Aharonov–Bohm effect in resonances for scattering by three solenoids

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Abstract: We study how the Aharonov–Bohm effect is reflected in the location of quantum resonances for scattering by three solenoids at large separation. We also discuss what happens in the case of four solenoids.

Key words: Aharonov–Bohm effect; magnetic Schrödinger operators; quantum resonances; scattering by solenoids.

1. Introduction. In quantum mechanics, a vector potential is said to have a direct significance to particles moving in a magnetic field. This quantum effect is known as the Aharonov–Bohm effect (AB effect) ([3]). We study this effect through resonances in scattering systems by three solenoids the centers of which are placed almost in line. The resonances are shown to be generated near the real axis by the trajectories trapped between these centers, when the centers are largely separated from one another. The location of the resonances depends on the ratio of the distances between the centers as well as on the magnetic fluxes. We also discuss the case of four solenoids.

We begin by fixing the basic notation to formulate the obtained results. We work in the two dimensional space $\boldsymbol{R}^2$ with generic point $x = (x_1, x_2)$ and write

$$H(A) = \sum_{j=1}^{2}(-i\partial_j - a_j)^2, \quad \partial_j = \partial/\partial x_j,$$

for the Schrödinger operator with $A = (a_1, a_2) : \boldsymbol{R}^2 \to \boldsymbol{R}^2$ as a vector potential. The magnetic field b associated with A is defined as

$$b = \nabla \times A = \partial_1 a_2 - \partial_2 a_1 : \boldsymbol{R}^2 \to \boldsymbol{R}$$

and the magnetic flux is defined by the integral $(2\pi)^{-1}\int b(x)\,dx$, where the integral with no domain attached is taken over the whole space.

We now take $A(x)$ to be

$$\text{(1)} \qquad A(x) = (-x_2/|x|^2, x_1/|x|^2) = (-\partial_2 \log|x|, \partial_1 \log|x|)$$

which generates the solenoidal field

$$\nabla \times A = (\partial_1^2 + \partial_2^2)\log|x| = 2\pi\delta(x)$$

with center at the origin. This vector potential is often called the Aharonov–Bohm potential in physics literatures. The scattering system by one solenoid is known as one of exactly solvable models in quantum mechanics ([1,3,6]). We consider the operator $H = H(\alpha A)$ associated with the solenoid $2\pi\alpha\delta(x)$, α being a magnetic flux. The operator formally defined is symmetric over $C_0^\infty(\boldsymbol{R}^2 \setminus \{0\})$, but it is not necessarily essentially self–adjoint in $L^2 = L^2(\boldsymbol{R}^2)$ because of the strong singularities at the origin. The Friedrichs extension is realized by imposing the boundary condition

$$\text{(2)} \qquad \lim_{|x|\to 0} |u(x)| < \infty$$

at the center. We use the same notation H to denote this self–adjoint operator. The operator admits the partial wave expansion. We denote by $f(\omega \to \theta; E)$ the amplitude for scattering from the incident direction $\omega \in S^1$ to the final one θ at energy $E > 0$. The amplitude is explicitly calculated as

$$f(\omega \to \theta; E) = c_0(E)\sin(\alpha\pi)e^{i[\alpha](\theta-\omega)}\frac{e^{i(\theta-\omega)}}{1 - e^{i(\theta-\omega)}}$$

with $c_0(E) = (2/\pi)^{1/2}e^{i\pi/4}E^{-1/4}$, where the Gauss notation $[\alpha]$ denotes the greatest integer not exceeding α and the coordinates θ, ω over the unit circle S^1 are identified with the azimuth angles from the positive x_1 axis. In particular, the backward amplitude takes the form

$$\text{(3)} \qquad f(\omega \to -\omega; E) = (2\pi)^{-1/2}e^{i\pi/4}(-1)^{[\alpha]+1}\sin(\alpha\pi)E^{-1/4}$$

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independent of ω. We note that the amplitude vanishes for integer flux α and that the forward amplitude $f(\omega \to \omega; E)$ diverges for non–integer flux α.

We define the Hamiltonian associated with the three centers

(4) $$d_\mp = (\mp\kappa_\mp d, 0), \quad d_0 = (0, \kappa d^{1/2})$$

labelled by the large parameter $d \gg 1$, where $\kappa_\pm > 0$ and $\kappa_- + \kappa_+ = 1$. By assumption, it follows that $|d_+ - d_-| = d$ for the distance between d_- and d_+. Let $A_d(x)$ be the potential defined by

(5) $$A_d(x) = \alpha_- A(x - d_-) + \alpha_0 A(x - d_0) + \alpha_+ A(x - d_+),$$

which generates the three solenoids $2\pi\alpha_\pm\delta(x - d_\pm)$ and $2\pi\alpha_0\delta(x - d_0)$. We consider the self–adjoint operator

(6) $$H_d = H(A_d)$$

under the boundary conditions like (2) at each center. It is known that H_d has no positive eigenvalues and the continuous spectrum occupied by $[0, \infty)$ is absolutely continuous. We can show that the resolvent

$$R(\zeta; H_d) = (H_d - \zeta)^{-1} : L^2 \to L^2,$$

with $\mathrm{Re}\,\zeta > 0$ and $\mathrm{Im}\,\zeta > 0$ is meromorphically continued from the upper half plane of the complex plane to the lower half plane $\{\zeta \in \boldsymbol{C} : \mathrm{Re}\,\zeta > 0,\ \mathrm{Im}\,\zeta < 0\}$ across the positive real axis where the continuous spectrum of H_d is located. Then $R(\zeta; H_d)$ with $\mathrm{Im}\,\zeta \le 0$ is well defined as an operator from $L^2_{\mathrm{comp}}(\boldsymbol{R}^2)$ to $L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$ in the sense that $\chi R(\zeta; H_d)\chi : L^2 \to L^2$ is bounded for every $\chi \in C_0^\infty(\boldsymbol{R}^2)$, where $L^2_{\mathrm{comp}}(\boldsymbol{R}^2)$ and $L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$ denote the spaces of square integrable functions with compact support and of locally square integrable functions, respectively. The resonances of H_d are defined as the poles of $R(\zeta; H_d)$ in the lower half plane (the second sheet or the unphysical plane).

The argument here is restricted only to a neighborhood of the positive axis. We fix $E_0 > 0$ and take a complex neighborhood

(7) $$D_d = \{\zeta : |\mathrm{Re}\,\zeta - E_0| < \delta_0 E_0,\ |\mathrm{Im}\,\zeta| < (1 + 2\delta_0)E_0^{1/2}((\log d)/d)\}$$

for δ_0, $0 < \delta_0 \ll 1$, small enough. We denote by $f_\pm(\omega \to \theta; E)$ the scattering amplitude by $2\pi\alpha_\pm\delta(x)$ and set

(8) $$f_0(\zeta) = f_-(-\omega_1 \to \omega_1; \zeta) f_+(\omega_1 \to -\omega_1; \zeta) = (2\pi)^{-1} i (-1)^{[\alpha_-]+[\alpha_+]} \sin(\alpha_-\pi)\sin(\alpha_+\pi) k^{-1}$$

for $\omega_1 = (1, 0)$ (see (3)), where the branch $k = \zeta^{1/2}$ is taken in such a way that $\mathrm{Re}\,k > 0$ for $\mathrm{Re}\,\zeta > 0$. Let $\kappa_\pm$ and κ be as in (4). We define the integral $I(\zeta)$ by

(9) $$I(\zeta) = (2/\pi)^{1/2} e^{-i\pi/4} \int_0^\tau e^{is^2/2}\, ds$$

with $\tau(\zeta) = \kappa(1/\kappa_- + 1/\kappa_+)^{1/2}\zeta^{1/4}$, and the two terms $\pi_\mp(\zeta)$ by

(10) $$\pi_\mp(\zeta) = p_0(\zeta) e^{\pm i\alpha_0\pi} + (1 - p_0(\zeta)) e^{\mp i\alpha_0\pi},$$

where the contour is taken to be the segment from 0 to τ and

$$p_0(\zeta) = (1 + I(\zeta))/2.$$

We further define

$$h(\zeta; d) = (e^{2ikd}/d) f_0(\zeta)\pi_-(\zeta)\pi_+(\zeta)$$

over D_d. If the fluxes α_- and α_+ are not an integer and if $\pi_\pm(E_0) \ne 0$, then $f_0(\zeta) \ne 0$ and $\pi_\pm(\zeta) \ne 0$ over the complex neighborhood D_d of E_0. We can show ([7, Lemma 4.6]) that the equation

(11) $$h(\zeta; d) = 1$$

has the solutions

$$\{\zeta_j(d)\}, \quad \zeta_j \in D_d, \quad \mathrm{Re}\,\zeta_1 < \mathrm{Re}\,\zeta_2 < \cdots < \mathrm{Re}\,\zeta_{N_d},$$

such that $\zeta_j(d)$ behaves like

$$\mathrm{Im}\,\zeta_j(d) \sim -E_0^{1/2}(\log d)/d,$$
$$\mathrm{Re}(\zeta_{j+1}(d) - \zeta_j(d)) \sim 2\pi E_0^{1/2}/d$$

for $d \gg 1$. Then the first result is formulated as the theorem below, and the second one is obtained as a consequence of this theorem.

Theorem 1. *Let the notation be as above. Assume that α_- and α_+ are not an integer and $\pi_\pm(E_0)$ does not vanish. Then we can take $\delta_0 > 0$ so small that the neighborhood D_d defined by* (7) *has the following property: For any $\varepsilon > 0$ small enough, there exists $d_\varepsilon \gg 1$ such that for $d > d_\varepsilon$, H_d has the resonances $\{\zeta_{\mathrm{res},j}(d)\}$, $\zeta_{\mathrm{res},j} \in D_d$, with*

$$\mathrm{Re}\,\zeta_{\mathrm{res},1}(d) < \cdots < \mathrm{Re}\,\zeta_{\mathrm{res},N_d}(d)$$

in the neighborhood $\{\zeta \in \boldsymbol{C} : |\zeta - \zeta_j(d)| < \varepsilon/d\}$, and $R(\zeta; H_d)$ is analytic as a function with values in operators from $L^2_{\mathrm{comp}}(\boldsymbol{R}^2)$ to $L^2_{\mathrm{loc}}(\boldsymbol{R}^2)$ over

$$D_d \setminus \{\zeta_{\mathrm{res},1}(d), \cdots, \zeta_{\mathrm{res},N_d}(d)\}.$$

Corollary 2. *Let $I = [a, b]$ with $0 < a < b < \infty$ and let*

$$D_d(I) = \{\zeta \in \boldsymbol{C} : \operatorname{Re}\zeta \in I, \\ |\operatorname{Im}\zeta| \le (1 + 2\delta_0)(\operatorname{Re}\zeta)^{1/2}((\log d)/d)\}$$

for $\delta_0 > 0$ small enough. Denote by $N_d(I)$ the number of resonances in $D_d(I)$. Assume that α_- and α_+ are not an integer and $\pi_\pm(E)$ does not vanish over the interval I. Then one can take $\delta_0 > 0$ so small that $N_d(I)$ obeys the asymptotic formula

$$N_d(I) = ((b^{1/2} - a^{1/2})/\pi)d + O(d^{1/2})$$

as $d \to \infty$.

The rigorous proof of the theorem is long. We are going to discuss the details elsewhere ([9]).

2. Heuristic arguments. We see from an intuitive point of view how reasonable (11) is as an approximate relation to determine the location of the resonances. For brevity, we consider the special case $\kappa = 0$ ($d_0 = (0, 0)$). In this case, the three centers are exactly placed in line and the integral $I(\zeta)$ defined by (9) vanishes, and hence $h(\zeta; d)$ is explicitly represented as

$$h(\zeta; d) = (e^{2ikd}/d)f_0(\zeta)\cos^2(\alpha_0\pi).$$

We denote by $\varphi_0(x; \omega, E) = \exp(iE^{1/2}x\cdot\omega)$ the plane wave with ω as an incident direction at energy $E > 0$ and write $x_\pm$ for $x_\pm = x - d_\pm$. The incident plane wave $\varphi_0(x_-; -\omega_1, E)$ takes the form

$$f_-(-\omega_1 \to \omega_1; E)(e^{iE^{1/2}|x_-|}/|x_-|^{1/2})$$

after scattered into the direction ω_1 by the solenoid $2\pi\alpha_-\delta(x_-)$, and the scattered wave hits the other solenoid $2\pi\alpha_+\delta(x_+)$. Since $|x_-|$ behaves like

$$|x_-| = |x - d_-| = |d_+ - d_- + x_+| \\ = d + \omega_1 \cdot x_+ + O(d^{-1})$$

around d_+, the scattered wave behaves like the plane wave

$$(e^{iE^{1/2}d}/d^{1/2})f_-(-\omega_1 \to \omega_1; E)\varphi_0(x_+; \omega_1, E)$$

when it arrives at d_+, provided that there is not the third solenoid $2\pi\alpha_0\delta(x)$ between d_- and d_+. If the potential $\alpha_0 A(x)$ associated with it is present, then the wave function undergoes a change of the phase factor by the AB effect. We consider particles moving from d_- to d_+ and distinguish between the trajectories passing over $x_2 > 0$ and $x_2 < 0$ to denote the former and latter trajectories by l_+ and l_-, respectively. The vector potential $A(x)$ defined by (1) satisfies the relation $A(x) = \nabla\gamma(x)$ for the azimuth angle $\gamma(x)$ from the positive x_1 axis. The change in the phase factor of the wave function is given by the line integral

$$\int_{l_\pm} \alpha_0 A(x)\cdot dx = \alpha_0 \int_{l_\pm} \nabla\gamma(x)\cdot dx = \mp\alpha_0\pi$$

along $l_\pm$. The contribution from l_+ and l_- is fifty–fifty, and hence $\cos(\alpha_0\pi)$ arises as the sum of the phase factors $\exp(\mp i\alpha_0\pi)$. Thus the scattered wave takes

$$(e^{iE^{1/2}d}/d^{1/2})f_-(-\omega_1 \to \omega_1; E) \\ \times \cos(\alpha_0\pi)\varphi_0(x_+; \omega_1, E)$$

as an approximate form, when it hits the center d_+. A similar argument applies to the plane wave $\varphi_0(x_+; \omega_1, E)$ after scattered into the direction $-\omega_1$ by the solenoid $2\pi\alpha_+\delta(x_+)$, so that it again returns to the center d_-, taking the approximate form $h(E; d)\varphi_0(x_-; -\omega_1, E)$. Hence the trapping phenomenon between d_- and d_+ is described by the series

$$\left(\sum_{n=1}^{\infty} h(E; d)^n\right)\varphi_0(x_-; -\omega_1, E).$$

For example, the term with $h(E; d)^n$ describes the contribution from the trajectory oscillating n times. This is the reason why the resonances are approximately determined by (11) and why the magnetic flux α_0 is related to their locations through the AB effect. If $\kappa \ne 0$, then the contribution from $l_\pm$ is not necessarily fifty–fifty, but it depends on the ratio of the distances between the centers, as is seen from (10). The coefficient $\pi_\mp(\zeta)$ describes the AB effect term along the trajectory from $d_\mp$ to $d_\pm$.

3. Strategy. The proof of the theorem is done by constructing the resolvent kernel $R(\zeta; H_d)(x, y)$ with the spectral parameter $\zeta \in D_d$. Here we mention only the basic strategy briefly. The idea is based on the results obtained by [4] and [8].

As already stated, the scattering system by one solenoid is known to be exactly solvable. We make a full use of the information from such a system. Let $K = H(\alpha_0 A(\cdot - d_0))$ with flux α_0 in (5). We know that the Hamiltonian with one solenoid has no resonances in $\boldsymbol{C} \setminus \{0\}$. The first step is to analyze the behavior as $|x - y| \to \infty$ of the resolvent kernel $R(\zeta; K)(x, y)$. The kernel is represented in terms of a complex integral and admits the decomposition

$$R(\zeta; K)(x, y) = R_{\mathrm{fr}}(\zeta; K)(x, y) + R_{\mathrm{sc}}(\zeta; K)(x, y),$$

where $R_{\mathrm{fr}}(\zeta;K)(x,y)$ corresponds to the free trajectory which goes directly from y to x without being scattered by the solenoid $2\pi\alpha_0\delta(x-d_0)$, while $R_{\mathrm{sc}}(\zeta;K)(x,y)$ corresponds to the scattered trajectory which goes from y to x after scattered at the center d_0. The first term on the right side behaves like

$$(12)\qquad R_{\mathrm{fr}}(\zeta;K)(x,y) \sim e^{i\alpha_0(\gamma(\hat{x};-\hat{y})-\pi)}(e^{ik|x-y|}/|x-y|^{1/2})$$

with $\hat{x}=x/|x|$ as $|x-y|\to\infty$, where $\gamma(\theta;\omega)$, $0\le\gamma(\theta;\omega)<2\pi$, denotes the azimuth angle from ω to θ and we skip some numerical constants. The second term takes the asymptotic form

$$(13)\quad R_{\mathrm{sc}}(\zeta;K)(x,y) \sim f_0(-\hat{y}\to\hat{x};\zeta)(e^{ik|x|}/|x|^{1/2})(e^{ik|y|}/|y|^{1/2}),$$

where $f_0(\omega\to\theta;E)$ denotes the scattering amplitude by $2\pi\alpha_0\delta(x)$.

The second step is to construct the resolvent kernel with two solenoids for the pair (d_-,d_0) by composing two resolvent kernels with one solenoid. In doing this, a difficulty comes from the exponential growth of the resolvent kernel. For magnetic fields compactly supported, the corresponding vector potentials can not be expected to fall off rapidly at infinity, because of the topological feature of the two dimensional space that $\boldsymbol{R}^2\setminus\{0\}$ is not simply connected. This is the case where the magnetic fluxes do not vanish. In fact, it is seen from (1) that vector potentials have the long range property. Thus the vector potentials can not be expected to be well separated, even if the supports of the two magnetic fields are largely separated from each other. In other words, cut–off functions used to separate the two centers do not have bounded supports. As is seen from (12) and (13), the resolvent kernels with spectral parameters in the lower half plane grow exponentially at infinity, and hence the composition of the resolvent kernels can not be controlled simply by integration by parts using oscillatory properties. We make use of gauge transformations and of a complex scaling method to construct the resolvent kernels for two solenoids. Then the asymptotic form (13) plays an important role in construction. We have already constructed resolvent kernels for two solenoids in [4]. The complex scaling method in the resonance problem has been initiated by [2]. We refer to the book [5] and literatures there for details and further developments.

The third step is to construct the resolvent kernel of the operator H_d in question from the two kernels corresponding to the two centers (d_-,d_0) and to one center d_+. The construction is again based on the complex scaling method. Then the asymptotic analysis on the behavior as $d=|d_+-d_-|\to\infty$ of $R(\zeta;K)(d_\pm,d_\mp)$ is crucial. We note from (12) and (13) that $R_{\mathrm{fr}}(\zeta;K)(x,y)$ and $R_{\mathrm{sc}}(\zeta;K)(x,y)$ are singular along the forward direction $\hat{x}=-\hat{y}$. In fact,

$$R_{\mathrm{fr}}(\zeta;K)(x,y)\sim e^{\mp i\alpha_0\pi}e^{ik|x-y|}|x-y|^{-1/2}$$

is not necessarily continuous along the direction $\hat{x}=-\hat{y}$, and the forward amplitude $f_0(-\hat{y}\to-\hat{y};\zeta)$ is divergent for $R_{\mathrm{sc}}(\zeta;K)(x,y)$. However, these singularities are canceled, and we have

$$R(\zeta;K)(d_\pm,d_\mp)\sim(e^{ikd}/d^{1/2})\pi_\mp(\zeta).$$

The AB effect term $\pi_\mp(\zeta)$ is obtained through this asymptotic form.

4. Four solenoids. We move to the case of four solenoids. Assume that the four centers are located at $d_\pm$ and at

$$d_1=(-\kappa_0 d,\kappa_1 d^{1/2}),\quad d_2=(\kappa_0 d,\kappa_2 d^{1/2}),$$

where $d_\pm$ are as in (4), and $\kappa_0>0$ satisfies $\kappa_0<\min(\kappa_-,\kappa_+)$. We use the same notation

$$A_d(x)=\alpha_-A(x-d_-)+\alpha_1A(x-d_1)+\alpha_2A(x-d_2)+\alpha_+A(x-d_+)$$

to denote the vector potential associated with these centers (see (5)). We also write $H_d=H(A_d)$ for the self–adjoint realization obtained by imposing the condition like (2) at each center. For the operator H_d with four solenoids, we can obtain a result similar to Theorem 1 in the two special cases: (1) $\kappa_1=\kappa_2=0$; (2) $\kappa_0=0$. The first case means that all the centers are horizontally placed along an identical direction, while the second one means that the two centers d_1 and d_2 are vertically placed to the trajectory trapped between d_- and d_+.

We discuss case (1). We define the angle ψ_0, $0<\psi_0<\pi/2$, through the relation

$$\cos\psi_0=\left(\frac{\kappa_--\kappa_0}{\kappa_-+\kappa_0}\right)^{1/2}\left(\frac{\kappa_+-\kappa_0}{\kappa_++\kappa_0}\right)^{1/2}<1$$

and set

$$\pi_0=(1-\psi_0/\pi)\cos(\beta_+\pi)+(\psi_0/\pi)\cos(\beta_-\pi)$$

with $\beta_+=\alpha_2+\alpha_1$ and $\beta_-=\alpha_2-\alpha_1$. The constant

π_0 describes the AB effect arising from the trajectories from $d_\mp$ to $d_\pm$. Let $f_0(\zeta)$ be as in (8). We further define

$$h_1(\zeta; d) = (e^{2ikd}/d) f_0(\zeta)\pi_0^2$$

over the neighborhood D_d defined by (7). If $\alpha_\pm$ is not an integer and $\pi_0 \neq 0$, then we can show that the location of the resonances in D_d of H_d is approximately determined by the solutions to the equation $h_1(\zeta; d) = 1$ as in Theorem 1.

Next we consider case (2). For brevity, we assume that $\kappa_1 < \kappa_2$. We define the integrals $I_j(\zeta)$, $j = 1, 2$, by

$$I_j(\zeta) = (2/\pi)^{1/2} e^{-i\pi/4} \int_0^{\tau_j} e^{is^2/2}\, ds$$

with $\tau_j(\zeta) = \kappa_j(1/\kappa_- + 1/\kappa_+)^{1/2}\zeta^{1/4}$, and we set

$$\rho_\mp(\zeta) = p_1(\zeta)e^{\pm i\beta_+\pi} + p_3(\zeta)e^{\pm i\beta_-\pi} + p_2(\zeta)e^{\mp i\beta_+\pi}$$

with $\beta_\pm$ as above, where

$$p_1(\zeta) = (1 + I_1(\zeta))/2, \quad p_2(\zeta) = (1 - I_2(\zeta))/2$$

and $p_3(\zeta) = 1 - p_1(\zeta) - p_2(\zeta)$. The term $\rho_-(\zeta)$ describes the AB effect arising from the trajectory from d_- to d_+, and $\rho_+(\zeta)$ corresponds to the trajectory from d_+ to d_-. We further define

$$h_2(\zeta; d) = (e^{2ikd}/d) f_0(\zeta)\rho_-(\zeta)\rho_+(\zeta)$$

for $\zeta \in D_d$. Assume that $\alpha_\pm$ is not an integer and $\rho_\pm(E_0) \neq 0$. Then we can show that the location of the resonances in D_d of H_d is specified by the equation $h_2(\zeta; d) = 1$ as in Theorem 1. To illustrate the vertical case, we end the note by discussing the particular case when the sum of the two fluxes vanishes ($\alpha_1 + \alpha_2 = 0$), and $\kappa_1 = -\kappa$ and $\kappa_2 = \kappa$ with $\kappa > 0$. We set $\alpha_1 = -\alpha$ and $\alpha_2 = \alpha$. Then we have $I_1(\zeta) = -I_2(\zeta) = -I(\zeta)$ for the integral $I(\zeta)$ defined by (9), and $\rho_\mp(\zeta)$ takes the form

$$\rho_\mp(\zeta) = (1 - I(\zeta)) + I(\zeta)e^{\pm 2i\alpha\pi}.$$

We add a brief comment. Loosely speaking, the AB effect is not observed, provided that $0 < \kappa \ll 1$ or $\kappa \gg 1$. If the width $2\kappa d^{1/2}$ between the two centers $d_1 = (0, -\kappa d^{1/2})$ and $d_2 = (0, \kappa d^{1/2})$ is small in comparison with the distance $d = |d_+ - d_-|$, then a large contribution comes from the closed trajectories enclosing the two centers d_1 and d_2, and the phase factor of the wave function along such trajectories is not changed. In fact, the integral $I(\zeta)$ goes to zero as the interval $[0, \tau]$ shrinks ($\kappa \to 0$), and hence $\rho_\mp(\zeta) = 1$. If, conversely, the width is large, then a large contribution comes from the closed trajectories passing between the two centers. In this case, the integral interval $[0, \tau]$ expands to $[0, \infty)$ ($\kappa \to \infty$), and $\rho_\mp$ is calculated as $\rho_\mp(\zeta) = e^{\pm 2i\alpha\pi}$ by making use of the formula $\int_0^\infty e^{is^2/2}\, ds = (\pi/2)^{1/2} e^{i\pi/4}$. As a result, $\rho_-(\zeta)\rho_+(\zeta) = 1$, and hence the AB effect term disappears.

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