Gröbner basis, Mordell-Weil lattices and deformation of singularities, I

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Abstract: We call a section of an elliptic surface to be everywhere integral if it is disjoint from the zero-section. The set of everywhere integral sections of an elliptic surface is a finite set under a mild condition. We pose the basic problem about this set when the base curve is $\mathbf{P}^1$. In the case of a rational elliptic surface, we obtain a complete answer, described in terms of the root lattice E_8 and its roots. Our results are related to some problems in Gröbner basis, Mordell-Weil lattices and deformation of singularities, which have served as the motivation and idea of proof as well.

Key words: Gröbner basis; integral section; Mordell-Weil lattice; deformation of singularities.

1. Introduction. Let S be a smooth projective surface having a relatively minimal elliptic fibration $f : S \to C$ with the zero-section O over a curve C, and let E be the generic fibre of f which is an elliptic curve over the function field $K = k(C)$ (k is a base field of any characteristic). Assume that S has at least one singular fibre. Then the group $M = E(K)$ of K-rational points is finitely generated (Mordell-Weil theorem). It can be identified with the group of sections of f. For each P in $E(K)$, we denote by (P) the image curve of the corresponding section $C \to S$; the curve (P) may be also called a "section" without confusion.

An element P of M is called *everywhere integral* [16] if (P) is disjoint from the zero-section (O). Let $\mathcal{P}$ be the set of all everywhere integral sections:

$$(1.1) \qquad \mathcal{P} = \{P \in M | (P) \cap (O) = \emptyset\}$$

Theorem 1.1. *The set $\mathcal{P}$ is a finite subset of the Mordell-Weil group M.*

Proof. By the height formula [11, Theorem 8.6], we have for any $P \in M$

$$(1.2) \qquad \langle P, P\rangle = 2\chi + 2(PO) - \sum_{w \in R_f} contr_w(P),$$

where the notation is as follows: χ is the arithmetic genus of S (a positive integer), (PO) is the intersection number of two irreducible curves (P) and (O) on S, and $contr_w(P)$ is the local contribution at w (a non-negative rational number); the summation is over the set R_f of the points $w \in C$ with $f^{-1}(w)$ reducible. If P belongs to the set $\mathcal{P}$, then it follows that $\langle P, P\rangle \le 2\chi$. Thus $\mathcal{P}$ forms a set of points with bounded height in M, and hence it is a finite set. (Recall that, by the theory of Mordell-Weil lattices [11], the height pairing is positive-definite on M modulo torsion.) □

Now consider the case: $C = \mathbf{P}^1$, $K = k(t)$. For the sake of simplicity, we assume in the following that the base field k is algebraically closed. Suppose that E/K is given by a generalized Weierstrass equation:

$$(1.3) \quad E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

and O is the point at infinity $(x : y : 1) = (0 : 1 : 0)$. Without loss of generality, we assume that the coefficients a_ν are polynomials in t and "minimal" in the sense that if, for some $l \in k[t]$, a_ν is divisible by l^ν for all ν, then l must be a constant (i.e. $l \in k$), and if furthermore this holds even after one makes a coordinate change of x, y. Then we have

$$(1.4) \qquad \deg a_\nu \le \nu\chi \quad (\nu = 1, 2, 3, 4, 6)$$

where χ is the arithmetic genus of S, which is known to be characterized as the smallest integer satisfying the above condition.

Lemma 1.2. *Let $P \in M = E(K)$. Then $P = (x, y)$ belongs to the set $\mathcal{P}$ if and only if x, y are polynomials in t such that*

$$(1.5) \qquad \deg(x) \le 2\chi, \quad \deg(y) \le 3\chi.$$

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Proof. See the proof of [16, Theorem 2]. □

Let

$$(1.6)\quad P=(x,y):\begin{cases} x=x_0+x_1t+\cdots+x_{2\chi}t^{2\chi}\\ y=y_0+y_1t+\cdots+y_{3\chi}t^{3\chi},\end{cases}$$

and let

$$(1.7)\quad z=z(P)=(x_0,x_1,\cdots,x_{2\chi},y_0,y_1,\cdots,y_{3\chi}).$$

Then, substituting (1.6) into (1.3), we obtain a polynomial identity in t:

$$(1.8)\quad y^2+\cdots-(x^3+\cdots+a_6)=\phi_0+\phi_1t+\cdots+\phi_{6\chi}t^{6\chi}.$$

Let us denote by I the ideal generated by the coefficients ϕ_d of t^d in the polynomial ring R:

$$(1.9)\quad I:=(\phi_0,\ldots,\phi_{6\chi})\subset R:=k[x_0,\cdots,x_{2\chi},y_0,\cdots,y_{3\chi}].$$

We call I the *defining ideal* of $\mathcal{P}$. Obviously we have

$$(1.10)\quad P=(x,y)\in\mathcal{P}\Leftrightarrow z=z(P)\in V(I)\subset\mathbf{A}^{5\chi+2}$$

with $V(I)$ denoting, as usual, the affine scheme of common zeroes of I in the affine space. The map $P\mapsto z(P)$ defines a bijection from $\mathcal{P}$ to the reduced part $V(I)_{red}$ of $V(I)$, and in particular, we have

$$(1.11)\quad n:=\#\mathcal{P}=\#V(I)_{red}.$$

Note that $V(I)_{red}=V(\sqrt{I})$ where $\sqrt{I}$ denotes the radical of I.

Now we consider the (irredundant) primary decomposition of the ideal I:

$$(1.12)\quad I=\mathbf{q}_1\cap\cdots\cap\mathbf{q}_n$$

and the associated prime decomposition of the radical $\sqrt{I}$:

$$(1.13)\quad \sqrt{I}=\mathbf{p}_1\cap\cdots\cap\mathbf{p}_n.$$

Here each $\mathbf{q}_i$ is a primary ideal in the polynomial ring R and $\mathbf{p}_i=\sqrt{\mathbf{q}_i}$ is a prime ideal. In fact, $\mathbf{p}_i$ is the maximal ideal of the point $z(P)\in V(I)$ defined by (1.7) for the corresponding $P=P_i\in\mathcal{P}$. Let us call

$$(1.14)\quad \mu(P_i):=\dim_k R/\mathbf{q}_i$$

the *multiplicity* of $P_i\in\mathcal{P}$ (cf. [3, Ch. 4], [9, Ch. 4], [19, Ch. VII].)

We study the following question:

Question 1.3. *Given an elliptic surface S over $\mathbf{P}^1$ of arithmetic genus χ, with the generic fibre E given by* (1.3) *and* (1.4) *as above, what are* (i) *the number of everywhere integral sections*: $n=\#\mathcal{P}$, (ii) *the linear dimension*: $\dim_k R/I$, *and* (iii) *the multiplicity $\mu(P_i)=\dim_k R/\mathbf{q}_i$ for each $i\le n$?*

Note that, by the Chinese Remainder theorem, we have

$$(1.15)\quad \dim_k R/I=\sum_{i=1}^{n}\dim_k R/\mathbf{q}_i=\sum_{i=1}^{n}\mu(P_i).$$

Hence (ii) will follow from (iii).

Before going further, we present an explicit example.

Example 1.4. *Let $E/k(t)$ be the elliptic curve*

$$(1.16)\quad y^2=x^3+t^5+1.$$

Here we assume k has characteristic 0 or $p>5$. Then (i) *the number of everywhere integral sections $n=\#\mathcal{P}$ is equal to* 240. (ii) *The linear dimension $\dim_k R/I$ is equal to* 240, *too.* (iii) *For all $P\in\mathcal{P}$, the multiplicity $\mu(P)$ is equal to* 1.

Proof. Let us show that $\dim_k R/I=240$ by a direct computation using the method of Gröbner basis. To simplify the notation, we replace the ideal

$$I\subset R=k[x_0,x_1,x_2,y_0,y_1,y_2,y_3]$$

by a similar ideal

$$I'\subset R'=k[u,x_0,x_1,y_0,y_1,y_2]$$

by letting $x_2=u^2, y_3=u^3$. (Note that $x_2^3-y_3^2$ is contained in I.) The Gröbner basis method yields a "shape basis" of I', i.e. a set of generators of I' of the form:

$$I'=(\Psi_{240}(u),x_i-\varphi_i(u),y_j-\psi_j(u)|i=0,1,j=0,1,2)$$

where Ψ,φ_i,ψ_j are polynomials of u and Ψ is a separable polynomial of degree 240. (The explicit form of the polynomial Ψ can be found in [13] or [15] if desired.) Therefore we have

$$\dim_k R/I=\dim_k R'/I'=\dim k[u]/(\Psi(u))=240.$$

Moreover the k-algebra $R/I\cong k[u]/(\Psi(u))$ is isomorphic to a direct sum of 240 copies of k, which shows that $I=\sqrt{I}$ and the primary decomposition of I is given by the 240 maximal ideals corresponding to the 240 roots of the polynomial $\Psi(u)$. In other words, $\mathcal{P}$ consists of $n=240$ elements and $\mu(P)=1$ for each P. □

In this paper, we give a complete answer to Question 1.3 in the case $\chi=1$, i.e. where S is a rational elliptic surface. The main theorem (Theorem 2.1) will be stated in the next section, whose proof will be given in the forthcoming Part II [17]. In §3, we study the behavior of the 240 roots in the

E_8-frame of a rational elliptic surface under specialization and establish a basic result (Theorem 3.4). As a by-product, we obtain a simple proof of the fact that the Mordell-Weil group M is generated by the set $\mathcal{P}$ of everywhere integral sections (Theorem 3.5), whose known proof depends on some case-by-case checking [10].

The plan of the part II is as follows: we prove the main theorem by applying Theorem 3.4 and some general arguments [4, 5, 8]. Then we exhibit a few examples to illustrate it (cf. [12–14]). Finally we discuss some open questions in the case of higher arithmetic genus $\chi > 1$.

As for the title of this paper, Gröbner basis computation is useful, as the above example shows, in dealing with Question 1.3 when S or E is explicitly given. We have made a helpful use of the software "Risa/asir" (developped by the authors of [9]) for some numerical experiments and for direct verification of our results based on the theory of Mordell-Weil lattices and geometry of elliptic surfaces. The idea from deformation of singularities (cf. [13], see also [17, §2.3]) is disguised as the specialization arguments in the proof of our main results.

Convention. Throughout the paper, we keep the notation of §1; we sometimes write $\mathcal{P}_S, I_S, \ldots$ to specify the dependence of $\mathcal{P}, I, \ldots$ on the elliptic surface S under consideration. We continue to assume that k is algebraically closed.

2. Answer in case $\chi = 1$. To state our main results, let us first recall some basic facts on rational elliptic surfaces, fixing the notation (cf. [10], [11, §10]).

Let $N = \mathrm{NS}(S)$ denote the Néron-Severi lattice of an elliptic surface S with a section. Let U be the rank two unimodular sublattice of N spanned by the classes of the zero-section (O) and any fibre F. Let $V = U^\perp$ be the orthogonal complement of U in N, which is called the *frame* of S; we have $N = U \oplus V$. If S is a rational elliptic surface (RES), the frame V is a negative-definite even unimodular lattice of rank 8, and hence it is isomorphic to E_8^-, the opposite lattice of the root lattice E_8 (cf. [2, Ch. 4]).

(2.1) $$\mathrm{NS}(S) = U \oplus V, \quad V \cong E_8^-.$$

Thus we call the frame V of a RES as the E_8-frame.

Let $\mathcal{D} = \mathcal{D}_S \subset V$ be the subset of "roots" in V:

(2.2) $$\mathcal{D} = \{cl(D) \in V | D^2 = -2\}.$$

By the above, it forms a root system of type E_8. In particular, we have

(2.3) $$\#\mathcal{D} = 240.$$

For any $P \in \mathcal{P} = \mathcal{P}_S$, we set

(2.4) $$D(P) := (P) - (O) - F.$$

Then we have $D(P) \perp U$ and $D(P)^2 = -2$, hence $D(P) \in \mathcal{D}$. (N.B. Here and in what follows, we sometimes write $D \in \mathcal{D}$ by abbreviating $cl(D) \in \mathcal{D}$, where $cl(D)$ denotes the class of a divisor D in N. We write $D_1 \equiv D_2$ if $cl(D_1) = cl(D_2)$ in N.)

On the other hand, each reducible fibre $f^{-1}(v) (v \in R_f)$ is decomposed as a sum of its irreducible components with positive integer coefficients:

(2.5) $$f^{-1}(v) = \Theta_{v,0} + \sum_{i=1}^{m_v - 1} k_{v,i} \Theta_{v,i}$$

where $\Theta_{v,0}$ is the unique component intersecting the zero-section (O) and where m_v denotes the number of the irreducible components. Let T_v denote the sublattice of N generated by $\Theta_{v,i} (1 \le i \le m_v - 1)$. It is known (see [6, 7, 18]) that each $\Theta_{v,i}$ has self-intersection number -2 (i.e. $\Theta_{v,i} \in \mathcal{D}$) and T_v is a (negative) root lattice of ADE-type determined by the type of the reducible fibre. Let T be the sublattice of the E_8-frame V defined by

(2.6) $$T = \oplus_{v \in R_f} T_v \subset V \cong E_8^-$$

which is called the *trivial lattice* of S.

Now our main theorem is the following

Theorem 2.1. *Assume that S is a rational elliptic surface. Then* (i) *the number of everywhere integral sections $n = \#\mathcal{P}$ is bounded by* 240:

(2.7) $$0 \le n \le 240,$$

and we have

(2.8) $$n = 240 \Longleftrightarrow T = 0.$$

(ii)

(2.9) $$\dim_k R/I = 240 - \nu(T)$$

where $\nu(T)$ is the number of roots in the trivial lattice T.

(iii) *For each $i \le n$, the multiplicity $\mu(P_i)$ (see* (1.14)) *is equal to the combinatorial multiplicity $m(P_i)$ to be defined below. In other words, we have*

(2.10) $$\mu(P) = m(P) \text{ for all } P \in \mathcal{P}.$$

Definition 2.2. For any $P \in \mathcal{P}$, let $R_f(P)$ denote the subset of $v \in R_f$ such that (P) intersects some non-identity component $\Theta_{v,i} (i \neq 0)$ of $f^{-1}(v)$. The *root graph associated with P*, denoted by $\Delta(P)$, is the connected graph with the vertices

(2.11) $D(P), \Theta_{v,i}$ $(v \in R_f(P), i \neq 0)$,

where two vertices α, β are connected by an edge iff the intersection number $\alpha \cdot \beta = 1$. By a *distinguished root* of $\Delta(P)$, we mean a linear combination of the vertices of the form:

$$D = D(P) + \sum_{v,i} n_{v,i}\Theta_{v,i} \ (n_{v,i} \in \mathbf{Z}, \geq 0) \tag{2.12}$$

satisfying $D^2 = -2$. Further we denote by $m(P)$ the number of distinguished roots in the root graph $\Delta(P)$, and call it the *combinatorial multiplicity* of P.

The proof will be postponed to the part II [17]. First we need to establish, in the next section, the fundamental relationship of the two sets $\mathcal{P}$ and $\mathcal{D}$ for a given RES (Theorem 3.4).

3. Relationship of $\mathcal{P}$ and $\mathcal{D}$. For a rational elliptic surface, the Mordell-Weil group $M = E(K)$ is isomorphic to the quotient group of the Néron-Severi group N by the subgroup $U \oplus T$, hence to the quotient group V/T:

$$M \cong N/(U \oplus T) \cong V/T \tag{3.1}$$

where V and $T = \oplus T_v$ are defined before in §2 (see [10, 11]).

Now we study the relation of $\mathcal{P}$ and $\mathcal{D}$, by restricting the natural projection $\pi : V \to V/T \cong M$, to the set of the roots $\mathcal{D} \subset V$:

$$\pi : \mathcal{D} \to M. \tag{3.2}$$

Lemma 3.1. *Assume $T = 0$. Then the Mordell-Weil lattice M is isomorphic to E_8, and $\mathcal{P}$ is equal to the set of sections $P \in M$ of height $\langle P, P\rangle = 2$. In this case, the map π gives a bijection: $\mathcal{D} \to \mathcal{P}$. The inverse map $\mathcal{P} \to \mathcal{D}$ is given by $P \mapsto D(P)$.*

Proof. If $T = 0$, the rational elliptic surface $f : S \to \mathbf{P}^1$ has no reducible fibres, and hence $M \cong E_8$ (see [10] or [11, §10]). Now the height formula (1.1) says that for any $P \in M$

$$\langle P, P\rangle = 2 + 2(PO)$$

where (PO) is the intersection number of (P) and (O). Hence P has height 2 iff $(PO) = 0$, i.e. iff $P \in \mathcal{P}$.

As the set of roots in E_8, both $\mathcal{P}$ and $\mathcal{D}$ have the same cardinality 240. Thus the map $P \mapsto D(P)$ gives a bijection $\mathcal{P} \to \mathcal{D}$, and it is clear that $\pi(D(P)) = P$ for any P. Hence the assertion. □

Lemma 3.2. *Suppose S is any rational elliptic surface. Let $\tilde{S}$ be a generic rational elliptic surface (cf. [17, §2]), and we consider a smooth specialization $\tilde{S} \to S$ preserving the elliptic fibration and the zero-section. Then it induces an isomorphism of the Néron-Severi lattices*

$$\sigma : \mathrm{NS}(\tilde{S}) \xrightarrow{\sim} \mathrm{NS}(S), \tag{3.3}$$

which gives rise to a bijection $\mathcal{D}_{\tilde{S}} \to \mathcal{D}_S$.

Proof. In general, a specialization of smooth projective surfaces $\tilde{S} \to S$ induces an injective homomorphism $\mathrm{NS}(\tilde{S}) \hookrightarrow \mathrm{NS}(S)$ preserving the intersection pairings. In the case of RES, it gives a lattice isomorphism of $\mathrm{NS}(\tilde{S})$ onto $\mathrm{NS}(S)$ in view of (2.1), which preserves the sublattices U, V by assumption. It is obvious that the set of roots $\mathcal{D}$ in V, (2.2), is also preserved, proving the last assertion. □

(N.B. This result may be called the *conservation law* of the E_8-roots on RES under specialization or deformation: cf. [13].)

Lemma 3.3. *For any $D \in \mathcal{D}_S$, $\pi(D) = P$ belongs to $\mathcal{P}_S$ unless $\pi(D) = O$. In this case, we have*

$$D \equiv D(P) + \gamma \quad (\gamma \in T) \tag{3.4}$$

where γ is a linear combination of $\Theta_{v,i}$ $(v \in R_f, i > 0)$ with non-negative integer coefficients.

Proof. Fix $D \in \mathcal{D}_S$, and assume that $\pi(D) = P \neq O$. We claim that $P \in \mathcal{P}_S$.

We may suppose that S is in the situation described in Lemma 3.2. Then there exists some $\tilde{D} \in \mathcal{D}_{\tilde{S}}$ such that $\sigma(\tilde{D}) = D$. Applying Lemma 3.1 to $\tilde{S}$ (which obviously has $T = 0$), we have

$$\tilde{D} = D(\tilde{P}) := (\tilde{P}) - (\tilde{O}) - \tilde{F} \tag{3.5}$$

for some $\tilde{P} \in \mathcal{P}_{\tilde{S}}$, where $\tilde{O}$ (or $\tilde{F}$) denotes the zero-section (or a fibre) of $\tilde{S}$.

Suppose that, under the specialization, the irreducible curve $\tilde{\Gamma} := (\tilde{P})$ on $\tilde{S}$ reduces to an effective divisor on S:

$$\Gamma = \sum_j \Gamma_j$$

with the irreducible components Γ_j. By the conservation of intersection numbers, we have

$$1 = (\tilde{\Gamma}\tilde{F}) = (\Gamma F) = \sum_j (\Gamma_j F)$$

with each $(\Gamma_j F) \geq 0$. Hence there exists a unique Γ_j, say $j = 1$, such that

$$(\Gamma_1 F) = 1, \quad (\Gamma_j F) = 0 \ (j \neq 1).$$

Then Γ_1 is a section of S, i.e. $\Gamma_1 = (P_1)$ for some $P_1 \in M$, and all other Γ_j are contained in fibres. Obviously P_1 is equal to $P = \pi(D)$.

Next, in the intersection number relation:

$$0 = (\tilde{\Gamma}(\tilde{O})) = (\Gamma(O)) = (PO) + \sum_{j>1}(\Gamma_j(O)),$$

observe that $(PO) \geq 0$ (because $P \neq O$ by assumption) and $(\Gamma_j O) \geq 0$. Hence we have $(PO) = 0$ and $(\Gamma_j O) = 0$. The former implies that $P \in \mathcal{P}_S$, as claimed, while the latter implies that the other components $\Gamma_j (j > 1)$, if any, are among the non-identity components $\Theta_{v,i}\,(i > 0)$ of reducible fibres. Therefore $\tilde{D}$ specializes via σ to the following

(3.6)
$$D^* = (P) - (O) - (F) + \gamma, \quad \gamma = \sum_{v,i>0} m_{v,i}\Theta_{v,i} \in T$$

where $m_{v,i}$ are some non-negative integers. On the other hand, since $\sigma(\tilde{D}) = D$, we have $D \equiv D^*$. This proves Lemma 3.3. □

Theorem 3.4. *For any rational elliptic surface S with a section, let $\mathcal{D}$ be the set of roots in the E_8-frame. Then the map $\pi : \mathcal{D} \to \mathcal{P} \cup \{O\}$ is a surjective map unless $T = 0$, and $\mathcal{D}$ is decomposed into the disjoint union:*

$$\mathcal{D} = \pi^{-1}(O) \bigsqcup \bigsqcup_{P\in\mathcal{P}} \pi^{-1}(P). \tag{3.7}$$

The inverse image $\pi^{-1}(O)$ is the set of roots in T (it is empty if $T = 0$). For any $P \in \mathcal{P}$, we have

(3.8)
$$\pi^{-1}(P) = \{D \in \mathcal{D} \mid D \equiv D(P) + \sum_{v,i>0} m_{v,i}\Theta_{v,i}\}$$

($m_{v,i} \geq 0$) which is equal to the set of distinguished roots in the root graph $\Delta(P)$ defined in §2. In particular, its cardinality is equal to the combinatorial multiplicity of P:

$$m(P) = \#\pi^{-1}(P), \tag{3.9}$$

and

$$\sum_{P\in\mathcal{P}} m(P) = 240 - \nu(T). \tag{3.10}$$

Proof. This is clear by Lemma 3.1 and 3.3. The decomposition (3.7) of $\mathcal{D}$ is just the union of the inverse images of π, and counting the cardinality gives the relation (3.10). □

As a by-product of the above proof, we obtain a conceptual proof of the following fact (see [9, Theorem 2.5], [11, Theorem 10.8]), which has been proven by using the classification of RES plus some case-by-case checking:

Theorem 3.5. *For any rational elliptic surface with a section (defined over an algebraically closed field of arbitarary characterisitic), the Mordell-Weil group is generated by the set $\mathcal{P}$ of everywhere integral sections.*

Proof. It is well-known that the root lattice E_8 is generated by a basis consisting of eight roots (see e.g. [1, 2]). Hence the E_8-frame V is generated by the set $\mathcal{D}$ of roots. Since we have $M \cong V/T$ by (3.1), M is generated by $\pi(\mathcal{D})$, hence by $\mathcal{P}$ by the first part of Lemma 3.3. □

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References

[1] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Hermann, Paris, 1968.
[2] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups*, Third edition, Springer, New York, 1999.
[3] D. Cox, J. Little and D. O'Shea, *Using algebraic geometry*, Springer, New York, 1998.
[4] A. Grothendieck, Éléments de Géometrie Algébrique, Publ. Math. IV, IHES.
[5] R. Hartshorne, *Algebraic geometry*, Springer, New York, 1977.
[6] K. Kodaira, On compact analytic surfaces. II, Ann. of Math. (2) **77** (1963), 563–626.
[7] K. Kodaira, On compact analytic surfaces. III, Ann. of Math. (2) **78** (1963), 1–40.
[8] D. Mumford, *Lectures on curves on an algebraic surface*, Princeton Univ. Press, Princeton, N.J., 1966.
[9] M. Noro and K. Yokoyama, Computational Foundations of Gröbner Bases, Tokyo Univ. Press (2003). (in Japanese).
[10] K. Oguiso and T. Shioda, The Mordell-Weil lattice of a rational elliptic surface, Comment. Math. Univ. St. Paul. **40** (1991), no. 1, 83–99.
[11] T. Shioda, On the Mordell-Weil lattices, Comment. Math. Univ. St. Paul. **39** (1990), no. 2, 211–240.
[12] T. Shioda, Construction of elliptic curves with high rank via the invariants of the Weyl groups, J. Math. Soc. Japan **43** (1991), no. 4, 673–719.
[13] T. Shioda, Mordell-Weil lattices of type E_8 and deformation of singularities, in: Lecture Notes in Math. **1468** (1991), 177–202.
[14] T. Shioda, Existence of a rational elliptic surface with a given Mordell-Weil lattice, Proc. Japan Acad. Ser. A Math. Sci. **68** (1992), no. 9, 251–255.

[15] T. Shioda, Cyclotomic analogue in the theory of algebraic equations of type E_6, E_7, E_8, in *Integral quadratic forms and lattices (Seoul, 1998)*, 87–96, Contemp. Math., 249, Amer. Math. Soc., Providence, RI.

[16] T. Shioda, Integral points and Mordell-Weil lattices, in *A panorama of number theory or the view from Baker's garden (Zürich, 1999)*, 185–193, Cambridge Univ. Press, Cambridge.

[17] T. Shioda, Gröbner Basis, Mordell-Weil Lattices and Deformation of Singularities, II. Proc. Japan Acad. Ser. A Math. Sci. **86** (2010), no. 2, 27–32.

[18] J. Tate, Algorithm for determining the type of a singular fiber in an elliptic pencil, in *Modular functions of one variable, IV (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, 33–52. Lecture Notes in Math., 476, Springer, Berlin.

[19] A. Weil, *Foundations of algebraic geometry*, Amer. Math. Soc., Providence, R.I., 1962.