

On the codimension-three conjecture

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Abstract: The codimension-three conjecture states that any regular holonomic module extends uniquely beyond an analytic subset with codimension equal to or larger than three. We give a sketch of a proof of this conjecture.

Key words: Codimension-three conjecture; regular holonomic modules; microlocal perverse sheaf.

1. Introduction. In this note we sketch a proof of a fundamental conjecture, *the codimension-three conjecture*, for microdifferential holonomic systems with regular singularities. This conjecture emerged at the end of the 1970's and is well-known among experts. However, as far as we know, it was never formally written down as a conjecture, perhaps because of lack of concrete evidence for it. As one can also view our result from the point of view of perverse sheaves we have written much of the introduction from that point of view.

Let X be a complex manifold. The notion of micro-support introduced by [KS1, KS2] allows us to study perverse sheaves micro-locally (i.e., locally on the cotangent bundle). Let us fix a conic Lagrangian subvariety $\Lambda \subset T^*X$. It is often important and interesting to understand the category $P_\Lambda(X)$ of perverse sheaves of \mathbf{C} -vector spaces on X with micro-support in Λ . Equivalently, via the Riemann-Hilbert correspondence, we can phrase this problem in terms of \mathcal{D}_X -modules. From that point of view we can view $P_\Lambda(X)$ as the category of regular holonomic \mathcal{D}_X -modules whose characteristic variety is contained in Λ . The basic structure of this category has been studied by several authors, for example, [Be,MV] and [KS2]. In [GMV1] it is shown how in principle one can describe this category: it is

equivalent to the category of finitely generated modules over an associative finitely presented algebra. However, it is perhaps more interesting to describe $P_\Lambda(X)$ in terms of the geometry of T^*X . This is the problem we consider here.

The category $P_\Lambda(X)$ gives rise to a stack $\mathcal{P}er_\Lambda$ on T^*X . From the point of view of perverse sheaves, it is the stack of microlocal perverse sheaves (see [W]). From the point of view of \mathcal{D}_X -modules the construction is more transparent: one simply passes from regular holonomic \mathcal{D}_X -modules to regular holonomic \mathcal{E}_X -modules; here \mathcal{E}_X stands for the ring of microdifferential operators on T^*X . One expects the microlocal description of $P_\Lambda(X)$, i.e., the description of $\mathcal{P}er_\Lambda(\Lambda)$ to be conceptually simpler.

Let us write

$$(1) \quad \Lambda = \Lambda^0 \sqcup \Lambda^1 \sqcup \Lambda^2 \sqcup \dots,$$

where Λ^i is the locus of codimension i singularities of Λ . We leave the appropriate notion of “singularity” vague for now. We set $\Lambda^{\geq i} = \cup_{k \geq i} \Lambda^k$. It is not difficult to show, either from the topological or from the analytic point of view, that the following two statements hold:

(2) The functor $\mathcal{P}er_\Lambda(\Lambda) \rightarrow \mathcal{P}er_\Lambda(\Lambda - \Lambda^{\geq 1})$ is faithful,

and

(3) The functor $\mathcal{P}er_\Lambda(\Lambda) \rightarrow \mathcal{P}er_\Lambda(\Lambda - \Lambda^{\geq 2})$ is fully faithful.

In particular the latter implies

If we have the Lagrangian $\Lambda = \Lambda_1 \cup \Lambda_2$ with each Λ_i Lagrangian and $\text{codim}_\Lambda(\Lambda_1 \cap \Lambda_2) \geq 2$, then $\mathcal{P}er_\Lambda(\Lambda) = \mathcal{P}er_{\Lambda_1}(\Lambda_1) \times \mathcal{P}er_{\Lambda_2}(\Lambda_2)$.

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In concrete terms, (3) means that beyond the codimension one singularities of Λ only conditions on objects are imposed. All the essential data are already given along Λ^0 and Λ^1 . Along the locus Λ^0 we specify a local system (cf. [K1]), and along Λ^1 we specify some “glue” between the local systems on various components of Λ^0 . Such a description of the stack $\mathcal{P}er_{\Lambda}(\Lambda - \Lambda^{\geq 2})$, in terms of Picard-Lefschetz/Morse theory is discussed in [GMV2].

In this paper we answer the question as to what happens beyond codimension two, i.e., we announce the following fundamental fact.

Theorem 1.1. *For an open subset U of Λ and a closed analytic subset Z of U of codimension at least 3, the functor $\mathcal{P}er_{\Lambda}(U) \rightarrow \mathcal{P}er_{\Lambda}(U - Z)$ is an equivalence of categories.*

We sketch an analytic proof of this result utilizing the ring of microdifferential operators. The detailed proof will appear in a forthcoming paper.

2. The set up. Let us recall the definition of the sheaves of rings of microdifferential operators, \mathcal{E}_X and $\hat{\mathcal{E}}_X$ (see [SKK,Sch,K2]). Let $\pi_X : T^*X \rightarrow X$ be the cotangent bundle to X . The \mathbf{C}^* -action on T^*X gives rise to the Euler vector field χ . We say that a function $f(x, \xi)$ on T^*X is homogeneous of degree j if $\chi f = jf$. Taking a local symplectic coordinate system $(x_1, \dots, x_n; \xi_1, \dots, \xi_n)$ of T^*X , we define the sheaf $\hat{\mathcal{E}}_X(m)$ for $m \in \mathbf{Z}$ by setting,

$$\hat{\mathcal{E}}_X(m)(U) = \left\{ \sum_{j=-\infty}^m p_j(x, \xi) \mid p_j(x, \xi) \in \mathcal{O}_{T^*X}(U) \right. \\ \left. \text{is homogeneous of degree } j \right\}$$

for an open subset U of T^*X , and then we set $\hat{\mathcal{E}}_X = \bigcup_{m \in \mathbf{Z}} \hat{\mathcal{E}}_X(m)$. The expression $\sum_{j=-\infty}^m p_j(x, \xi)$ is to be viewed as a formal expression. The formal expressions are multiplied using the Leibniz rule. In this manner $\hat{\mathcal{E}}_X$ becomes a sheaf of rings.

We define \mathcal{E}_X to be the subsheaf of rings of $\hat{\mathcal{E}}_X$ consisting of symbols $\sum_{j=-\infty}^m p_j(x, \xi)$ which satisfy the following growth condition:

$$(5) \quad \text{for every compact } K \subset U \text{ there exists a} \\ C > 0 \text{ such that} \\ \sum_{j=-\infty}^0 \|p_j(x, \xi)\|_K \frac{C^{-j}}{(-j)!} < \infty;$$

here $\|p_j(x, \xi)\|_K$ stands for the sup norm on K . Standard estimates can be used to show that \mathcal{E}_X is

indeed closed under multiplication and hence constitutes a subring of $\hat{\mathcal{E}}_X$.

In the study of both \mathcal{E} -modules and $\hat{\mathcal{E}}$ -modules, we can make use of canonical transformations. In particular, any holonomic module \mathcal{M} can locally be put in general position. We say that \mathcal{M} is in general position if its support Λ is such that the fibers of the projection $\Lambda \rightarrow X$ are at most one dimensional. We have the following basic fact:

Theorem 2.1. *Let us assume that a holonomic \mathcal{E}_X -module \mathcal{M} is in generic position at the point $p \in T^*X$. Then the $\mathcal{E}_{X,p}$ -module \mathcal{M}_p , the module \mathcal{M} localized at the point p , is a holonomic $\mathcal{D}_{X, \pi_X(p)}$ -module, and the canonical morphism $\mathcal{E}_{X,p} \otimes_{\mathcal{D}_{X, \pi_X(p)}} \mathcal{M}_p \rightarrow \mathcal{M}_p$ is an isomorphism.*

A proof of this result is given in [Bj], theorem 8.6.3, for example. The proof uses the same reduction that we utilize later in this paper combined with Fredholm theory. The estimates in the definition (5) are precisely the ones so that this theorem holds.

Let us recall the notion of regular singularities. For a coherent \mathcal{E}_X -module \mathcal{M} , a coherent $\mathcal{E}_X(0)$ -submodule \mathcal{N} is called an $\mathcal{E}_X(0)$ -lattice if $\mathcal{E}_X \otimes_{\mathcal{E}_X(0)} \mathcal{N} \rightarrow \mathcal{M}$ is an isomorphism. A holonomic \mathcal{E} -module with support Λ is said to have *regular singularities* if locally near any point on the support of \mathcal{M} the module \mathcal{M} has an $\mathcal{E}(0)$ -lattice \mathcal{N} which is invariant under $\mathcal{E}_{\Lambda}(1)$, the subsheaf of order 1 operators whose principal symbol vanishes on Λ . Kashiwara and Kawai show, using their notion of order:

Theorem 2.2. *A regular holonomic \mathcal{E} -module possesses a globally defined $\mathcal{E}(0)$ -lattice invariant under $\mathcal{E}_{\Lambda}(1)$. The analogous result holds for $\hat{\mathcal{E}}$ -modules.*

For a proof see [KK], Theorem 5.1.6. In the rest of the paper we make use of the (global) existence of an $\mathcal{E}(0)$ -lattice. Its invariance under $\mathcal{E}_{\Lambda}(1)$ will play no role.

Recall that the stack $\mathcal{P}er_{\Lambda}$ is equivalent to the stack of regular holonomic \mathcal{E}_X -modules with support in Λ . We prove the extension theorem 1.1 in the context of regular holonomic \mathcal{E}_X -modules.

3. Definition of sheaves of rings. In the next section we reduce the codimension-three conjecture to an analogous conjecture about another related ring. We will define this ring below.

Let us consider the formal power series ring $\widehat{A} = \mathbf{C}[[t]]$. It is, of course, a DVR (discrete valuation ring). We will define a subring A of \widehat{A} in the

following manner. For any $C > 0$ we define a norm $\|\cdot\|_C$ on \widehat{A} by the formula

$$(6) \quad \left\| \sum_{j=0}^{\infty} a_j t^j \right\|_C = \sum_{j=0}^{\infty} |a_j| \frac{C^j}{j!}.$$

We write A_C for the subring consisting of elements of \widehat{A} with finite norm. The ring A_C is a Banach DVR. Finally, we set

$$(7) \quad A = \varprojlim_{C \rightarrow 0} A_C.$$

The ring A is DNF (Dual Nuclear Frechet) DVR. We write K for the fraction field of A and \widehat{K} for the field of Laurent series.

Let X be a complex manifold. We write \mathcal{A}_X for the sheaf of holomorphic functions on X with values in A and similarly, for $\widehat{\mathcal{A}}_X$, \mathcal{K}_X , and $\widehat{\mathcal{K}}_X$. We can also view \mathcal{A}_X as a topological tensor product $\mathcal{A}_X = A \tilde{\otimes}_{\mathbb{C}} \mathcal{O}_X$, and similarly for $\widehat{\mathcal{A}}_X$, \mathcal{K}_X , and $\widehat{\mathcal{K}}_X$.

4. The reduction. In this section we make a basic reduction of the conjecture. In the sequel, we consider the projective cotangent bundle $P^*X := \mathring{T}^*X/\mathbb{C}^*$ where $\mathring{T}^*X := T^*X - X$. Since \mathcal{E}_X and $\widehat{\mathcal{E}}_X$ are constant along the fibers of $\mathring{T}^*X \rightarrow P^*X$, we regard \mathcal{E}_X and $\widehat{\mathcal{E}}_X$ as sheaves of rings on P^*X .

Let Λ be a locally closed Lagrangian subvariety of P^*X , let $\mathring{\Lambda} \subset \Lambda$ be an open subset such that $\Lambda - \mathring{\Lambda}$ is a closed analytic subset of codimension at least three in Λ . We assume that we are given a regular holonomic \mathcal{E}_X -module \mathcal{M} with support Λ which is defined on $\mathring{\Lambda}$, and an $\mathcal{E}_X(0)|_{\mathring{\Lambda}}$ -lattice \mathcal{N} of \mathcal{M} . As the extension problem is local, we can work in a neighborhood of a particular point $p \in \Lambda - \mathring{\Lambda}$ which we might as well assume, working by induction, to be a smooth point of $\Lambda - \mathring{\Lambda}$. We now consider the extension problem in the vicinity of this point and we put Λ , via a canonical transformation, in generic position at p . In the neighborhood of $\pi_X(p)$ we take a coordinate system (x_1, \dots, x_n) such that $\pi_X(p)$ corresponds to the origin and the point p corresponds to dx_n at the origin. Then we have a finite map, defined locally in the neighborhood of p :

$$(8) \quad \rho : \Lambda \rightarrow Y \subset \mathbf{C}^{n-1} \quad \text{given by } x_1, \dots, x_{n-1}.$$

where Y is an open subset of \mathbf{C}^{n-1} . We may assume that there exists an open subset \mathring{Y} of Y such that $\rho^{-1}(\mathring{Y}) = \mathring{\Lambda}$ and $Y - \mathring{Y}$ is an analytic subset of codimension ≥ 3 . Let us now consider the $\rho_* \mathcal{E}_X$ -module $\rho_* \mathcal{M}$ and the $\rho_* \mathcal{E}_X(0)$ -module $\rho_* \mathcal{N}$. Also, we

write t for the variable $\partial_{x_n}^{-1}$. Using this notation \mathcal{A}_Y is a subsheaf of rings of $\rho_* \mathcal{E}_X(0)$ and \mathcal{K}_Y is a subsheaf of rings of $\rho_* \mathcal{E}_X$ and similarly in the formal case. By a standard argument using division theorems (see [SKK]), we conclude that

Proposition 4.1. *The sheaf $\rho_* \mathcal{M}|_{\mathring{Y}}$ is coherent over $\mathcal{K}_Y|_{\mathring{Y}}$ and $\rho_* \mathcal{N}|_{\mathring{Y}}$ is coherent over $\mathcal{A}_Y|_{\mathring{Y}}$.*

Let $i : \mathring{\Lambda} \hookrightarrow \Lambda$ and $j : \mathring{Y} \hookrightarrow Y$ be the open inclusions. Then a standard argument shows

- The \mathcal{E}_X -module $i_* \mathcal{M}$ is a coherent \mathcal{E}_X -module if and only if $j_* \rho_* \mathcal{M}$ is a coherent \mathcal{K}_Y -module.

We now recall that holonomic modules are Cohen-Macaulay, i.e., we have

$$(10) \quad \mathcal{E}xt_{\mathcal{E}_X}^k(\mathcal{M}, \mathcal{E}_X)|_{\mathring{\Lambda}} = 0 \quad \text{unless } k = \dim X.$$

Since we have a Serre's duality type isomorphism $\rho_* \mathcal{E}xt_{\mathcal{E}_X}^k(\mathcal{M}, \mathcal{E}_X) \simeq \mathcal{E}xt_{\mathcal{K}_Y}^{k-\dim X}(\rho_* \mathcal{M}, \mathcal{K}_Y)$, we obtain:

$$(11) \quad \mathcal{E}xt_{\mathcal{K}_Y}^k(\rho_* \mathcal{M}, \mathcal{K}_Y)|_{\mathring{Y}} = 0 \quad \text{unless } k = 0.$$

In particular, we conclude

$$(12) \quad \rho_* \mathcal{M}|_{\mathring{Y}} \text{ is locally free as a } \mathcal{K}_Y|_{\mathring{Y}}\text{-module.}$$

Note that statement (11) does not imply (12) immediately. However, we can conclude (12) by using a result of Popescu, Bhatwadekar, and Rao [P]; for a nice discussion, see also [S]. They show:

- Let R be a regular local ring containing a field with maximal ideal \mathfrak{m} and $t \in \mathfrak{m} - \mathfrak{m}^2$. Then every finitely generated projective module over the localized ring R_t is free.

This result is related to Serre's conjecture and was conjectured by Quillen in [Q].

5. The formal case. In this section we discuss the proof of the theorem in the formal case. In the previous section we reduced the proof to the following situation. Let Y be a complex manifold and \mathring{Y} an open subset of Y such that $Y - \mathring{Y}$ is an analytic subset of codimension ≥ 3 . We write $j : \mathring{Y} \rightarrow Y$ for the open inclusion. We assume that we are given a locally free $\widehat{\mathcal{K}}_Y|_{\mathring{Y}}$ -module $\widehat{\mathcal{M}}$ of finite rank together with an $\widehat{\mathcal{A}}_Y|_{\mathring{Y}}$ -lattice $\widehat{\mathcal{N}}$ in $\widehat{\mathcal{M}}$. Let us write

$$(14) \quad \widehat{\mathcal{N}}^* = \mathcal{H}om_{\mathcal{A}_{\mathring{Y}}}(\widehat{\mathcal{N}}, \mathcal{A}_{\mathring{Y}}).$$

By replacing $\hat{\mathcal{N}}$ with $\hat{\mathcal{N}}^{**}$, we may assume that $\hat{\mathcal{N}}$ is reflexive (i.e., that $\hat{\mathcal{N}} \rightarrow \hat{\mathcal{N}}^{**}$ is an isomorphism).

We claim:

Theorem 5.1. *If $\hat{\mathcal{N}}$ is a reflexive coherent $\hat{\mathcal{A}}_Y$ -module, then $j_*\hat{\mathcal{N}}$ is a coherent $\hat{\mathcal{A}}_Y$ -module.*

From this theorem we immediately conclude that $j_*\hat{\mathcal{M}}$ is a coherent $\hat{\mathcal{K}}_Y$ -module and hence obtain our theorem in the formal case.

As to the proof of the theorem above, we reduce it, using [KS3, Proposition 1.2.18], to the following analogous classical result in several complex variables. We state this result, due to Trautman, Frisch-Guenot, and Siu, in a special case suitable for us.

Theorem 5.2. *If \mathcal{F} is a reflexive coherent \mathcal{O}_Y -module, then $j_*\mathcal{F}$ is a coherent \mathcal{O}_Y -module.*

For a discussion of results of this type, see the Séminaire Bourbaki talk by Douady [D].

6. Convergent case. In this section we will argue that formal case implies the convergent case. Just as in the previous section we are reduced to the following situation. Let Y be a complex manifold and \mathring{Y} an open subset of Y such that $Y - \mathring{Y}$ is an analytic subset of codimension ≥ 3 . We write $j : \mathring{Y} \rightarrow Y$ for the open inclusion. We assume the we are given a locally free $\mathcal{K}_{\mathring{Y}}$ -module \mathcal{M} of finite rank on \mathring{Y} . We assume that it is locally free as a \mathcal{K}_Y -module. As the corresponding $\hat{\mathcal{E}}_X$ -module extends across $\Lambda - \hat{\Lambda}$, we obtain the additional piece of information that the sheaf $j_*\hat{\mathcal{M}}$ is locally free. As we work locally along $Y - \mathring{Y}$, we can then assume that $j_*\hat{\mathcal{M}}$ is free. Thus, we have, locally along Y the following situation:

$$(15) \quad \begin{array}{ccccc} \hat{\mathcal{M}} & \cong & \hat{\mathcal{K}}_{\mathring{Y}}^{\oplus r} & \longleftarrow & \hat{\mathcal{A}}_Y^{\oplus r} \\ \uparrow & & \square & & \uparrow \\ \mathcal{M} & \longrightarrow & (\mathcal{M} \cap \hat{\mathcal{A}}_Y^{\oplus r}) = \mathcal{N}. & & \end{array}$$

As \mathcal{M} is locally free on \mathring{Y} we conclude that the $\mathcal{K}_{\mathring{Y}}$ -lattice \mathcal{N} is also locally free on \mathring{Y} . We will now conclude using the following result.

Theorem 6.1. *Let Y be a complex manifold and \mathring{Y} an open subset of Y such that $Y - \mathring{Y}$ is a closed submanifold of codimension ≥ 2 . Then any locally free \mathcal{A}_Y -module \mathcal{N} whose formal completion $\hat{\mathcal{N}}$ extends as a locally free module on Y also extends to a locally free module on Y .*

In the proof of this theorem we make use of the following fact:

Let B be a Banach algebra and let \mathcal{O}_X^B (resp. \mathcal{C}_X^B) be the sheaf of rings of B -valued holo-morphic (resp. continuous) functions

- (16) on a Stein manifold X . Then two locally free \mathcal{O}_X^B -modules of finite rank are isomorphic as soon as they are isomorphic after tensoring by \mathcal{C}_X^B .

This an extension of the Oka principle to infinite dimensional bundles. These issues are discussed in the paper [Bu].

The following result on extensions of submodules also follows from theorem 6.1:

Theorem 6.2. *Let \mathcal{M} be a holonomic \mathcal{E}_X -module defined on an open set Ω of T^*X and let Z be a closed analytic subset of Ω of codimension at least $\dim X + 2$. Then any holonomic submodule \mathcal{N} of $\mathcal{M}|_{\Omega - Z}$ extends to a holonomic $\mathcal{E}_X|_{\Omega}$ -submodule of \mathcal{M} .*

Note that if an extension exists it is unique by (3).

7. Open problems. In this section we discuss open problems which are closely related to our main result.

We have the following result:

Proposition 7.1. *The category of regular holonomic \mathcal{E}_X -modules is a full subcategory of the category of regular holonomic $\hat{\mathcal{E}}_X$ -modules.*

Thus, it is natural to conjecture:

Conjecture 7.2. *The category of regular holonomic \mathcal{E}_X -modules is equivalent to the category of regular holonomic $\hat{\mathcal{E}}_X$ -modules.*

If we fix supports and consider the subcategories where the objects have a fixed conic Lagrangian support Λ , one can check that the two categories coincide in codimension zero by direct verification. Then, by theorem 6.1, the problem reduces to the case of pure codimension one locus.

Microlocal perverse sheaves can be defined with arbitrary coefficients, for example, with integral coefficients. This gives us a stack $\mathcal{P}er_{\Lambda}(\mathbf{Z})$ on a Lagrangian Λ . We conclude with:

Conjecture 7.3. *The codimension-three conjecture holds for the stack $\mathcal{P}er_{\Lambda}(\mathbf{Z})$.*

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