Takayasu cofibrations revisited

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Abstract: This short note gives a new proof for the existence of the cofibrations constructed by S. Takayasu [16], using techniques in the category of unstable modules over the mod two Steenrod algebra.

Key words: Steinberg modules; Takayasu cofibrations; unstable modules.

1. Introduction. Given a natural number n, let $\widetilde{\rho}_n$ be the reduced real regular representation of the elementary abelian 2-group $V_n := (\mathbf{Z}/2)^n$. Let $BV_n^{k\widetilde{\rho}_n}$, $k \in \mathbf{N}$, denote the Thom space over the classifying space BV_n associated to the direct sum of k copies of the representation $\widetilde{\rho}_n$. Following S. Takayasu [16], let $M(n)_k$ denote the stable summand of $BV_n^{k\widetilde{\rho}_n}$ which corresponds to the Steinberg module of the general linear group $GL_n(\mathbf{F}_2)$ [14].

Takayasu constructed in [16] a cofibration of the following form:

$$\Sigma^k M(n-1)_{2k+1} \to M(n)_k \to M(n)_{k+1}.$$

The spectra $M(n)_0$ and $M(n)_1$ are respectively the spectra $M(n)$ and $L(n)$ considered by Mitchell and Priddy, and the splitting $M(n) \simeq L(n) \vee L(n-1)$ [14] corresponds to the cofibration above for $k = 0$. Takayasu also considered the spectra $M(n)_k$ associated to the virtual representations $k\widetilde{\rho}_n$, $k < 0$, and proved that the above cofibrations are still valid for these spectra. Here and below, all spectra are implicitly completed at the prime two.

We note also that the spectra $M(n)_k$, $k \geq 0$, are used in the description of layers of the Goodwillie tower of the identity functor evaluated at spheres [2,1], and the above cofibrations can be deduced from the Goodwillie calculus and the James fibrations. This was in fact given implicitly in [2, Propositions 4.6, 4.7] and was described more explicitly in [3] and [4, Chapter 2].

The purpose of this note is to give another proof for the existence of the above cofibrations for the cases $k \in \mathbf{N}$. This will be carried out by employing techniques in the category of unstable modules over the mod two Steenrod algebra $\mathscr{A}$ [15]. Especially, the formula for the action of Lannes' T-functor on the Steinberg unstable modules $\bar{\mathrm{T}}(L_n) \cong H \otimes L_{n-1}$ ([5, 6.1], [8, 4.19]) will play a crucial role in studying the vanishing of some extension groups of modules over the Steenrod algebra.

2. Algebraic short exact sequences. In this section, we recall the linear structure of the mod 2 cohomology of $M(n)_k$ and the short exact sequences relating these $\mathscr{A}$-modules.

The linear group $GL_n := GL_n(\mathbf{F}_2)$ acts from the left on $H^*V_n \cong \mathbf{F}_2[x_1, \dots, x_n]$ by the rule:

$$(gF)(x_1, \dots, x_n) := F\left(\sum_{i=1}^n g_{i,1}x_i, \dots, \sum_{i=1}^n g_{i,n}x_i\right),$$

where $g = (g_{i,j}) \in GL_n$ and $F(x_1, \dots, x_n) \in \mathbf{F}_2[x_1, \dots, x_n]$. This action commutes with the action of the Steenrod algebra on $\mathbf{F}_2[x_1, \dots, x_n]$.

By definition, the Euler class of the vector bundle associated to the reduced regular representation $\widetilde{\rho}_n$ is given by the top Dickson invariant:

$$\omega_n = \omega_n(x_1, \dots, x_n) := \prod_{0 \neq x \in \mathbf{F}_2\langle x_1, \dots, x_n\rangle} x.$$

Recall also that the Steinberg idempotent e_n of $\mathbf{F}_2[GL_n]$ is given by

$$e_n := \sum_{b \in B, \sigma \in \Sigma_n} b\sigma,$$

where B_n is the subgroup of upper triangular matrices in GL_n and Σ_n the subgroup of permutation matrices.

Let $M_{n,k}$ denote the mod 2 cohomology of the spectrum $M(n)_k$. By the Thom isomorphism, we have an isomorphism of $\mathscr{A}$-modules:

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$$M_{n,k} \cong \operatorname{Im}[\omega_n^k H^* BV_n \xrightarrow{e_n} \omega_n^k H^* BV_n].$$

Proposition 2.1 ([6]). *A basis for the graded vector space $M_{n,k}$ is given by the classes $e_n(\omega_1^{i_1-2i_2} \cdots \omega_{n-1}^{i_{n-1}-2i_n} \omega_n^{i_n})$ in which $i_j > 2i_{j+1}$ for $1 \leq j \leq n-1$ and $i_n \geq k$.*

We note that since we work with the left action of e_n on H^*V_n, the $\mathscr{A}$-module $M_{n,k}$ is invariant under the action of the group B_n, and not invariant under the action of the symmetric group Σ_n as in [14] and [16].

Theorem 2.2 (cf. [16]). *Let $\alpha : M_{n,k+1} \to M_{n,k}$ be the natural inclusion and let $\beta : M_{n,k} \to \Sigma^k M_{n-1,2k+1}$ be the map given by*

$$\beta(\omega_{i_1,\cdots,i_n}) = \begin{cases} 0, & i_n > k, \\ \Sigma^k \omega_{i_1,\cdots,i_{n-1}}, & i_n = k. \end{cases}$$

Then

$$0 \to M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \to 0$$

is a short exact sequence of $\mathscr{A}$-modules.

The exactness of the sequence can be proved by using the following

Lemma 2.3 ([7, Proposition 1.2]). *We have $\omega_{i_1,\cdots,i_n} = \omega_{i_1,\cdots,i_{n-1}} x_n^{i_n}$ + terms $\omega_{j_1,\cdots,j_{n-1}} x_n^j$ with $j > j_n$.*

We note also that a minimal generating set for the $\mathscr{A}$-module $M_{n,k}$ was constructed in [6], generalising the work of Inoue [9].

3. Existence of the cofibrations. A spectrum X is said to be of finite type if its mod 2 cohomology, H^*X, is finite-dimensional in each degree. Recall that given a sequence $X \to Y \to Z$ of spectra of finite type, if the composite $X \to Z$ is homotopically trivial and the induced sequence $0 \to H^*Z \to H^*Y \to H^*X \to 0$ is a short exact sequence of $\mathscr{A}$-modules, then $X \to Y \to Z$ is a cofibration.

We wish to realise the algebraic short sequence

$$0 \to M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \to 0,$$

by a cofibration of spectra

$$\Sigma^k M(n-1)_{2k+1} \to M(n)_k \to M(n)_{k+1}.$$

The inclusion of $k\widetilde{\rho}_n$ into $(k+1)\widetilde{\rho}_n$ induces a natural map of spectra

$$i : M(n)_k \to M(n)_{k+1}.$$

It is clear that this map realises the inclusion of $\mathscr{A}$-modules $\alpha : M_{n,k+1} \to M_{n,k}$. We wish now to realise the $\mathscr{A}$-linear map $\beta : M_{n,k} \to \Sigma^k M_{n-1,2k+1}$ by a map of spectra

$$j : \Sigma^k M(n-1)_{2k+1} \to M(n)_k$$

such that the composite $i \circ j$ is homotopically trivial. The existence of such a map is an immediate consequence of the following result.

Theorem 3.1. *For all $k \geq 0$, we have*

(a) *The natural map $[\Sigma^k M(n-1)_{2k+1}, M(n)_k] \to \operatorname{Hom}_{\mathscr{A}}(M_{n,k}, \Sigma^k M_{n-1,2k+1})$ is onto.*

(b) *The group $[\Sigma^k M(n-1)_{2k+1}, M(n)_{k+1}]$ is trivial.*

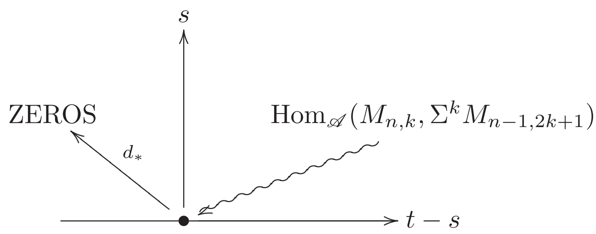
The theorem is proved by using the Adams spectral sequence

$$\operatorname{Ext}_{\mathscr{A}}^s(H^*Y, \Sigma^t H^*X) \Longrightarrow [\Sigma^{t-s}X, Y].$$

For the first part, it suffices to prove that

$$\operatorname{Ext}_{\mathscr{A}}^s(M_{n,k}, \Sigma^{k+t} M_{n-1,2k+1}) = 0 \tag{1}$$

for $s \geq 0$ and $t - s < 0$, so that the non-trivial elements in $\operatorname{Hom}_{\mathscr{A}}(M_{n,k}, \Sigma^k M_{n-1,2k+1})$ are permanent cycles:



For the second part, it suffices to prove that

$$\operatorname{Ext}_{\mathscr{A}}^s(M_{n,k+1}, \Sigma^{k+s} M_{n-1,2k+1}) = 0, \tag{2}$$

for $s \geq 0$. Here and below, $\mathscr{A}$-linear maps are of degree zero, and so $\operatorname{Ext}_{\mathscr{A}}^s(M, \Sigma^t N)$ is the same as the group denoted by $\operatorname{Ext}_{\mathscr{A}}^{s,t}(M, N)$ in the traditional notation.

The vanishing of the above extension groups will be proved in the next section.

4. On the vanishing of $\operatorname{Ext}_{\mathscr{A}}^s(M_{n,k}, \Sigma^{i+s} M_{m,j})$. In this section, we establish a sufficient condition for the vanishing of the extension groups $\operatorname{Ext}_{\mathscr{A}}^s(M_{n,k}, \Sigma^{i+s} M_{m,j})$. Note that we will always consider the modules $M_{n,k}$ with $k \geq 0$.

Below we consider separately two cases for the vanishing of the extension groups $\operatorname{Ext}_{\mathscr{A}}^s(M_{n,k}, \Sigma^{i+s} M_{m,j})$: Proposition 4.1 gives a condition for the case $j = 0$ and Proposition 4.2 gives a condition for the case $j > 0$.

Proposition 4.1. *Suppose $n > m \geq 0$ and $-\infty < i < |M_{n-m,k}|$. Put $M_m := M_{m,0}$. Then*

$$\operatorname{Ext}_{\mathscr{A}}^s(M_{n,k}, \Sigma^{i+s} M_m) = 0, \quad s \geq 0.$$

Here and below, $|M|$ denotes the connectivity of M, i.e. the minimal degree in which M is nontrivial.

To consider the case $j > 0$, put $\varphi(j) = 2j - 1$ and

$$F(i, j, q) = i + j + \varphi(j) + \varphi^2(j) + \cdots + \varphi^{q-1}(j),$$

where φ^t is the t-fold composition of φ. Explicitly,

$$F(i, j, q) = i + (j - 1)(2^q - 1) + q.$$

Note that $F(i + j, 2j - 1, q) = F(i, j, q + 1)$ and $F(i, j', q) \leq F(i, j, q)$ if $j' \leq j$.

Proposition 4.2. *Suppose $n > m \geq 0$, $j > 0$ and $F(i, j, q) < |M_{n-m+q,k}|$ for $0 \leq q \leq m$. Then*

$$\operatorname{Ext}^s_{\mathscr{A}}(M_{n,k}, \Sigma^{i+s} M_{m,j}) = 0, \quad s \geq 0.$$

Recall that Lannes' T-functor is left adjoint to the tensoring with $H := H^*B\mathbf{Z}/2$ in the category $\mathscr{U}$ of unstable modules over the Steenrod algebra [11]. We need the following result, observed by Harris and Shank [8], to prove Proposition 4.1.

Proposition 4.3 (Carlisle-Kuhn [5, 6.1] combined with Harris-Shank [8, 4.19])**.** *There is an isomorphism of unstable modules*

$$\mathrm{T}(L_n) \cong L_n \oplus (H \otimes L_{n-1}).$$

Here $L_n = M_{n,1}$.

Corollary 4.4. *For $n \geq m$, we have $|\mathrm{T}^m(M_{n,k})| = |M_{n-m,k}|$.*

Proof. By iterating the action of T on L_n, we see that there is an isomorphism of unstable modules

$$\mathrm{T}^m(L_n) \cong \bigoplus_{i=0}^{m} [H^{\otimes i} \otimes L_{n-i}]^{\oplus a_i},$$

where a_i are certain positive integers depending only on m. By using the exactness of T^m and the short exact sequences

$$0 \to M_{n,k+1} \xrightarrow{\alpha} M_{n,k} \xrightarrow{\beta} \Sigma^k M_{n-1,2k+1} \to 0,$$

it is easy to prove by induction that there is an isomorphism of graded vector spaces

$$\mathrm{T}^m(M_{n,k}) \cong \bigoplus_{i=0}^{m} [H^{\otimes i} \otimes M_{n-i,k}]^{\oplus a_i}.$$

The corollary follows. □

Proof of Proposition 4.1. Fix i, s and take a positive integer q big enough such that $i + s + q$ is positive. We have

$$\operatorname{Ext}^s_{\mathscr{A}}(M_{n,k}, \Sigma^{i+s} M_m) = \operatorname{Ext}^s_{\mathscr{A}}(\Sigma^q M_{n,k}, \Sigma^{i+s+q} M_m).$$

Using the Grothendieck spectral sequence, we need to prove that

$$\operatorname{Ext}^{s-j}_{\mathscr{U}}(\mathbf{D}_j \Sigma^q M_{n,k}, \Sigma^{i+s+q} M_m) = 0, \quad 0 \leq j \leq s.$$

Here $\mathbf{D}_j$ is the jth-derived functor of the destabilisation functor

$$\mathbf{D} : \mathscr{A}\text{-mod} \to \mathscr{U}$$

from the category of $\mathscr{A}$-modules to the category of unstable $\mathscr{A}$-modules [13].

As M_n is $\mathscr{U}$-injective, it is easily seen that $\Sigma^\ell M_m$ has a $\mathscr{U}$-injective resolution $I^\bullet$ where I^t is a direct sum of $M_m \otimes J(a)$ with $a \leq \ell - t$, where $J(a)$ is the Brown-Gitler module [12]. So we need to prove that, for $a \leq (i + s + q) - (s - j) = i + j + q$, we have

$$\operatorname{Hom}_{\mathscr{U}}(\mathbf{D}_j \Sigma^q M_{n,k}, M_m \otimes J(a)) = 0.$$

By Lannes-Zarati [13, Thm. 1.5], we have

$$\mathbf{D}_j \Sigma^q M_{n,k} = \Sigma R_j \Sigma^{j-1+q} M_{n,k} \subset \Sigma^{j+q} H^{\otimes j} \otimes M_{n,k},$$

where R_j is the Singer functor. It follows that $\operatorname{Hom}_{\mathscr{U}}(\mathbf{D}_j \Sigma^q M_{n,k}, M_m \otimes J(a))$ is a quotient of

$$\operatorname{Hom}_{\mathscr{U}}(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}, M_m \otimes J(a))$$

which is in turn a subgroup of $\operatorname{Hom}_{\mathscr{U}}(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}, H^{\otimes m} \otimes J(a)) \cong \operatorname{Hom}_{\mathbf{F}_2}((\mathrm{T}^m(\Sigma^{j+q} H^{\otimes j} \otimes M_{n,k}))^a, \mathbf{F}_2)$. This group is trivial because, by Corollary 4.4, we have $|\mathrm{T}^m(\Sigma^{j+q} H_i \otimes M_{n,k})| = |\Sigma^{j+q} M_{n-m,k}| = |M_{n-m,k}| + j + q > i + j + q \geq a$. The proposition follows. □

Proof of Proposition 4.2. We prove the proposition by induction on $m \geq 0$. By noting that $M_{0,j} = \mathbf{Z}/2$, the case $m = 0$ is a special case of Proposition 4.1.

Suppose $m > 0$. For simplicity, put $E^s(\Sigma^i M_{m,j}) = \operatorname{Ext}^s_{\mathscr{A}}(M_{n,k}, \Sigma^{i+s} M_{m,j})$. The short exact sequence of $\mathscr{A}$-modules $M_{m,j} \hookrightarrow M_{m,j-1} \twoheadrightarrow \Sigma^{j-1} M_{m-1,2j-1}$ induces a long exact sequence in cohomology $\cdots \to E^{s-1}(\Sigma^{i+j} M_{m-1,2j-1}) \to E^s(\Sigma^i M_{m,j}) \to E^s(\Sigma^i M_{m,j-1}) \to \cdots$. So from the cofiltration of $\Sigma^i M_{m,j}$

$$\begin{tikzcd}
\Sigma^i M_{m,j} \arrow[r, hook] & \Sigma^i M_{m,j-1} \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & \Sigma^i M_{m,1} \arrow[r, hook] \arrow[d, two heads] & \Sigma^i M_m \arrow[d, two heads] \\
 & \Sigma^{i+j-1} M_{m-1,2j-1} & & \Sigma^{i+1} M_{m-1,3} & \Sigma^i M_{m-1,1}
\end{tikzcd}$$

we see that, in order to prove $E^s(\Sigma^i M_{m,j}) = 0$, it suffices to prove that the groups $E^{s-1}(\Sigma^{i+j'} M_{m-1,2j'-1})$, $1 \le j' \le j$, and $E^s(\Sigma^i M_m)$, are trivial.

By Proposition 4.1, $E^s(\Sigma^i M_m)$ is trivial since $i = F(i, j, 0) < |M_{n,k}|$. For $1 \le j' \le j$ and $0 \le q \le m-1$, we have $F(i + j', 2j' - 1, q) = F(i, j', q + 1) \le F(i, j, q + 1) < |M_{n-m+1+q,k}|$. By inductive hypothesis for $m - 1$, we have $E^{s-1}(\Sigma^{i+j'} M_{m-1,2j'-1}) = 0$. The proposition is proved. □

We are now ready to prove Theorem 3.1. Recall that the connectivity of $M_{n,k}$ is given by

$$|M_{n,k}| = 1 + 3 + \cdots + (2^{n-1} - 1) + (2^n - 1)k.$$

Proof of Theorem 3.1 (*1*). Using the Adams spectral sequence, it suffices to prove that $\mathrm{Ext}^s_{\mathscr{A}}(M_{n,k}, \Sigma^{k+t} M_{n-1,2k+1}) = 0$ for $s \ge 0$ and $t - s < 0$. For $q \ge 0$, we have $F(k + t - s, 2k + 1, q) = k + t - s + 2k(2^q - 1) + q < (2^{q+1} - 1)k + q \le |M_{q+1,k}|$. The vanishing of the extension groups follows from Proposition 4.2. □

Proof of Theorem 3.1 (*2*). Using the Adams spectral sequence, it suffices to prove that $\mathrm{Ext}^s_{\mathscr{A}}(M_{n,k+1}, \Sigma^{k+s} M_{n-1,2k+1}) = 0$ for $s \ge 0$. For $q \ge 0$, we have $F(k, 2k + 1, q) = k + 2k(2^q - 1) + t = (2^{q+1} - 1)k + q < |M_{q+1,k+1}|$. The vanishing of the extension groups follows from Proposition 4.2. □

5. Unstable realisation of the cofibrations. The spectrum $\Sigma M(n)_k$, $n, k \in \mathbf{N}$, is a retract of a suspension spectrum by its original construction as a telescope. It is natural to ask whether the cofibration

$$\Sigma^k M(n-1)_{2k+1} \to M(n)_k \to M(n)_{k+1}$$

can be realised as a cofibration of spaces after one suspension.

To do this one can use the following classical result ([10] page 36). Given two vector bundles E and F over a base B, there is, up to homotopy, a cofibration of Thom spaces

$$S(F)^{q^*E} \to B^E \to B^{E \oplus F},$$

where $q : S(F) \to B$ denotes the projection on the associated sphere-bundle and q^*E the pullback by q of the bundle E over B.

Apply this result to the case where E is the vector bundle over BV_n, $V_n := (\mathbf{Z}/2)^n$, associated to the direct sum of k copies of the representation $\widetilde{\rho}_n$, and F the vector bundle over BV_n associated to $\widetilde{\rho}_n$. Suspend the cofibration one time and, as the situation is equivariant, apply the Steinberg idempotent, it is realised by unstable maps:

$$e_n \Sigma S(\widetilde{\rho}_n)^{q^*(k\widetilde{\rho}_n)} \to e_n \Sigma BV_n^{k\widetilde{\rho}_n} \to e_n \Sigma BV_n^{(k+1)\widetilde{\rho}_n}.$$

The second and the third terms are spaces which, by abuse of notation, are denoted also by $\Sigma M(n)_k$ and $\Sigma M(n)_{k+1}$. It remains to identify the first one, this is more delicate. As a spectrum, it is equivalent to $\Sigma\Sigma^k M(n-1)_{2k+1}$. However it is not immediately clear that the telescope of the Steinberg idempotent e_n on $\Sigma S(\widetilde{\rho}_n)^{q^*(k\widetilde{\rho}_n)}$ is equivalent to $\Sigma\Sigma^k M(n-1)_{2k+1}$. We leave to the reader to check it by first producing a map from $\Sigma\Sigma^k M(n-1)_{2k}$ to the telescope, then extending it to $\Sigma\Sigma^k M(n-1)_{2k+1}$ using an appropriate induction hypothesis.

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