On Nagumo's theorem

By Adrian CONSTANTIN

University of Vienna, Faculty of Mathematics, Nordbergstrasse 15, A-1090 Vienna, Austria

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Abstract: We present a different perspective on Nagumo's uniqueness theorem and its various generalizations. This allows us to improve these generalizations.

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1. Introduction. Nagumo's theorem [7] is one of the most remarkable uniqueness results for the solutions of the differential equation

$$x'(t) = f(t, x(t)) \tag{1.1}$$

with initial data

$$x(0) = 0, \tag{1.2}$$

where $a > 0$ and $f : [0, a] \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous with $f(t, 0) = 0$ for $t \in [0, a]$. It ensures that $x(t) \equiv 0$ is the unique solution to (1.1)–(1.2) if

$$|f(t, x) - f(t, y)| \le \frac{|x - y|}{t} \tag{1.3}$$

for $t \in (0, a]$ and $x, y \in \mathbf{R}^n$ with $|x|, |y| \le M$ for some $M > 0$. This result is more general than the classical Lipschitz condition and the growth of the coefficient $\frac{1}{t}$ as $t \downarrow 0$ is optimal: for any $\alpha > 1$ there exist continuous functions f satisfying (1.3) with the right-hand side multiplied by α but for which (1.1)–(1.2) has nontrivial solutions [1]. Throughout the last decades several generalizations appeared (see the discussion in [1] as well as [4, 5] and references therein). The most far-reaching generalization [1] is that if the continuous function $f : [0, a] \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies $f(t, 0) = 0$ for $t \in [0, a]$, if

$$|f(t, x) - f(t, y)| \le \frac{u'(t)}{u(t)}\, |x - y|, \tag{1.4}$$

for $t \in (0, a]$ and $x, y \in \mathbf{R}^n$ with $|x|, |y| \le M$ for some $M > 0$, where u is an absolutely continuous function on $[0, a]$ with $u(0) = 0$ and $u'(t) > 0$ a.e. on $[0, a]$, and if

$$\frac{f(t, x)}{u'(t)} \to 0 \tag{1.5}$$

as $t \downarrow 0$, uniformly in $|x| \le M$, then uniqueness holds. Appropriate choices of the function u (e.g. $u(t) = t^\alpha$ with $\alpha > 1$) in the conditions (1.4)–(1.5) yield the various generalizations of Nagumo's theorem that appeared throughout the research literature (see [1]).

The object of this note is to present Nagumo's theorem from a different perspective. A simple change of variables in the integral formulation of (1.1)–(1.2) will elucidate in Section 2 the somewhat peculiar character of the Lipschitz time-variable in (1.4) or in (1.3). Namely, (1.3) simply specifies the appropriate asymptotic behaviour of the solution to the new integral equation form of (1.1)–(1.2). In Section 3 we show that our approach yields an improvement of the uniqueness result provided by (1.4)–(1.5).

2. Alternative formulation. In this section we present an alternative proof of the uniqueness result in [1] for solutions to (1.1)–(1.2) under the conditions (1.4)–(1.5). Recall that the integral equation

$$x(t) = \int_0^t f(s, x(s))\, ds \tag{2.1}$$

is an equivalent formulation of (1.1)–(1.2) and plays a central role in the study of the Cauchy problem for ordinary differential equations [2, 6], being also useful in the investigation of the asymptotic behavior of global solutions (see the discussion in [3]). It turns out that a different integral formulation of (1.1)–(1.2) allows us to understand better the connection between the conditions (1.4) and (1.5), providing us with a setting suitable for an improvement of these conditions that will be the object of the next section.

The change of variables

$$\tau = -\ln u(t) \tag{2.2}$$

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transforms the time variable $t \in (0, a]$ into $\tau \in [-\ln u(a), \infty)$, and (2.1) into

$$y(\tau) = \int_\tau^\infty F(\xi, y(\xi))\, d\xi, \tag{2.3}$$

where

$$y(\tau) = x(t)$$

and

$$F(\tau, y) = \frac{u(t)}{u'(t)}\, f(t, x). \tag{2.4}$$

Notice that by l'Hospital's rule and (1.5), a solution to (2.1) satisfies

$$\lim_{t \downarrow 0} \frac{x(t)}{u(t)} = 0.$$

Thus we seek solutions to (2.3) such that

$$\lim_{\tau \to \infty} \Big(e^\tau\, |y(\tau)| \Big) = 0. \tag{2.5}$$

Assuming the existence of a nontrivial solution to (2.1) we will reach a contradiction. Indeed, assume that there is a nontrivial solution. By (1.5) there exists some $\delta \in (0, \min\{1,\, a,\, M\})$ such that

$$|f(t, x)| \le u'(t) \tag{2.6}$$

for $|x|,\, t \in [0, \delta]$, and such that the nontrivial solution is defined on $[0, \delta]$. In view of (2.4), the relation (2.6) means that if we set

$$\tau_0 = -\ln \delta > 0,$$

then

$$|F(\tau, y)| \le u(t) = e^{-\tau} \tag{2.7}$$

for $\tau \ge \tau_0$ and $|y| \le \delta$. Moreover, since $f(t, 0) = 0$, we have

$$|F(\tau, y)| \le |y| \tag{2.8}$$

in view of (2.4) and (1.4). By (2.7) the operator

$$(\mathbf{T}y)(\tau) = \int_\tau^\infty F(\xi, y(\xi)\, d\xi$$

is well-defined on the bounded subset

$$Y_\delta = \{ y \in Y :\ \|y\| \le \delta \}$$

of the normed space Y of continuous functions $y : [\tau_0, \infty) \to \mathbf{R}^n$ which satisfy

$$\sup_{\tau \ge \tau} \{ |y(\tau)|\, e^\tau \} < \infty,$$

endowed with the norm

$$\|y\| = \sup_{\tau \ge \tau_0} \{ |y(\tau)|\, e^\tau \}.$$

If $y \in Y_\delta$ is a nontrivial solution to the integral equation (2.3), let

$$\varepsilon = \sup_{\tau \ge \tau_0} \{ |y(\tau)|\, e^\tau \} > 0.$$

In view of (2.5), there is $\tau_1 \ge \tau_0$ with

$$\varepsilon = |y(\tau_1)|\, e^{\tau_1} > |y(\tau)|\, e^\tau \quad \text{for} \quad \tau > \tau_1.$$

But then (2.8) yields

$$\begin{aligned}
\varepsilon\, e^{-\tau_1} = |y(\tau_1)| &= \Big| \int_{\tau_1}^\infty F(\xi, y(\xi))\, d\xi \Big| \\
&\le \int_{\tau_1}^\infty \Big| F(\xi, y(\xi)) \Big|\, d\xi \\
&\le \int_{\tau_1}^\infty |y(\xi)|\, d\xi \\
&= \int_{\tau_1}^\infty \Big(|y(\xi)|\, e^\xi \Big)\, e^{-\xi}\, d\xi \\
&< \varepsilon \int_{\tau_1}^\infty e^{-\xi}\, d\xi = \varepsilon\, e^{-\tau_1}.
\end{aligned}$$

The obtained contradiction shows that the trivial solution to (2.1) is unique.

3. An improved generalization. While the generalization of Nagumo's theorem provided by (1.4)–(1.5) appears to be quite satisfactory with respect to the growth in the t-variable as $t \downarrow 0$, the approach of Section 2 can be used to show that it is possible to weaken the requirement on the modulus of continuity of f in the spatial variable in (1.4). More precisely, given $M > 0$, define the class $\mathcal{F}_M$ of strictly increasing functions $\omega : [0, M] \to [0, \infty)$ with $\omega(0) = 0$ and such that

$$\int_0^r \frac{\omega(s)}{s}\, ds \le r, \qquad r \in (0, M]. \tag{3.1}$$

The simplest example of such a function is $\omega(s) = s$. Using the mean-value theorem in (3.1) we notice that if $\omega \in \mathcal{F}_M$, then there is a sequence $r_n \downarrow 0$ along which $\omega(r_n) \le r_n$. It is less obvious, but nevertheless true, that there exist functions $\omega \in \mathcal{F}_M$ such that $\omega(r_n) > r_n$ along a sequence $r_n \downarrow 0$. An explicit example will be given in Remark 3.2. Thus there are functions $\omega \in \mathcal{F}_M$ that oscillate around $s \mapsto s$ in any neighborhood of $s = 0$. This shows that a slight improvement of (1.4), in terms of the modulus of continuity of f in the spatial variable, is possible.

We will now prove the following result. Given $a,\, M > 0$, let the continuous function $f : [0, a] \times$

$\mathbf{R}^n \to \mathbf{R}^n$ satisfy $f(t,0) = 0$ for $t \in [0,a]$, and let $u : [0,a] \to [0,\infty)$ be an absolutely continuous function with $u(0) = 0$ and $u'(t) > 0$ a.e. on $[0,a]$.

Theorem 3.1. *Assume that for $t \in (0,a]$ and $|x| \le M$ we have*

$$|f(t,x)| \le \frac{u'(t)}{u(t)}\,\omega(|x|), \tag{3.2}$$

where $\omega \in \mathcal{F}_M$, and that

$$\frac{f(t,x)}{u'(t)} \to 0 \tag{3.3}$$

as $t \downarrow 0$, uniformly in $|x| \le M$. Then the problem (1.1)–(1.2) *has only the trivial solution.*

Proof. As in Section 2, a nontrivial solution defined on $[0,\delta]$ with $\delta \in (0, \min\{1,\, a,\, M\})$ chosen so that (2.6) holds for $|x|, t \in [0,\delta]$, would yield a nontrivial solution $y : [\tau_0, \infty) \to \mathbf{R}^n$ to the integral equation (2.3), with the asymptotic behaviour (2.5). As before, τ is related to t by means of (2.2). While (2.7) will continue to hold, instead of (2.8) we now have

$$|F(\tau, y)| \le \omega(|y|). \tag{3.4}$$

By our assumption,

$$\varepsilon = \sup_{\tau \ge \tau_0} \{|y(\tau)|\, e^{\tau}\} > 0,$$

the supremum being finite by (2.7) and (2.3). In view of (2.5), there is $\tau_1 \ge \tau_0$ with

$$\varepsilon = |y(\tau_1)|\, e^{\tau_1} > |y(\tau)|\, e^{\tau} \tag{3.5}$$

for all $\tau > \tau_1$. Notice that by (2.2), if $t_1 \in (0,\delta]$ is such that

$$u(t_1) = e^{-\tau_1},$$

then

$$\int_{\tau_1}^{\infty} \omega(\varepsilon\, e^{-\xi})\, d\xi = \int_0^{t_1} \omega(\varepsilon\, u(s))\, \frac{u'(s)}{u(s)}\, ds$$

and the change of variables $r = \varepsilon\, u(s)$ transforms the latter integral into

$$\int_0^{\varepsilon\, u(t_1)} \frac{\omega(r)}{r}\, dr = \int_0^{\varepsilon\, e^{-\tau_1}} \frac{\omega(r)}{r}\, dr.$$

Using this in combination with (3.4) and (3.5), we infer that

$$\begin{aligned}
\varepsilon\, e^{-\tau_1} = |y(\tau_1)| &= \Big| \int_{\tau_1}^{\infty} F(\xi, y(\xi))\, d\xi \Big| \\
&\le \int_{\tau_1}^{\infty} \Big| F(\xi, y(\xi)) \Big|\, d\xi \\
&\le \int_{\tau_1}^{\infty} \omega(|y(\xi)|)\, d\xi \\
&= \int_{\tau_1}^{\infty} \omega\Big(\Big\{ |y(\xi)|\, e^{\xi} \Big\}\, e^{-\xi} \Big)\, d\xi \\
&< \int_{\tau_1}^{\infty} \omega\Big(\varepsilon\, e^{-\xi} \Big)\, d\xi \\
&= \int_0^{\varepsilon\, e^{-\tau_1}} \frac{\omega(r)}{r}\, dr \le \varepsilon\, e^{-\tau_1},
\end{aligned}$$

the last inequality being valid by (3.1). The obtained contradiction proves our claim. □

Remark 3.2. For $M \in \left(0, \dfrac{1}{23\pi}\right)$ the function

$$\omega(s) = s + \frac{1}{2}\, s^2 \sin\frac{1}{s} - \frac{1}{3}\, s^2 \tag{3.6}$$

belongs to the class $\mathcal{F}_M$, and the function $s \mapsto \omega(s) - s$ oscillates around 0 in any interval $[0,\varepsilon]$ with $\varepsilon > 0$. Indeed, the monotonocity of ω follows since $\omega'(s) > 0$ for $s \in (0,M]$. As for its oscillatory character, notice that for any $n \ge 12$ we have

$$\omega(s_n) = s_n - \frac{1}{3}\, s_n^2 \quad \text{for} \quad s_n = \frac{1}{2n\pi},$$

while

$$\omega(r_n) = r_n + \frac{1}{6}\, r_n^2 \quad \text{for} \quad r_n = \frac{2}{(4n+1)\pi}.$$

It remains to verify (3.1). Notice that for $s \in (0,M]$ we have $\sin\dfrac{1}{s} \ge 0$ only if

$$\frac{1}{(2n+1)\pi} \le s \le \frac{1}{2n\pi}$$

for some integer $n \ge 12$. Since for any fixed $r \in (0,M]$ there is some integer $N \ge 12$ with

$$\frac{1}{(2N+1)\pi} \le r < \frac{1}{(2N-1)\pi},$$

we deduce that

$$\begin{aligned}
\int_0^r s \sin\frac{1}{s}\, ds &\le \sum_{n \ge N} \int_{\frac{1}{(2n+1)\pi}}^{\frac{1}{2n\pi}} s \sin\frac{1}{s}\, ds \\
&\le \sum_{n \ge N} \int_{\frac{1}{(2n+1)\pi}}^{\frac{1}{2n\pi}} s\, ds \\
&= \frac{1}{8\pi^2} \sum_{n \ge N} \frac{4n+1}{n^2 (2n+1)^2}
\end{aligned}$$

$$< \frac{1}{8\pi^2} \sum_{n \geq N} \frac{1}{n^3} < \frac{1}{8\pi^2} \int_{N-1}^{\infty} \frac{1}{s^3}\, ds$$

$$= \frac{1}{16\pi^2 (N-1)^2} < \frac{1}{3\pi^2 (2N+1)^2} \leq \frac{1}{3}\, r^2$$

which proves the validity of (3.1).

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