-dimensional representation ρ_{2k} case.

By Theorem 0.2 in [4], $H^*(M_K, \rho_{2k}) = 0$ for $k \geq 1$. Then, by Theorem 2.5, we can prove that $\mathrm{tor}(M_K, \rho_{2k}) = \mathrm{tor}(M_K, \alpha\otimes\rho_{2k})|_{t=1} = \Delta_{K,\rho_{2k}}(1)$ from the map at the chain level $C_*(M_K, \alpha\otimes\rho_{2k}) \to C_*(M_K, \rho_{2k})$ induced by evaluation $t = 1$. Then, we have:

$$\mathcal{T}_{2k}(M_K) = \frac{\mathrm{tor}(M_K, \rho_{2k})}{\mathrm{tor}(M_K, \rho_2)} = \frac{\Delta_{K,\rho_{2k}}(1)}{\Delta_{K,\rho_2}(1)} = \mathcal{A}_{K,2k}(1).$$

Hence we have done in the case of ρ_{2k} in Theorem 1.1: $\lim_{k\to\infty} \frac{\log|\mathcal{A}_{K,2k}(1)|}{(2k)^2} = \frac{\mathrm{Vol}(K)}{4\pi}$ by Theorem 4.3.

Case 2. Odd-dimensional representation ρ_{2k+1} case.

Although the idea of the proof is the same as Yamaguchi's one in [12,13], I think it is worth outlining it here for the convenience of readers. He investigated the case of the adjoint representation of $\mathrm{SL}(2,\mathbf{C})$, which is essentially equivalent to ρ_3 in our setting.

The homology group $H_*(M_K;\mathbf{Z}) = H_0(M_K;\mathbf{Z}) \oplus H_1(M_K;\mathbf{Z})$ has the basis $\{[p],[\mu]\}$, where $[p]$ is the homology class of a point and $[\mu]$ is that of the meridian of K. Further, $H_1(\partial M_K;\mathbf{Z})$ has the basis $\{[\mu],[\lambda]\}$, where $[\lambda]$ is the homology class of a longitude of K. By Proposition 4.1, we may define $h^1 = i_*([v\otimes\widetilde{\lambda}])$, $h^2 = i_*([v\otimes\widetilde{T}])$ and $\mathbf{h} = \{h^1, h^2\}$.

It is known that M_K collapses to a 2-dimensional CW-complex W with only one vertex. We call φ this deformation. Thus M_K is simple homotopy equivalent to W. It is enough to prove the theorem for W since a Reidemeister torsion is a simple homotopy invariant.

By Proposition 3.5 in [1], we have $H_0(W, \alpha\otimes\rho_{2k+1}) = 0$. Further, we have the next lemma by the same argument as Proposition 7 in [12] or Proposition 3.1.1 in [13].

Lemma 5.1. *For* $* = 1, 2$, *we have*: $H_*(M_K, \alpha\otimes\rho_{2k+1}) = 0$.

Proposition 5.2. $\mathrm{tor}(M_K, \alpha\otimes\rho_{2k+1})$ *has a simple zero at* $t = 1$. *Moreover the following holds*:

$$\mathrm{tor}(M_K, \rho_{2k+1}, \mathbf{h}) = \lim_{t\to 1} \frac{\mathrm{tor}(M_K, \alpha\otimes\rho_{2k+1})}{t-1}.$$

Proof. We define the subchain complex $C'_*(W,\rho_{2k+1})$ of the chain complex $C_*(W,\rho_{2k+1})$ by

$$C'_2(W,\rho_{2k+1}) = \mathrm{span}_{\mathbf{C}}\langle v\otimes\widetilde{\varphi(T)}\rangle,$$
$$C'_1(W,\rho_{2k+1}) = \mathrm{span}_{\mathbf{C}}\langle v\otimes\widetilde{\varphi(\lambda)}\rangle$$

and $C'_i(W,\rho_{2k+1}) = 0$ ($i \neq 1,2$). Note that v is fixed by $\rho_{2k+1}(G)$, and the boundary operators of $C'_*(W,\rho_{2k+1})$ are zero by the definition. The modules of this subchain complex are lifts of homology groups $H_*(W,\rho_{2k+1})$. Similarly, we define the subcomplex $C'_*(W,\alpha\otimes\rho_{2k+1})$ of $C_*(W,\alpha\otimes\rho_{2k+1})$ by

$$C'_2(W,\alpha\otimes\rho_{2k+1}) = \mathrm{span}_{\mathbf{C}(t)}\langle 1\otimes v\otimes\widetilde{\varphi(T)}\rangle,$$
$$C'_1(W,\alpha\otimes\rho_{2k+1}) = \mathrm{span}_{\mathbf{C}(t)}\langle 1\otimes v\otimes\widetilde{\varphi(\lambda)}\rangle$$

and $C'_i(W,\alpha\otimes\rho_{2k+1}) = 0$ for $i \neq 1,2$. Since v is an invariant vector of $\rho_{2k+1}(G)$, we have:

$$\begin{aligned}
\partial(1\otimes v\otimes\widetilde{\varphi(T)}) &= 1\otimes v\otimes\partial(\widetilde{\varphi(T)}) \\
&= 1\otimes v\otimes(\mu\cdot\widetilde{\varphi(\lambda)}) - 1\otimes v\otimes\widetilde{\varphi(\lambda)} \\
&= t\otimes\rho_{2k+1}^{-1}(\mu)v\otimes\widetilde{\varphi(\lambda)} \\
&\quad - 1\otimes v\otimes\widetilde{\varphi(\lambda)} \\
&= t\otimes v\otimes\widetilde{\varphi(\lambda)} - 1\otimes v\otimes\widetilde{\varphi(\lambda)} \\
&= (t-1)(1\otimes v\otimes\widetilde{\varphi(\lambda)}).
\end{aligned}$$

Thus the boundary operators of $C'_*(W, \alpha\otimes\rho_{2k+1})$ are given by

$$0 \to C'_2(W, \alpha\otimes\rho_{2k+1}) \xrightarrow{t-1} C'_1(W, \alpha\otimes\rho_{2k+1}) \to 0.$$

This means that the homology of $C'_*(W, \alpha\otimes\rho_{2k+1})$ is zero.

By the definition, the chain complex $C'_*(W, \rho_{2k+1})$ has the natural basis:

$$\mathbf{c}' = \{v\otimes\widetilde{\varphi(T)}, v\otimes\widetilde{\varphi(\lambda)}\}.$$

Let $C''_*(W, \rho_{2k+1})$ be the quotient of $C_*(W, \rho_{2k+1})$ by $C'_*(W, \rho_{2k+1})$, $\mathbf{c}''$ a basis of $C''_*(W, \rho_{2k+1})$, and $\bar{\mathbf{c}}''$ a lift of $\mathbf{c}''$ to $C_*(W, \rho_{2k+1})$. By Lemma 5.1, we can apply Proposition 3.3.1 in [13] to this setting, then we have:

$$\begin{aligned}\lim_{t\to 1}&\frac{\mathrm{tor}(C_*(W, \alpha\otimes\rho_{2k+1}), \mathbf{1}\otimes\mathbf{c}'\sqcup\mathbf{1}\otimes\bar{\mathbf{c}}'')}{\mathrm{tor}(C'_*(W, \alpha\otimes\rho_{2k+1}), \mathbf{1}\otimes\mathbf{c}')}\\ &= \mathrm{tor}(C_*(W, \rho_{2k+1}), \mathbf{c}'\sqcup\bar{\mathbf{c}}'', \mathbf{h}).\end{aligned}$$

By the calculation above, we have $\mathrm{tor}(C'_*(W, \alpha\otimes\rho_{2k+1}), \mathbf{1}\otimes\mathbf{c}') = t-1$, thus we have this proposition. □

Proof of Theorem 1.1. By Theorem 2.5 and Lemma 5.1, we have $\mathrm{tor}(M_K, \alpha\otimes\rho_{2k+1}) = \Delta_{K,\rho_{2k+1}}(t)$. We also have $\Delta_{K,\rho_{2k+1}}(t) = (t-1)\tilde{\Delta}_{K,\rho_{2k+1}}(t)$ and $\mathrm{tor}(M_K, \rho_{2k+1}, \mathbf{h}) = \tilde{\Delta}_{K,\rho_{2k+1}}(1)$ by Proposition 5.2, where $\tilde{\Delta}_{K,\rho_{2k+1}}(t)$ is a rational function. Then,

$$\begin{aligned}\mathcal{A}_{K,2k+1}(1) &= \frac{\tilde{\Delta}_{K,\rho_{2k+1}}(1)}{\tilde{\Delta}_{K,\rho_3}(1)} = \frac{\mathrm{tor}(M_K, \rho_{2k+1}, \mathbf{h})}{\mathrm{tor}(M_K, \rho_3, \mathbf{h})}\\ &= \mathcal{T}_{2k+1}(M_K).\end{aligned}$$

Thus we have Theorem 1.1 by Theorem 4.3. □

6. Some calculations on the figure eight knot complement. Let K be the figure eight knot 4_1. Note that it is known that the volume of K is $2.02988\cdots$. The knot group $G(K)$ has the following presentation:

$$G(K) = \langle a, b \mid ab^{-1}a^{-1}ba = bab^{-1}a^{-1}b\rangle,$$

where a and b correspond to the meridians of K. Consider the representation of this fundamental group:

$$\rho(a) = \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix},\ \rho(b) = \begin{pmatrix}1 & 0\\ -u & 1\end{pmatrix},$$

where u is a complex value satisfying $u^2+u+1=0$. This representation is the holonomy representation of $G(K)$. By the definition, we have $p\left(\rho(a)^{-1}\begin{pmatrix}x\\ y\end{pmatrix}\right) = p\begin{pmatrix}x-y\\ y\end{pmatrix}$, and $(x-y)^2 = x^2-2xy+y^2$, $(x-y)y = xy-y^2$. Hence, we have:

$$\rho_3(a) = \begin{pmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ 1 & -1 & 1\end{pmatrix}.$$

By the same calculations, we have:

$$\rho_3(b) = \begin{pmatrix}1 & u & u^2\\ 0 & 1 & 2u\\ 0 & 0 & 1\end{pmatrix},$$

$$\rho_4(a) = \begin{pmatrix}1 & 0 & 0 & 0\\ -3 & 1 & 0 & 0\\ 3 & -2 & 1 & 0\\ -1 & 1 & -1 & 1\end{pmatrix},$$

$$\rho_4(b) = \begin{pmatrix}1 & u & u^2 & u^3\\ 0 & 1 & 2u & 3u^2\\ 0 & 0 & 1 & 3u\\ 0 & 0 & 0 & 1\end{pmatrix}, \ldots.$$

Set $A = \rho_2(a) = {}^t\rho(a)^{-1} = \begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$ and $B = \rho_2(b) = {}^t\rho(b)^{-1} = \begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}$. Via Fox calculus for $G(K)$, we obtain the denominator of $\Delta_{K,\rho_2}(t) = \det(tB-I) = (t-1)^2$. On the other hand, the numerator of $\Delta_{K,\rho_2}(t) = \det(I - t^{-1}AB^{-1}A^{-1} + AB^{-1}A^{-1}B - tB + BAB^{-1}A^{-1}) = \frac{1}{t^2}(t-1)^2(t^2-4t+1)$. Here we use the value $u = \frac{-1+\sqrt{-3}}{2}$. Continuing in this way, we have obtained the following data:

$$\Delta_{K,\rho_2}(t) = \frac{1}{t^2}(t^2-4t+1),$$

$$\Delta_{K,\rho_3}(t) = -\frac{1}{t^3}(t-1)(t^2-5t+1),$$

$$\Delta_{K,\rho_4}(t) = \frac{1}{t^4}(t^2-4t+1)^2,$$

$$\Delta_{K,\rho_5}(t) = -\frac{1}{t^5}(t-1)(t^4-9t^3+44t^2-9t+1),$$

$$\begin{aligned}\frac{4\pi\log|\mathcal{A}_{K,4}(t)|}{4^2} &= \frac{\pi\log|t^2-4t+1|}{4} \xrightarrow{t=1} \frac{\pi\log 2}{4}\\ &\approx 0.544397\cdots,\end{aligned}$$

$$\begin{aligned}\frac{4\pi\log|\mathcal{A}_{K,5}(t)|}{5^2} &= \frac{4\pi\log|\frac{t^4-9t^3+44t^2-9t+1}{t^2-5t+1}|}{5^2}\\ &\xrightarrow{t=1} \frac{4\pi\log\frac{28}{3}}{5^2} \approx 1.12273\cdots.\end{aligned}$$

n	$\frac{4\pi \log \lvert \mathcal{A}_{K,n}(1) \rvert}{n^2}$	n	$\frac{4\pi \log \lvert \mathcal{A}_{K,n}(1) \rvert}{n^2}$
6	$1.35850\cdots$	7	$1.58331\cdots$
8	$1.66441\cdots$	9	$1.76436\cdots$
10	$1.79618\cdots$	11	$1.85105\cdots$
12	$1.86678\cdots$	13	$1.90158\cdots$
14	$1.91009\cdots$	15	$1.93361\cdots$

These calculations were done by using Wolfram Mathematica.

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