A quantitative result on polynomials with zeros in the unit disk

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Abstract: On Sendov's conjecture, M. J. Miller states the following in his paper [10,11]; if a zero β of a polynomial which has all the zeros in the closed unit disk is sufficiently close to the unit circle, then the distance from β to the closest critical point is less than or equal to 1. It is desirable to quantify this assertion. In this paper, we estimate the radius of the disk with center at 0 containing all the critical points and estimate the range of the zero β satisfying the above for the first step. This result, moreover, implies that if Sendov's conjecture is false, then the polynomial must be close to an extremal one.

Key words: Sendov's conjecture; polynomial; zero; critical point.

1. Introduction. Let $0 \leq \beta < 1$. Define $S(n, \beta)$ to be the set of complex polynomials of degree n with all the zeros in the closed unit disk and at least one zero at β and let $|P|_\beta$ be the distance from β to the closest zero of P'. Under the notation, Sendov's conjecture (see [7, p. 25 Problem 4.5]) is stated as

Conjecture (Sendov)**.** *For* $P \in S(n, \beta)$, $|P|_\beta \leq 1$.

In addition, we believe that an extremal polynomial should be of the form $P(z) = c(z^n - e^{i\theta})$, where $c \neq 0$ and θ are any complex and real number, respectively. Now a polynomial P of degree n is extremal if the maximum value of the distance from a zero to the closetst critical point of P is larger than or equal to those of any polynomials of degree n.

Sendov's conjecture is true in the case $2 \leq n \leq 8$ (see [1–4,9,12,13], for example). This is also true in the case when β is close to 0 even if $n > 8$. We should, therefore, show that in the opposite case, that is, the case when β is close to 1. In this situation, M. J. Miller proved the following.

Theorem A (Miller [10])**.** *There are constants* $K_n > 0$ *so that, if* β *is sufficiently close to* 1 *and* $P \in S(n+1, \beta)$*, then* $|P|_\beta \leq 1 - K_n(1-\beta)$*. Furthermore, one can choose the* K_n *so that* $\lim_{n\to\infty} K_n = 1/3$.

This implies that Sendov's conjecture is true if β is sufficiently close to 1. We, however, do not know how close β is to 1 nor how large K_n is. In the proof, Miller uses the fact that under the assumption $|P|_\beta \geq \beta$, if β is close to 1, then the polynomial must be close to the extremal one, while it is not constructed as any proposition. Nevertheless, it is a key to Theorem A. In this paper, our aim is to quantify this proposition for the first step, though our intended objective is to quantify Theorem A.

Let A_n be a unique positive root (note that $0 < A_n < 1$) of $x^{2n-2} + C_n x - C_n = 0$ with

$$C_n = \frac{2^{2n-2}(n-1)^2(2n-3)^{2n-3}\left(1 + 4\sin^2(\pi/n)\right)}{(n-2)^{2n-4}\sin^2(\pi/n)}.$$

Then our result is stated as

Main Theorem. *Suppose that* $A_n < a < 1$ *and that* $P \in S(n, a)$*. If* $\{z \in \mathbf{C}\,;\, |z-a| \leq 1\}$ *contains no zero of* $P'(z)$*, then all zeros* ζ_j $(j = 1, \ldots, n-1)$ *of* $P'(z)$ *satisfy* $|\zeta_j| < \delta + 1 - a$*, where* $\delta = (C_n(1-a))^{\frac{1}{2n-2}}$.

Since A_n and C_n are constants depending only on n, δ converges to 0 as a tends to 1, which implies that all the critical points are near the origin if a is sufficiently close to 1. The result, therefore, also implies that for a sufficiently close to 1, if we deny Sendov's conjecture, namely, assume that $|P|_a > 1$, then the polynomial must be close to the extremal one. The result will be used in the author's forthcoming paper [5]. We will quantify Theorem A in the paper, where the result of this paper will play a vital role.

2. Preliminary results. Let $P \in S(n, a)$, where a is real (by rotation) and satisfies $0 < a < 1$. If $|P|_a > \rho$ for $\rho \in [a, 1)$, then we know the following two results.

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Lemma A (Dieudonné [6]). *Let $|P|_a > \rho$ for $P \in S(n,a)$. Then, for $|z| \le 1$,*

$$P'(\rho z + a) = P'(a)(1 + zf(z))^{n-1}$$

for an f analytic in the closed unit disk and less than one in modulus.

Lemma B (Kumar and Shenoy [8]). *If $|P|_a > \rho$ for $P \in S(n,a)$, then $P(z)$ has no zero in $\{z \in \mathbf{C}\,;\, |z-a| \le 2\rho\sin(\pi/n)\}$.*

In Lemma A, we need an estimate of $f(z)$. Putting $P(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k) = (z-a)Q(z)$, we obtain

$$\frac{Q'(a)}{Q(a)} = \frac{1}{a-z_1} + \frac{1}{a-z_2} + \cdots + \frac{1}{a-z_{n-1}} \tag{1}$$

and

$$\frac{Q'(a)}{Q(a)} = \frac{P''(a)}{2P'(a)} = \frac{n-1}{2\rho} f(0) \tag{2}$$

by simple calculations. From this point on, suppose that $|P|_a > \rho \ge a$ for the real zero $a < 1$ and $n \ge 4$.

Lemma 1. *If $a > 1/(1+2\sin(\pi/n))$, then $1 - \varepsilon < \operatorname{Re} f(0) < 1$ with*

$$\varepsilon = \frac{1-a^2}{4a^2\sin^2(\pi/n)}. \tag{3}$$

Proof. Since the second inequality is trivial by Lemma A, it is sufficient to show the first one. Lemma B implies $|a - z_k| > 2\rho\sin(\pi/n)$ for each k. If $a > 1/(1+2\sin(\pi/n))$, then $\{z \in \mathbf{C}\,;\, |z-a| = 2\rho\sin(\pi/n)\}$ intersects $\{z \in \mathbf{C}\,;\, |z| = 1\}$. The real part x_0 of the image of the intersection by $1/(a-z)$ is

$$x_0 := \frac{4\rho^2\sin^2(\pi/n) - (1-a^2)}{8a\rho^2\sin^2(\pi/n)}. \tag{4}$$

Hence, $\operatorname{Re} 1/(a-z_k) > x_0$ for each k. Thus, since

$$\operatorname{Re}\frac{Q'(a)}{Q(a)} = \operatorname{Re}\sum_{k=1}^{n-1}\frac{1}{a-z_k} > (n-1)x_0$$

from (1),

$$\operatorname{Re} f(0) > 2\rho x_0$$

from (2). If $\rho \ge a$, then

$$\begin{aligned} 2\rho x_0 &= \frac{4\rho^2\sin^2(\pi/n) - (1-a^2)}{4a\rho\sin^2(\pi/n)} \\ &\ge \frac{4a^2\sin^2(\pi/n) - (1-a^2)}{4a^2\sin^2(\pi/n)} \\ &= 1 - \varepsilon. \end{aligned}$$

□

Remark 1. If $1/\sqrt{1+4\sin^2(\pi/n)} < a < 1$, then

$$0 < \frac{1-a^2}{4a^2\sin^2(\pi/n)} < 1,$$

namely, $0 < \varepsilon < 1$, which implies that $0 < 2\rho x_0 < 1$. Here, $1/\sqrt{1+4\sin^2(\pi/n)}$ is always greater than $1/(1+2\sin(\pi/n))$ for $n \ge 4$.

3. The estimate of $\operatorname{Re} f(z)$.

We need the following lemma to estimate $\operatorname{Re} f(z)$.

Lemma 2. *For $0 < x < 1$, $(1-x)^2 < k < 1$ and $0 < r < 1$,*

$$\frac{1 - r + kr - r^2}{1 - kr^2} - \frac{1-r^2}{1-kr^2}x > 1 - \frac{1+r}{1-r}x.$$

Proof. Since the derivative of the left hand side with respect to k is

$$\frac{r(1+(1-x)r)(1-r^2)}{(1-kr^2)^2} > 0,$$

$$\begin{aligned} &\frac{1 - r + kr - r^2}{1 - kr^2} - \frac{1-r^2}{1-kr^2}x \\ &\quad > \frac{1 - r + (1-x)^2 r - r^2}{1 - (1-x)^2 r^2} - \frac{1-r^2}{1-(1-x)^2 r^2}x \\ &\quad = 1 - \frac{1+r}{1-(1-x)r}x \\ &\quad > 1 - \frac{1+r}{1-r}x. \end{aligned}$$

□

Lemma 3. *If $a > 1/\sqrt{1+4\sin^2(\pi/n)}$, then $1 - (1+r)\varepsilon/(1-r) < \operatorname{Re} f(z) < 1$ for $|z| = r$.*

Proof. The Schwarz-Pick lemma implies

$$\left|\frac{f(z) - f(0)}{1 - \overline{f(0)}f(z)}\right| \le |z|$$

on the open unit disk. Put $w = f(z)$ and $c = f(0)$ for the sake of simplicity. Then $|(w-c)/(1-\bar{c}w)| \le |z| = r$ on $\{z \in \mathbf{C}\,;\, |z| = r\}$. This implies

$$\left|w - \frac{1-r^2}{1-|c|^2r^2}c\right| \le \frac{r\left(1-|c|^2\right)}{1-|c|^2r^2}. \tag{5}$$

Here, since by Lemma 1

$$\begin{aligned} \left|w - \frac{1-r^2}{1-|c|^2r^2}c\right| &\ge -\operatorname{Re}\left(w - \frac{1-r^2}{1-|c|^2r^2}c\right) \\ &> -\operatorname{Re} w + \frac{1-r^2}{1-|c|^2r^2}(1-\varepsilon), \end{aligned}$$

from (5) we obtain

$$\begin{aligned}\operatorname{Re} w &> \frac{1-r+|c|^2 r-r^2}{1-|c|^2 r^2}-\frac{1-r^2}{1-|c|^2 r^2}\varepsilon \\ &> 1-\frac{1+r}{1-r}\varepsilon\end{aligned}$$

via Lemma 2 with $x=\varepsilon$ and $k=|c|^2$. □

4. The proof of Main Theorem. We now estimate the radius of the disk with center at 0 containing all the critical points by proving.

Lemma 4. *Suppose that* $A_n < a \le \rho < 1$, *where* A_n *is a unique positive root of*

$$x^{2n-2}+C_n x - C_n = 0 \tag{6}$$

with

$$C_n = \frac{2^{2n-2}(n-1)^2(2n-3)^{2n-3}\left(1+4\sin^2(\pi/n)\right)}{(n-2)^{2n-4}\sin^2(\pi/n)}.$$

If $\{z \in \mathbf{C}\,;\,|z| \le 1\}$ *contains no zero of* $P'(\rho z+a)$, *then all zeros* ζ_j $(j=1,\dots,n-1)$ *of* $P'(z)$ *satisfy* $|(\zeta_j - a)/\rho + 1| < \delta$ *with*

$$\delta = (C_n(1-a))^{\frac{1}{2n-2}}. \tag{7}$$

Proof. Put

$$p(z) = (1+zf(z))^{n-1} \quad \text{and} \quad p_0(z) = (1+z)^{n-1}$$

for $f(z)$ in Lemma A, and the zeros of $p(z)$ are those of $P'(\rho z+a)$ from Lemma A. Let δ be given by (7). Note that by Lemma 3 and a simple calculation $|f(z)-1| < \tilde{\varepsilon} := \sqrt{2(1+r)\varepsilon/(1-r)}$ for $|z|=r$. On $\{z \in \mathbf{C}\,;\,|z|=r\}$,

$$\begin{aligned}&|p(z)-p_0(z)| \\ &\quad = |(1+zf(z))^{n-1}-(1+z)^{n-1}| \\ &\quad = \left| z(f(z)-1)\sum_{k=0}^{n-2}(1+zf(z))^k(1+z)^{n-2-k}\right| \\ &\quad < r\tilde{\varepsilon}\sum_{k=0}^{n-2}(1+|z|)^{n-2} \\ &\quad = r\tilde{\varepsilon}(n-1)(1+r)^{n-2}.\end{aligned}$$

Hence, we obtain

$$|p(z)-p_0(z)| < r\tilde{\varepsilon}(n-1)(1+r)^{n-2}\left(\frac{1+\delta}{r}\right)^{n-1}$$

on $\{z \in \mathbf{C}\,;\,|z|=1+\delta\}$ via

Lemma C (Rahman and Schmeisser [12, p. 406 Remark 12.1.5]). *Let* f *be a polynomial of degree at most* n *and* ρ *any positive number less than* 1. *Then*

$$\max_{|z|=\rho}|f(z)| \ge \max_{|z|=1}|f(z)|\rho^n \quad (0 \le \rho < 1),$$

where equality holds for any $\rho \in (0,1)$ *if and only if* $f(z) \equiv cz^n$ *for some complex number* c.

Moreover, by the maximum modulus principle,

$$|p(z)-p_0(z)| < r\tilde{\varepsilon}(n-1)(1+r)^{n-2}\left(\frac{1+\delta}{r}\right)^{n-1}$$

on $|z+1|=\delta$, too. Letting $r=(n-2)/(n-1)$ to minimize the right hand side,

$$\begin{aligned}&r\tilde{\varepsilon}(n-1)(1+r)^{n-2}\left(\frac{1+\delta}{r}\right)^{n-1} \\ &\quad = \tilde{\varepsilon}(n-1)\left(\frac{1+r}{r}\right)^{n-2}(1+\delta)^{n-1} \\ &\quad = \tilde{\varepsilon}(n-1)\left(\frac{2n-3}{n-2}\right)^{n-2}(1+\delta)^{n-1} \\ &\quad = \sqrt{2(2n-3)\varepsilon}\,(n-1)\left(\frac{2n-3}{n-2}\right)^{n-2}(1+\delta)^{n-1} \\ &\quad = \sqrt{2}\,\frac{\sqrt{1-a^2}}{2a\sin(\pi/n)}(n-1)\frac{(2n-3)^{n-\frac{3}{2}}}{(n-2)^{n-2}}(1+\delta)^{n-1} \\ &\quad < 2^{n-1}\frac{\sqrt{1-a}}{a\sin(\pi/n)}(n-1)\frac{(2n-3)^{n-\frac{3}{2}}}{(n-2)^{n-2}} \\ &\quad < \left(\frac{2^{n-1}(n-1)(2n-3)^{n-\frac{3}{2}}\sqrt{1+4\sin^2(\pi/n)}}{(n-2)^{n-2}\sin(\pi/n)}\right) \\ &\qquad \cdot (1-a)^{\frac{1}{2}} \\ &\quad = \sqrt{C_n(1-a)} \\ &\quad = \delta^{n-1} \\ &\quad = |p_0(z)|\end{aligned}$$

on $\{z \in \mathbf{C}\,;\,|z+1|=\delta\}$ by making use of $\delta<1$ and $1/\sqrt{1+4\sin^2(\pi/n)} < A_n < a < 1$. Therefore, since

$$|p(z)-p_0(z)| < |p_0(z)| \quad \text{on } \{z \in \mathbf{C}\,;\,|z+1|=\delta\},$$

Rouché's theorem implies that $p(z)$ and $p_0(z)$ have the same number of zeros, counted according to their multiplicities, inside $|z+1|=\delta$. Now, since $p_0(z)$ has $n-1$ zeros inside $|z+1|=\delta$, so does $p(z)$. □

Remark 2. $F(x) := x^{2n-2}+C_n x - C_n$ is monotone increasing if $x>0$ since $F'(x) = (2n-2)x^{2n-3}+C_n > 0$ and $F(x)$ satisfies that $F(0)<0$ and $F(1)>0$. The equation (6), therefore, has the unique root A_n in the interval $(0,1)$. A numerical computation gives the approximate values of $1-A_n$;

n	$1 - A_n$
4	$1.481468313 \times 10^{-6}$
5	$3.134606298 \times 10^{-8}$
6	$8.259759265 \times 10^{-10}$
7	$2.492744903 \times 10^{-11}$
8	$8.267420496 \times 10^{-13}$
9	$2.942709427 \times 10^{-14}$
10	$1.107262403 \times 10^{-15}$

On the other hand, the disk $\{z \in \mathbf{C}\,;\, |z+1| < \delta\}$ is contained in the disk $\{z \in \mathbf{C}\,;\, |z + a/\rho| < \delta + 1 - a/\rho\}$ and since

$$\begin{aligned}\delta &= (C_n(1-a))^{\frac{1}{2n-2}} \\ &< \left(a^{2n-2}\right)^{\frac{1}{2n-2}} \\ &= a \;<\; \frac{1+a-\rho}{\rho}\end{aligned}$$

for $a > A_n$, $\rho\delta + \rho - a < 1$. This implies

Corollary 1. *Suppose that $A_n < a \le \rho < 1$, where A_n is given in Lemma 4 and that $P \in S(n,a)$. If $\{z \in \mathbf{C}\,;\, |z| \le 1\}$ contains no zero of $P'(\rho z + a)$, then all zeros ζ_j $(j = 1, \ldots, n-1)$ of $P'(z)$ satisfy $|(\zeta_j - a)/\rho + a/\rho| < \delta + 1 - a/\rho$ with δ in (7).*

Obviously, the following theorem is equivalent to Corollary 1.

Theorem 1. *Suppose that $A_n < a \le \rho < 1$, where A_n is given in Lemma 4 and that $P \in S(n,a)$. If $\{z \in \mathbf{C}\,;\, |z - a| \le \rho\}$ contains no zero of $P'(z)$, then all zeros ζ_j $(j = 1, \ldots, n-1)$ of $P'(z)$ satisfy $|\zeta_j| < \rho\delta + \rho - a$, where δ is given by (7).*

Letting ρ tend to 1, we obtain Main Theorem.

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