Time-weighted energy method for quasi-linear hyperbolic systems of viscoelasticity

By Priyanjana M. N. DHARMAWARDANE[*)], Tohru NAKAMURA[**)] and Shuichi KAWASHIMA[**)]

(Communicated by Masaki KASHIWARA, M.J.A., May 12, 2011)

Abstract: The aim in this paper is to develop the time-weighted energy method for quasi-linear hyperbolic systems of viscoelasticity. As a consequence, we prove the global existence and decay estimate of solutions for the space dimension $n \geq 2$, provided that the initial data are small in the L^2-Sobolev space.

Key words: Viscoelasticity; time-weighted energy method; global existence; decay estimate.

1. Introduction. We consider the second order quasi-linear hyperbolic systems of viscoelasticity

$$u_{tt} - \sum_j b^j(\partial_x u)_{x_j} + \sum_{j,k} K^{jk} * u_{x_j x_k} + L u_t = 0, \tag{1}$$

with the initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{2}$$

Here u is an m-vector function of $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ $(n \geq 1)$ and $t \geq 0$; $b^j(v)$ are smooth m-vector functions of $v = (v_1, \ldots, v_n) \in \mathbf{R}^{mn}$, where $v_j \in \mathbf{R}^m$ corresponds to u_{x_j}; $K^{jk}(t)$ are smooth $m \times m$ real matrix functions of $t \geq 0$ satisfying $K^{jk}(t)^T = K^{kj}(t)$ for each j, k, and $t \geq 0$, and L is an $m \times m$ real symmetric constant matrix; the symbol "$*$" denotes the convolution with respect to t.

We assume that there exists a smooth function $\phi(v)$ (the free energy) such that

$$b^j(v) = D_{v_j}\phi(v), \tag{3}$$

where $D_{v_j}\phi(v)$ denotes the Fréchet derivative of $\phi(v)$ with respect to v_j. We define

$$B^{jk}(v) = D_{v_k} b^j(v) = D_{v_k} D_{v_j}\phi(v). \tag{4}$$

It then follows that $B^{jk}(v)^T = B^{kj}(v)$ for each j, k, and $v \in \mathbf{R}^{mn}$. Notice that (1) is written as

$$u_{tt} - \sum_{j,k} B^{jk}(0) u_{x_j x_k} + \sum_{j,k} K^{jk} * u_{x_j x_k} + L u_t = \sum_j g^j(\partial_x u)_{x_j}. \tag{5}$$

where $g^j(\partial_x u) := b^j(\partial_x u) - b^j(0) - \sum_k B^{jk}(0) u_{x_k} = O(|\partial_x u|^2)$. We introduce the following symbols of the differential operators associated with (5):

$$B_\omega(0) := \sum_{j,k} B^{jk}(0)\omega_j\omega_k,$$

$$K_\omega(t) := \sum_{j,k} K^{jk}(t)\omega_j\omega_k$$

for $\omega = (\omega_1, \ldots, \omega_n) \in S^{n-1}$. We see that $B_\omega(0)$ and $K_\omega(t)$ are real symmetric matrices. Using these symbols, we impose the following structural conditions.

[**A1**]. $B_\omega(0)$ is positive definite for each $\omega \in S^{n-1}$, while $K_\omega(t)$ is nonnegative definite for each $\omega \in S^{n-1}$ and $t \geq 0$, and L is real symmetric and nonnegative definite.

[**A2**]. $B_\omega(0) - \mathcal{K}_\omega(t)$ is positive definite for each $\omega \in S^{n-1}$ uniformly in $t \geq 0$, where $\mathcal{K}_\omega(t) := \int_0^t K_\omega(s)\,ds$.

[**A3**]. $K_\omega(0) + L$ is (real symmetric and) positive definite for each $\omega \in S^{n-1}$.

[**A4**]. $K_\omega(t)$ is smooth in $t \geq 0$ and decays exponentially as $t \to \infty$. Precisely, there are positive constants C_0 and c_0 such that $-C_0 K_\omega(t) \leq \dot{K}_\omega(t) \leq -c_0 K_\omega(t)$ and $-C_0 K_\omega(t) \leq \ddot{K}_\omega(t) \leq C_0 K_\omega(t)$ for $\omega \in S^{n-1}$ and $t \geq 0$, where $\dot{K}_\omega(t) := \partial_t K_\omega(t)$ and $\ddot{K}_\omega(t) := \partial_t^2 K_\omega(t)$.

Notations. For a nonnegative integer s, $H^s = H^s(\mathbf{R}^n)$ denotes the Sobolev space of L^2 functions

2000 Mathematics Subject Classification. Primary 35L51, 74D10.

[*)] Graduate School of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan.
[**)] Faculty of Mathematics, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan.

doi: 10.3792/pjaa.87.99

on $\mathbf{R}^n$, equipped with the norm $\|\cdot\|_{H^s}$. For a nonnegative integer l, ∂_x^l denotes the totality of all the l-th order derivatives with respect to $x \in \mathbf{R}^n$. Also, for an interval I and a Banach space X, $C^l(I;X)$ denotes the space of l-times continuously differential functions on I with values in X. Throughout the paper, C denotes various generic positive constants.

2. Time-weighted energy estimate and decay estimate. In this section, we first state our result on the time-weighted energy estimate for small solutions to the problem (1), (2). Then, as a corollary, we prove the global existence and quantitative decay of small solutions. For this purpose, we introduce the time-weighted energy norm $E(t)$ and the corresponding dissipation norm $D(t)$:

$$E(t)^2 := \sum_{m=0}^{s} E_m(t)^2,$$

$$D(t)^2 := \sum_{m=0}^{s} \tilde{D}_m(t)^2 + \sum_{m=0}^{s-1} D_m(t)^2,$$

where

$$E_m(t)^2 = \sup_{0\le\tau\le t} (1+\tau)^m \times \left(\|(\partial_x^m u_t, \partial_x^{m+1} u)(\tau)\|_{H^{s-m}}^2 + \sum_{l=m}^{s} Q_K[\partial_x^{l+1} u](\tau) \right),$$

$$\tilde{D}_m(t)^2 = \int_0^t (1+\tau)^m \left(\|(I-P)\partial_x^m u_t(\tau)\|_{H^{s-m}}^2 + \sum_{l=m}^{s} Q_K[\partial_x^{l+1} u](\tau) \right) d\tau,$$

$$D_{m-1}(t)^2 = \int_0^t (1+\tau)^{m-1} \times \left(\|(\partial_x^m u_t, \partial_x^{m+1} u)(\tau)\|_{H^{s-m}}^2 + \sum_{l=m}^{s} Q_K[\partial_x^{l+1} u](\tau) \right) d\tau.$$

Here I and P are the identity matrix and the orthogonal projection matrix onto $\ker(L)$, respectively. Also, the quantity Q_K is defined as

$$Q_K[\partial_x u] := Q_K^\sharp[\partial_x u] + Q_K^\flat[\partial_x u],$$

$$Q_K^\sharp[\partial_x u] := \sum_{j,k} \int_{\mathbf{R}^n} K^{jk}[u_{x_j}, u_{x_k}]\, dx,$$

$$Q_K^\flat[\partial_x u] := \sum_{j,k} \int_{\mathbf{R}^n} \langle K^{jk} u_{x_j}, u_{x_k} \rangle\, dx,$$

where

$$K^{jk}[\psi_j, \psi_k](t) = \int_0^t \langle\, K^{jk}(t-\tau)(\psi_j(t) - \psi_j(\tau)),\ \psi_k(t) - \psi_k(\tau)\,\rangle\, d\tau.$$

Our time-weighted energy estimate involves the following time-weighted L^∞ norm $N(t)$:

$$N(t) := \sup_{0\le\tau\le t} \Big\{ \|(\partial_x u(\tau)\|_{L^\infty} + (1+\tau)\|(\partial_x u_t, \partial_x^2 u)(\tau)\|_{L^\infty} \Big\} \tag{6}$$

and is given as follows:

Proposition 1 (Time-weighted energy estimate)**.** *Suppose that all the conditions* [A1]–[A4] *are satisfied. Let* $n \ge 1$ *and* $s \ge [n/2]+2$. *Assume that* $(u_1, \partial_x u_0) \in H^s$ *and put* $E_0 = \|(u_1, \partial_x u_0)\|_{H^s}$. *Let* u *be a solution to the problem* (1), (2) *satisfying* $(u_t, \partial_x u) \in C^0([0,T]; H^s)$ *for* $T>0$ *such that* $N_0(T) = \sup_{0\le\tau\le T} \|(\partial_x u, \partial_x u_t, \partial_x^2 u)(\tau)\|_{L^\infty}$ *is suitably small. Then we have the following time-weighted energy estimate for* $t \in [0,T]$:

$$E(t)^2 + D(t)^2 \le CE_0^2 + CN(t)D(t)^2. \tag{7}$$

As a simple corollary, we can show the global existence and quantitative decay estimate of small solutions when $n \ge 2$. In fact, using the Gagliardo-Nirenberg inequality $\|v\|_{L^\infty} \le C\|\partial_x^{s_0} v\|_{L^2}^{\theta} \|v\|_{L^2}^{1-\theta}$ with $s_0 = [n/2]+1$ and $\theta = n/(2s_0)$, we can estimate $\|(\partial_x u, \partial_x u_t, \partial_x^2 u)(t)\|_{L^\infty}$ in terms of the time-weighted energy norm $E(t)$ as

$$\|\partial_x u(t)\|_{L^\infty} \le CE(t)(1+t)^{-n/4},$$

$$\|(\partial_x u_t, \partial_x^2 u)(t)\|_{L^\infty} \le CE(t)(1+t)^{-n/4-1/2},$$

where we have used $s \ge s_0+1$. This shows that $N(t) \le CE(t)$ for $n \ge 2$. Consequently, the energy inequality (7) is reduced to $E(t)^2 + D(t)^2 \le CE_0^2 + CE(t)D(t)^2$, from which we can deduce $E(t)^2 + D(t)^2 \le CE_0^2$, provided that E_0 is suitably small and $n \ge 2$. Thus we obtain the following result on the global existence and quantitative decay estimate of solutions.

Theorem 1 (Global existence and decay estimate)**.** *Suppose that all the conditions* [A1]–[A4] *are satisfied. Let* $n \ge 2$ *and* $s \ge [n/2]+2$. *Assume that* $(u_1, \partial_x u_0) \in H^s$ *and put* $E_0 = \|(u_1, \partial_x u_0)\|_{H^s}$. *Then there is a positive constant* δ_0 *such that if* $E_0 \le \delta_0$, *then the problem* (1), (2) *has a unique global solution* u *verifying* $(u_t, \partial_x u) \in C^0([0,\infty); H^s)$. *The solution satisfies the time-weighted energy estimate*

$$E(t)^2 + D(t)^2 \le CE_0^2$$

for $t \geq 0$. In particular, we have the following decay estimates:

$$\|(\partial_x^m u_t, \partial_x^{m+1} u)(t)\|_{L^2} \leq CE_0(1+t)^{-m/2} \tag{8}$$

for $t \geq 0$, where $0 \leq m \leq s$.

In our previous paper [1], we have proved the global existence and asymptotic decay (without decay rate) of small solutions to the problem (1), (2) for all space dimensions $n \geq 1$. The above theorem gives the quantitative decay estimate of solutions obtained in [1] for $n \geq 2$. For more detailed decay estimate of solutions to the corresponding linearized system (*i.e.*, (5) with $g^j \equiv 0$), we refer the reader to [3]. Also, we refer to [5,8] for related results for simpler equations of viscoelasticity.

3. Time-weighted energy method. In this section, we develop the time-weighted energy method for the system (1) and give the outline of the proof of Proposition 1; the detailed proof will be given in our forthcoming paper [2]. The time-weighted energy method was first effectively used by Matsumura [7] in the study of the compressible Navier-Stokes equation. Then similar time-weighted energy methods were used for many other nonlinear systems of partial differential equations, such as hyperbolic systems of balance laws [6], the dissipative Timoshenko system [4], the compressible Euler-Maxwell system [9], and so on. Our time-weighted energy method developed below is quite similar to the one employed in [6,7].

We apply ∂_x^l to (1) to obtain

$$\begin{aligned}\partial_x^l u_{tt} - \sum_{j,k} B^{jk}(\partial_x u)\partial_x^l u_{x_j x_k} \\ + \sum_{j,k} K^{jk} * \partial_x^l u_{x_j x_k} + L\partial_x^l u_t = f^{(l)},\end{aligned} \tag{9}$$

where $f^{(l)} = \sum_{j,k}[\partial_x^l, B^{jk}(\partial_x u)]u_{x_j x_k}$, and $[\,\cdot\,,\cdot\,]$ denotes the commutator. As the first step of our time-weighted energy method, we take the inner product of (9) with $\partial_x^l u_t$ and integrate in x over $\mathbf{R}^n$. Then we multiply the resulting equation by $(1+t)^m$, integrate with respect to t, and add for l with $m \leq l \leq s$. After tedious computations as in [1], we arrive at the basic energy estimate of the form

$$\begin{aligned}&E_m(t)^2 + \tilde{D}_m(t)^2 \\ &\quad \leq CE_0^2 + CN(t)D(t)^2 + mCD_{m-1}(t)^2,\end{aligned} \tag{10}$$

where $0 \leq m \leq s$; the last term on the right-hand side of (10) is absent if $m = 0$.

In the second step, we produce a part of the dissipation in $D(t)$. We take the inner product of (9) with $\sum_{j,k}(K^{jk} * \partial_x^l u_{x_j x_k})_t$ and integrate over $\mathbf{R}^n$. Moreover, we multiply the result by $(1+t)^m$, integrate with respect to t, and add for l with $m \leq l \leq s-1$. Then the technical computations in [1] yield

$$\begin{aligned}&\int_0^t (1+\tau)^m \|\partial_x^{m+1} u_t(\tau)\|_{H^{s-m-1}}^2 \, d\tau \\ &\quad \leq CE_0^2 + CN(t)D(t)^2 \\ &\qquad + \alpha \int_0^t (1+\tau)^m \|\partial_x^{m+2} u(\tau)\|_{H^{s-m-1}}^2 \, d\tau \\ &\qquad + C_\alpha(E_m(t)^2 + \tilde{D}_m(t)^2) + mCD_{m-1}(t)^2\end{aligned} \tag{11}$$

for any $\alpha > 0$, where $0 \leq m \leq s-1$ and C_α is a constant depending on α; the last term on the right-hand side of (11) is absent if $m = 0$. In the third step, we create the remaining part of the dissipation in $D(t)$. We apply ∂_x^{l+1} to (5), take the inner product with $\partial_x^{l+1} u$, and integrate over $\mathbf{R}^n$. Moreover, we multiply the result by $(1+t)^m$, integrate with respect to t, and add for l with $m \leq l \leq s-1$. Then the technical computations as in [1] give

$$\begin{aligned}&\int_0^t (1+\tau)^m \|\partial_x^{m+2} u(\tau)\|_{H^{s-m-1}}^2 \, d\tau \\ &\quad \leq CE_0^2 + CN(t)D(t)^2 \\ &\qquad + C\int_0^t (1+\tau)^m \|\partial_x^{m+1} u_t(\tau)\|_{H^{s-m-1}}^2 \, d\tau \\ &\qquad + C(E_m(t)^2 + \tilde{D}_m(t)^2) + mCD_{m-1}(t)^2,\end{aligned} \tag{12}$$

where $0 \leq m \leq s-1$; the last term on the right-hand side of (12) is absent if $m = 0$. Now we combine (11) and (12). Taking $\alpha > 0$ suitably small, we have

$$\begin{aligned}D_m(t)^2 &\leq CE_0^2 + CN(t)D(t)^2 \\ &\quad + C(E_m(t)^2 + \tilde{D}_m(t)^2) + mCD_{m-1}(t)^2,\end{aligned} \tag{13}$$

where $0 \leq m \leq s-1$. Moreover, substituting (10) into (13), we obtain

$$D_m(t)^2 \leq CE_0^2 + CN(t)D(t)^2 + mCD_{m-1}(t)^2 \tag{14}$$

for $0 \leq m \leq s-1$, where the last term on the right-hand side of (14) is absent if $m = 0$.

Finally, we apply to (10) and (14) the induction argument with respect to m, and conclude that

$$E_m(t)^2 + \tilde{D}_m(t)^2 \leq CE_0^2 + CN(t)D(t)^2,$$

$$D_m(t)^2 \leq CE_0^2 + CN(t)D(t)^2$$

for $0 \leq m \leq s$ and $0 \leq m \leq s-1$, respectively. This gives the desired estimate (7). Thus the proof of Proposition 1 is complete.

References

[1] P. M. N. Dharmawardane, T. Nakamura and S. Kawashima, *Global solutions to quasi-linear hyperbolic systems of viscoelasticity*. (to appear in Kyoto J. Math).

[2] P. M. N. Dharmawardane, T. Nakamura and S. Kawashima, *Decay estimates of solutions for quasi-linear hyperbolic systems of viscoelasticity*. (Preprint).

[3] P. M. N. Dharmawardane, J. E. Muñoz Rivera and S. Kawashima, Decay property for second order hyperbolic systems of viscoelastic materials, J. Math. Anal. Appl. **366** (2010), no. 2, 621–635.

[4] K. Ide and S. Kawashima, Decay property of regularity-loss type and nonlinear effects for dissipative Timoshenko system, Math. Models Methods Appl. Sci. **18** (2008), no. 7, 1001–1025.

[5] S. Kawashima and Y. Shibata, Global existence and exponential stability of small solutions to nonlinear viscoelasticity, Comm. Math. Phys. **148** (1992), no. 1, 189–208.

[6] S. Kawashima and W.-A. Yong, Decay estimates for hyperbolic balance laws, Z. Anal. Anwend. **28** (2009), no. 1, 1–33.

[7] A. Matsumura, An energy method for the equations of motion of compressible viscous and heat-conductive fluids, MRC Technical Summary Report, Univ. of Wisconsin-Madison, #**2194** (1981).

[8] J. E. Muñoz Rivera, Asymptotic behaviour in linear viscoelasticity, Quart. Appl. Math. **52** (1994), no. 4, 628–648.

[9] Y. Ueda and S. Kawashima, *Decay property of regularity-loss type for the Euler-Maxwell system*. (Preprint).