Entire functions sharing an entire function of smaller order with their shifts

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Abstract: We study the growth of solutions of a certain difference equations, and study the uniqueness question of entire functions of finite orders sharing an entire function of smaller order with their shifts. The uniqueness results in this paper also extend and improve Theorem 1 [11].

Key words: Difference Nevanlinna theory; uniqueness of entire functions; shared values.

1. Introduction and main results. In 1977, Rubel and Yang [20] proved that if an entire function f shares two distinct finite values CM with its derivative f', then $f = f'$. What can be said about the relationship between f and f', if an entire function f shares one finite value a CM with its derivative f'? In 1996, Brück [3] made the conjecture that if f is a nonconstant entire function of hyper-order $\rho_2(f) < \infty$, where $\rho_2(f)$ is not a positive integer, and if f and f' share one finite value a CM, then $f - a = c(f' - a)$ for some constant $c \neq 0$. If $a = 0$, the above conjecture was proved by Brück [3]. Brück [3] also proved the above conjecture is true, provided that $a \neq 0$ and $N(r, 1/f') = S(r, f)$, where f is a nonconstant entire function. In 2005, Al-Khaladi [1] showed that the conjecture remains true for a nonconstant meromorphic function f, provided that $N(r, 1/f') = S(r, f)$. But the conjecture is still an open question by now.

Recently the value distribution theory of difference polynomials, Nevanlinna characteristic of $f(z+\eta)$, Nevanlinna theory for the difference operator and the difference analogue of the lemma on the logarithmic derivative has been established (see [4,7,8,13,14]). Using these theories, uniqueness questions of meromorphic functions sharing values with their shifts have been recently treated as well (see [10,11,16,17,22]).

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Throughout this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations of the Nevanlinna theory of meromorphic functions as explained in [9,12,21]. Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-points with the same multiplicities. We say that f and g share the value a IM, provided that f and g have the same a-points ignoring multiplicities (see [21]). Suppose that b is a meromorphic function. If $f - b$ and $g - b$ share 0 CM, we say that f and g share b CM. If $f - b$ and $g - b$ share 0 IM, we say that f and g share b IM. In addition, we denote by $\mu(f)$, $\rho(f)$, $\rho_2(f)$ and $\lambda(f)$ the lower order of f, the order of f, the hyper-order of f and the exponent of convergence of zeros of f respectively (see [9,12,21]). If $\mu(f) = \rho(f)$, we say that f is of regular growth. We also need the following definition:

Definition 1.1 ([21, Theorem 2.1 and Definition 2.1]). Let f be a transcendental meromorphic function in the complex plane such that $\rho(f) = \rho \leq \infty$. A complex number a is said to be a Borel exceptional value if

$$\limsup_{r\longrightarrow\infty} \frac{\log^+ n\left(r, \frac{1}{f-a}\right)}{\log r} = \limsup_{r\longrightarrow\infty} \frac{\log^+ N\left(r, \frac{1}{f-a}\right)}{\log r} < \rho.$$

In this paper, we will consider the growth of entire functions sharing an entire function of smaller order with their shifts, and study the uniqueness question of entire functions sharing an entire function of smaller order with their shifts.

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We recall the following result which was given by Heittokangas, Korhonen, Laine, Rieppo and Zhang:

Theorem A ([11, Theorem 1]). *Let f be a nonconstant meromorphic function of finite order*

$$\rho(f) := \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r} < 2, \tag{1.1}$$

and let η be a nonzero complex number. If $f(z+\eta)$ and $f(z)$ share a finite complex value a CM, then $f(z+\eta) - a = c(f(z) - a)$ for all $z \in \mathbf{C}$, where c is some nonzero complex number.

The following example shows that the assumption (1.1) of Theorem A is necessary:

Example A ([11]). Let $f(z) = e^{z^2} + 1$, and let η be a nonzero finite complex value. Then it immediately yields that $f(z+\eta) - 1 = (f(z) - 1)e^{2\eta z + \eta^2}$ for all $z \in \mathbf{C}$. Moreover, we find that $N(r, 1/f - 1) = 0$ and so $\lambda(f - 1) < \rho(f)$.

Regarding Theorem A and Example A, we now consider the growth of solutions of the difference equation

$$f(z+\eta) - a(z) = (f(z) - a(z))e^{\alpha(z)}, \tag{1.2}$$

where α and a are entire functions, a is such that $\rho(a) < \rho(f)$ and $\lambda(f - a) < \rho(f)$? In this direction, we will prove the following result:

Theorem 1.1. *Suppose that f is a nonconstant entire solution of the difference equation*

$$f(z+\eta) - a(z) = (f(z) - a(z))e^{P(z)}, \tag{1.3}$$

where a is an entire function such that $\rho(a) < \rho(f)$, P is a nonconstant polynomial. If $\lambda(f - a) < \rho(f)$, then $\rho(f) = 1 + \deg(P)$.

We also get the following result to improve Theorem A:

Theorem 1.2. *Let f be a nonconstant entire function such that $\rho(f) < 2$, let $a \not\equiv 0$ be an entire function such that $\rho(a) < \rho(f)$, and let η be a nonzero complex number. If $f(z) - a(z)$ and $f(z+\eta) - a(z)$ share 0 CM, then $f(z+\eta) - a(z) = c(f(z) - a(z))$ for some nonzero constant c.*

Proceeding as in the proof of Theorem 1.2, we can get the following result:

Theorem 1.3. *Let f be a transcendental entire function such that $\rho(f) < 2$, let η be a nonzero complex number, and let Q be a nonzero polynomial. If $f(z) - Q(z)$ and $f(z+\eta) - Q(z)$ share 0 CM, then $f(z+\eta) - Q(z) = c(f(z) - Q(z))$ for some nonzero constant c.*

2. Some lemmas. The following lemmas will be used in the proof of main results in this paper:

Lemma 2.1 ([4, Theorem 2.1]). *Let f be a transcendental meromorphic function of a order $\rho(f) < \infty$, and let η be a nonzero complex number. Then*

$$T(r, f(z+\eta)) = T(r, f(z)) + O(r^{\rho(f)-1+\varepsilon}) + O(\log r),$$

as $r \to \infty$, where ε is any positive number.

Lemma 2.2 ([21, Theorem 2.11]). *Let f be a transcendental meromorphic function in the complex plane such that $\rho(f) > 0$. If f has two distinct Borel exceptional values in the extended complex plane, then $\mu(f) = \rho(f)$ and $\rho(f)$ is a positive integer or ∞.*

Lemma 2.3 ([5, Theorem 2.1]). *Let f be a nonconstant meromorphic function of an order $\rho(f) = \rho < \infty$, and let λ_1 and λ_2 be, respectively, the exponent of convergence of the zeros and poles of f. Then for any $\varepsilon > 0$, there exists a set $E \subset (1,\infty)$ of $|z| = r$ of a finite logarithmic measure, such that*

$$\frac{f(z+\eta)}{f(z)} = \exp\left\{\eta \frac{f'(z)}{f(z)} + O(r^{\beta+\varepsilon})\right\}$$

holds for $r \notin E \cup [0,1]$, where $\beta = \max\{\rho - 2, 2\lambda - 2\}$ if $\lambda < 1$ and $\beta = \max\{\rho - 2, \lambda - 1\}$ if $\lambda \geq 1$ and $\lambda = \max\{\lambda_1, \lambda_2\}$.

Lemma 2.4 ([6, Corollary 1]). *Let f be transcendental entire function of finite order ρ, let $\Gamma = \{(k_1, j_1), (k_2, j_2), \cdots, (k_m, j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > j_i$ for $i = 1, 2, \cdots, m$, and let $\varepsilon > 0$ be a given constant. Then, there exists a subset $E \subset (1,\infty)$ that has a finite logarithmic measure, such that for all z satisfying $|z| \notin E \cup [0,1]$ and for all $(k,j) \in \Gamma$, we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leq |z|^{(k-j)(\rho-1+\varepsilon)}.$$

Lemma 2.5 ([18]). *Let $Q(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, where n is a positive integer and $a_n = \alpha_n e^{i\theta_n}$, $\alpha_n > 0$, $\theta_n \in [0, 2\pi)$. For any given positive number ε satisfying $0 < \varepsilon < \frac{\pi}{4n}$, consider $2n$ angles:*

$$S_j : -\frac{\theta_n}{n} + (2j-1)\frac{\pi}{2n} + \varepsilon < \theta < -\frac{\theta_n}{n} + (2j+1)\frac{\pi}{2n} - \varepsilon,$$

where $j = 0, 1, \cdots, 2n-1$. Then there exists a positive number $R = R(\varepsilon)$ such that for $|z| = r > R$, $\mathrm{Re}\{Q(z)\} > \alpha_n(1-\varepsilon)r^n \sin(n\varepsilon)$ if $z \in S_j$ where j is even, while $\mathrm{Re}\{Q(z)\} < -\alpha_n(1-\varepsilon)r^n \sin(n\varepsilon)$ if $z \in S_j$ where j is odd.

Lemma 2.6 ([15, Lemma 2.5]). *Suppose that f is a transcendental entire function of a finite order $\rho(f) = \rho < \infty$, and that a set $E_r \subset \mathbf{R}^+$ has a finite logarithmic measure. Then, there exists a sequence of positive numbers $r_k \notin E_r$, $r_k \longrightarrow \infty$ such that for given $\varepsilon > 0$, as r_k sufficiently large, we have $r_k^{\rho-\varepsilon} < \nu(r_k, f) < r_k^{\rho+\varepsilon}$ and $\exp r_k^{\rho-\varepsilon} < M(r_k, f) < \exp r_k^{\rho+\varepsilon}$.*

Lemma 2.7 ([19, Corollary 1]). *Let f be an entire function of finite order and let $\{w_n\}$ be an unbounded sequence. Assume that $\bigcup_{n=1}^{\infty}\{z : f(z) = w_n\}$ has only $k < \infty$ distinct limiting directions, then $f(z)$ is a polynomial of degree at most k.*

Lemma 2.8 ([4, Theorem 8.2]). *Let f be a meromorphic function, let η be a nonzero complex number, and let $\gamma > 1$ and $\varepsilon > 0$ be given real constants, then there exists a subset $E \subset (1, \infty)$ of a finite logarithmic measure, such that for all $|z| \notin E \cup [0, 1]$ we have*

$$\left|\log\left|\frac{f(z+\eta)}{f(z)}\right|\right| \le A\left(\frac{T(\gamma r, f)}{r} + \frac{n(\gamma r)}{r}\log(\gamma r)\log^+ n(\gamma r)\right),$$

where A is a positive constant which depends only on γ and η, $n(t) = n(t, f) + n(t, 1/f)$.

Lemma 2.9 ([21, Theorem 2.1]). *Let f be a transcendental meromorphic function with infinitely many zeros and let λ be the exponent of convergence of zeros of f. Then*

$$\limsup_{r\longrightarrow\infty} \frac{\log N(r, 1/f)}{\log r} = \limsup_{r\longrightarrow\infty} \frac{\log n(r, 1/f)}{\log r} = \lambda.$$

3. Proof of theorems.

Proof of Theorem 1.1. First of all, by (1.3) and the assumption $\rho(a) < \rho(f) < \infty$ we deduce $\rho(e^P) \le \rho(f) < \infty$. Therefore P is a polynomial with degree $\deg(P) \le \rho(f)$. Next we set

$$P(z) = p_m z^m + p_{m-1}z^{m-1} + \cdots + p_1 z + p_0, \tag{3.1}$$

where $p_m, p_{m-1}, \cdots, p_1, p_0$ are complex numbers and $p_m = \alpha_m e^{i\theta_m} \ne 0$, $\alpha_m > 0$, $\theta_m \in [0, 2\pi)$, $m \ge 1$ is a positive integer. From (1.3), (3.1) and Lemma 2.1 we can deduce $\rho(f) \ge \rho(e^P) = m \ge 1$. From $\rho(a) < \rho(f)$ we have $\rho(f) = \rho(f-a)$. Suppose that $\rho(f) > 1$. Then, by Lemma 2.2, the assumption $\lambda(f-a) < \rho(f) < \infty$ and $\rho(f) \ge 2$ implies that $\rho(f-a) = \rho(f) =: l \ge 2$ is a positive integer. By Lemma 2.3 and $\rho(f) \ge 2$ we know that there exists some set $E \subset \mathbf{R}^+$ of finite logarithmic measure such that

$$\frac{f(z+\eta) - a(z+\eta)}{f(z) - a(z)}\{1 + o(1)\} = \exp\left\{\eta\frac{f'(z) - a'(z)}{f(z) - a(z)} + O(r^{\beta+\varepsilon})\right\}, \tag{3.2}$$

as $|z| = r \notin E$ and $r \to \infty$, where β is a constant satisfying $\beta = \max\{\rho(f-a) - 2, \lambda(f-a) - 1\}$. Let $z_1, z_2, \cdots, z_n$ be nonzero zeros of $f(z) - a(z)$, and each zero is repeated as many times as its multiplicity. Then

$$f(z) - a(z) = e^{g(z)}z^{m_0}\prod_{n=1}^{\infty} E_{m_n}\left(\frac{z}{z_n}\right) =: h(z)e^{g(z)}, \tag{3.3}$$

where $m_0 \ge 0$ is an integer, $\prod_{n=1}^{\infty} E_{m_n}\left(\frac{z}{z_n}\right)$ is the canonical product of $f(z)$ (see [12, pp. 6–7]), $g(z)$ is an entire function, $h(z) = z^{m_0}\prod_{n=1}^{\infty} E_{m_n}\left(\frac{z}{z_n}\right)$. From Ash [2, Theorem 4.3.6] we have $\lambda(h) = \rho(h) = \lambda(f-a)$. This together with (3.3) and the assumptions $\lambda(f-a) < \rho(f) < \infty$ and $\rho(a) < \rho(f)$ implies that $\rho(f) = \rho(f-a) = \rho(e^g) = l$. Therefore, $g(z)$ is a nonconstant polynomial with degree $\deg(g) = l$. Let

$$g(z) =: q_l z^l + q_{l-1}z^{l-1} + \cdots + q_1 z + q_0, \tag{3.4}$$

where $q_l, q_{l-1}, \cdots, q_1, q_0$ are complex numbers and $q_l \ne 0$. From (3.3) and (3.4) we have

$$\frac{f'(z) - a'(z)}{f(z) - a(z)} = \frac{h'(z)}{h(z)} + lq_l z^{l-1}(1 + o(1)), \tag{3.5}$$

as $|z| \to \infty$, $h(z) \ne 0$ and $|z| \notin E$. From (3.2) and (3.5) we have

$$\frac{f(z+\eta) - a(z+\eta)}{f(z) - a(z)}\{1 + o(1)\} = \exp\left\{\eta\frac{h'(z)}{h(z)} + l\eta q_l z^{l-1}(1 + o(1))\right\}, \tag{3.6}$$

as $|z| \to \infty$, $h(z) \ne 0$ and $|z| \notin E$. Noting that $\rho(h) = \lambda(h) = \lambda(f-a) < \rho(f) = l$, we can see from Lemma 2.4 that there exists some subset $\tilde{E} \subset (1, \infty)$ that has finite logarithmic measure, we assume w. l. o. g. $E \subset \tilde{E}$ such that for all z satisfying $|z| \notin \tilde{E} \cup [0, 1]$, we have

$$\left|\eta \frac{h'(z)}{h(z)}\right| \le |\eta||z|^{\lambda(f-a)-1+\varepsilon} = o\{l\eta q_l z^{l-1}\}. \tag{3.7}$$

From (3.7) we can see that (3.2) can be rewritten as

$$\begin{aligned}\frac{f(z+\eta)-a(z+\eta)}{f(z)-a(z)}&\{1+o(1)\}\\ &= \exp\{l\eta q_l z^{l-1}(1+o(1))\},\end{aligned} \tag{3.8}$$

as $|z| \notin \tilde{E} \cup [0,1]$. Next we set $\eta q_l = \beta_l e^{i\vartheta_l} \neq 0$, $\beta_l > 0$. For given sufficiently small positive number ε satisfying $0 < \varepsilon < \frac{\pi}{4l}$, we consider $2l$ angles:

$$\begin{aligned} S_j : -\frac{\vartheta_l}{l} + (2j-1)\frac{\pi}{2l} + \varepsilon &< \theta \\ &< -\frac{\vartheta_l}{l} + (2j+1)\frac{\pi}{2l} - \varepsilon, \end{aligned} \tag{3.9}$$

where $j = 0, 1, \cdots, 2l-1$. Then, from (3.9) and Lemma 2.5 we can see that there exists some positive number $R_0 = R_0(\varepsilon)$ such that for every z satisfying $z \in S_{j_1}$ and $|z| > R_0$, where and in what follows, $j_1 \in \{0, 1, \cdots, 2l-1\}$ denotes an even integer, we have

$$\mathrm{Re}\{g(z)\} > \beta_l(1-\varepsilon)r^l \sin(l\varepsilon), \tag{3.10}$$

and such that for every z satisfying $z \in S_{j_2}$ and $|z| > R_0$, where and in what follows, $j_2 \in \{0, 1, \cdots, 2l-1\}$ denotes an odd integer, we have

$$\mathrm{Re}\{g(z)\} < -\beta_l(1-\varepsilon)r^l \sin(l\varepsilon). \tag{3.11}$$

Noting that a is an entire function, we can get from Lemma 2.1 that $\rho(a(z+\eta)) = \rho(a(z)) < \rho(f)$. Combining this with $\rho(f) = l \ge 2$, we have

$$\rho(a(z+\eta) - a(z)) \le \rho(a(z)) < l - \varepsilon,$$

where ε is a sufficiently small positive number. Hence

$$\lim_{\substack{|z|\to\infty \\ z\in S_{j_1}}} \frac{M(|z|, a(z)) + M(|z|, a(z+\eta))}{\exp |z|^{l-\varepsilon}} = 0. \tag{3.12}$$

From (3.10) and (3.12) we have

$$\frac{f(z+\eta)-a(z)}{f(z)-a(z)} = \frac{f(z+\eta)-a(z+\eta)}{f(z)-a(z)} + o(1), \tag{3.13}$$

as $z \in S_{j_1}$ and $|z| \to \infty$. From (1.3), (3.8) and (3.13) we deduce

$$\begin{aligned} e^{P(z)} &= \frac{f(z+\eta)-a(z)}{f(z)-a(z)} \\ &= \exp\{l\eta q_l z^{l-1}(1+o(1))\} + o(1), \end{aligned} \tag{3.14}$$

as $z \in S_{j_1}$, $|z| \notin E \cup [0,1]$ and $|z| \to \infty$. Next we let $\eta q_l = \beta_{l-1} e^{i\vartheta_{l-1}}$, where $\beta_{l-1} > 0$. For given sufficiently small positive number ε satisfying $0 < \varepsilon < \frac{\pi}{4l-4}$, we consider $2(l-1)$ angles $\hat{S}_j$:

$$\begin{aligned} -\frac{\vartheta_{l-1}}{l-1} + \frac{(2j-1)\pi}{2l-2} + \varepsilon &< \theta \\ &< -\frac{\vartheta_{l-1}}{l-1} + \frac{(2j+1)\pi}{2l-2} - \varepsilon, \end{aligned} \tag{3.15}$$

where $0 \le j \le 2l-3$. Then, from (3.15) and Lemma 2.5 we can see that there exists some positive number $R_0 = R_0(\varepsilon)$ such that for every z satisfying $z \in \hat{S}_{k_1}$ and $|z| > R_0$, where and in what follows, $k_1 \in \{0, 1, \cdots, 2l-3\}$ denotes an even integer, we have

$$\mathrm{Re}\{\eta q_l z^{l-1}\} > \beta_{l-1}(1-\varepsilon)r^{l-1} \sin((l-1)\varepsilon), \tag{3.16}$$

and such that for every z satisfying $z \in \hat{S}_{k_2}$ and $|z| > R_0$, where and in what follows, $k_2 \in \{0, 1, \cdots, 2l-3\}$ denotes an odd integer, we have

$$\mathrm{Re}\{\eta q_l z^{l-1}\} < -\beta_{l-1}(1-\varepsilon)r^{l-1} \sin((l-1)\varepsilon). \tag{3.17}$$

We discuss the following two cases:

Case 1. Suppose that there exist some $j_1 \in \{0, 1, \cdots, 2l-1\}$ and some $k_1 \in \{0, 1, \cdots, 2l-3\}$ such that

$$S_{j_1} \cap \hat{S}_{k_1} \neq \emptyset. \tag{3.18}$$

Then, from (3.10) and (3.16) we deduce from (3.14) and $l \ge 2$ that $\deg(P) \ge l-1$. On the other hand, from (3.1), (3.10), (3.14) and (3.16) we have

$$\begin{aligned} |e^{P(z)}| &= |e^{P_m z^m(1+o(1))}| \\ &= e^{\alpha_m r^m \cos(\theta_m + m\theta)(1+o(1))} \to \infty, \end{aligned} \tag{3.19}$$

as $r \to \infty$ and $z \in S_{j_1} \cap \hat{S}_{k_1}$, where $z = re^{i\theta} \in S_{j_1} \cap \hat{S}_{k_1}$. Then, for fixed θ such that $z = re^{i\theta} \in S_{j_1} \cap \hat{S}_{k_1}$, as $r \to \infty$, we have

$$\begin{aligned} e^{\frac{\alpha_m r^m \cos(\theta_m + m\theta)}{2}} &\le |e^{P(z)}| \\ &\le 2\exp\{l|\eta||q_l||z|^{l-1}(1+o(1))\} \\ &\le 2\exp\{2l|\eta||q_l||z|^{l-1}\}. \end{aligned} \tag{3.20}$$

From (3.20) we have $m \le l-1$. This together with $m = \deg(P)$ and $\deg(P) \ge l-1$ reveals the conclusion of Theorem 1.1.

Case 2. Suppose that for every j_1 satisfying $j_1 \in \{0, 1, \cdots, 2l-1\}$ and every $k_1 \in \{0, 1, \cdots, 2l-3\}$ we have

$$S_{j_1} \cap \hat{S}_{k_1} = \emptyset. \tag{3.21}$$

Then, it follows from (3.21) that for every j_1

satisfying $j_1 \in \{0,1,\cdots,2l-1\}$, there exists some $k_2 \in \{0,1,\cdots,2l-3\}$ such that

$$S_{j_1} \subset \hat{S}_{k_2}. \tag{3.22}$$

By (3.9), (3.10), (3.14), (3.15), (3.17) and Phragmén-Lindelöf principle (see [18, p. 270]) we can deduce that $l\eta q_l z^{l-1}$ is a constant, which is impossible. Finally we suppose that $\rho(f) = l = 1$. Then, in the same manner as in the proof of Theorem 1.1 we have (3.2)–(3.8). From (1.3), (3.8) and $\rho(f) = l = 1$ we can find that e^P is a nonzero constant, which reveals the conclusion of Theorem 1.1. This proves Theorem 1.1.

Proof of Theorem 1.2. By the assumptions of Theorem 1.2 we have (1.2). From (1.2), Lemma 2.1 and the assumption $\rho(a) < \rho(f) < \infty$ we can deduce $\rho(e^\alpha) \le \rho(f) < \infty$, which implies that $\alpha =: P$ is a polynomial, and so (1.2) can be rewritten as (1.3). From (1.3) and Lemma 2.1 we have

$$\begin{aligned} T(r, e^{P(z)}) &\le T(r,f(z)) + T(r,f(z+\eta)) + 2T(r,a(z)) \\ &\quad + O(1) \\ &\le T(r,f(z)) + T(r,f(z+\eta)) + 2r^{\rho(a)+\varepsilon} \\ &\le 2T(r,f(z)) + O(r^{\rho(f)-1+\varepsilon}) + 2r^{\rho(a)+\varepsilon} \\ &\quad + O(\log r), \end{aligned} \tag{3.23}$$

as $r \to \infty$. Noting that $\rho(a) < \rho(f)$, we can get from (3.23) that $\rho(e^P) \le \rho(f) < \infty$, which implies that P is a polynomial. If P is a constant, from (1.3) we get the conclusion of Theorem 1.2. Next we suppose that P is a nonconstant polynomial. Then we have (3.1). By Lemma 2.6 we know that there exist some infinite sequence of points $z_{r_k} = r_k e^{i\theta_k}$, where $\theta_k \in [0, 2\pi)$, such that $|f(z_{r_k})| = M(r_k, f)$, and such that for any given positive number ε, as $r_k \to \infty$ and $r_k \notin E$, where $E \subset \mathbf{R}^+$ is a subset with finite logarithmic measure, we have

$$\exp r_k^{\rho(f)-\varepsilon} < |f(z_{r_k})| < \exp r_k^{\rho(f)+\varepsilon}. \tag{3.24}$$

Noting that $\rho(a) < \rho(f) < \infty$, we can get from (3.24) that

$$\lim_{\substack{r_k\to\infty \\ r_k \notin E}} \frac{M(r_k, a)}{|f(z_{r_k})|} = 0. \tag{3.25}$$

From (1.3), (3.24) and (3.25) we have

$$\begin{aligned} e^{P(z_{r_k})} &= \frac{f(z_{r_k}+\eta) - a(z_{r_k})}{f(z_{r_k}) - a(z_{r_k})} \\ &= \frac{f(z_{r_k}+j\eta)}{f(z_{r_k})}\{1+o(1)\}, \end{aligned} \tag{3.26}$$

as $r_k \notin E$ and $r_k \to \infty$. Given a positive number ε, we set

$$T_\varepsilon = \bigcup_{j=0}^{m-1} \{z : |\arg z - \theta_j| < \varepsilon\}, \tag{3.27}$$

where

$$\theta_j = \left(\frac{2j}{m} + \frac{1}{2m}\right)\pi - \frac{\theta_m}{m}, \quad 0 \le j \le m-1. \tag{3.28}$$

Next we let $w_k = f(z_{r_k})$, $k = 1, 2, \cdots$. Then $\{w_k\}$ is an unbounded sequence. We discuss the following two cases:

Case 1. Suppose that T_ε are the only m distinct limiting directions of $\bigcup_{k=1}^{\infty}\{z : f(z) = w_k\}$. Then, from Lemma 2.7 and the assumption $\rho(f) < \infty$ we can see that f is a nonconstant polynomial, which contradicts the assumption $\rho(f) > 0$.

Case 2. Suppose that there exists some sufficiently small positive number ε_0 and there exist some infinite subsequence of the points z_{r_k}, say itself such that

$$\{z_{r_k}\} \subset \mathbf{C} \setminus T_{\varepsilon_0}. \tag{3.29}$$

Noting that $\cos(\theta_m + m\theta_j) = 0$ for $0 \le j \le m-1$, we can deduce from (3.26)–(3.29) that there exists a positive number $\beta(m, \varepsilon_0) \in (0, \alpha_m)$ that depends only upon m and ε_0 such that

$$|\operatorname{Re} P(z_{r_k})| \ge \beta(m,\varepsilon_0) r_k^m \tag{3.30}$$

or

$$|\operatorname{Re} P(z_{r_k})| \le -\beta(m,\varepsilon_0) r_k^m, \tag{3.31}$$

as $z_{r_k} \in \mathbf{C} \setminus T_{\varepsilon_0}$, $r_k \notin E$ and $r_k \longrightarrow \infty$. Noting that $\rho_2(f) = 0$, we can get from (3.26), (3.30), (3.31), Lemmas 2.8 and 2.9 that

$$\begin{aligned} \beta(m,\varepsilon_0) r_k^m &\le |\log|e^{P(z_{r_k})}|| \\ &\le \left|\log\left|\frac{f(z_{r_k}+j\eta)}{f(z_{r_k})}\right|\right| + o(1) \end{aligned}$$

and

$$\begin{aligned} &\left|\log\left|\frac{f(z_{r_k}+j\eta)}{f(z_{r_k})}\right|\right| + o(1) \\ &\le \frac{A_1 n\left(\gamma r_k, \frac{1}{f(z)}\right)}{r_k} \log(\gamma r_k) \log^+ n\left(\gamma r_k, \frac{1}{f(z)}\right) \end{aligned}$$

$$+\frac{A_1 T(\gamma r_k, f)}{r_k}$$
$$\leq A_1(\rho(f)+\varepsilon)\gamma^{\rho(f)+\varepsilon} r_k^{\rho(f)-1+\varepsilon}\log^2(\gamma r_k)$$
$$+ A_1\gamma^{\rho(f)+\varepsilon} r_k^{\rho(f)-1+\varepsilon},$$

as $z_{r_k} \in \mathbf{C}\setminus T_{\varepsilon_0}$, $r_k \notin E$ and $r_k \longrightarrow \infty$, where $\gamma > 1$ is some positive number, A_1 is some positive constant that depends on γ and η. Combining this with (1.3), we deduce

$$\rho(e^P) = m \leq \rho(f) - 1. \tag{3.32}$$

From (3.32) and $\rho(f) < 2$ we can see that P is a constant, which contradicts the above supposition.

Theorem 1.2 is thus completely proved.

4. Concluding remarks. Regarding Theorem 1.1, we now give the following question:

Question 4.1. What can be said about the relationship between $f(z)$ and $f(z+\eta)$, if we remove the assumption "$\lambda(f-a) < \rho(f)$" in Theorem 1.1?

Remark 4.1. If the assumption "$\lambda(f-a) < \rho(f)$" in Theorem 1.1 can indeed be removed, then Theorem 1.2 follows directly from Theorem 1.1.

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