Coarse fixed point theorem

By Tomohiro FUKAYA

Department of Mathematics, Kyoto University, Kitashirakawa Oiwake-cho, Sakyo-ku, Kyoto 606-8502, Japan

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Abstract: We study group actions on a coarse space and the induced actions on the Higson corona from a dynamical point of view. Our main theorem states that if an action of an abelian group on a proper metric space satisfies certain conditions, the induced action has a fixed point in the Higson corona. As a corollary, we deduce a coarse version of Brouwer's fixed point theorem.

Key words: Coarse geometry; Higson corona; fixed point theorem.

1. Introduction. A metric space X is *proper* if closed, bounded set in X are compact. Let X and Y be proper metric spaces and let $f: X \to Y$ be a map (not necessarily continuous). We define:

(a) The map f is *proper* if for each bounded subset B of Y, $f^{-1}(B)$ is a bounded subset of X.

(b) The map f is *bornologous* if for every $R > 0$ there exists $S > 0$ such that for each $x, y \in X$, $d(x, y) < R$ implies $d(f(x), f(y)) < S$.

(c) The map f is *coarse* if it is proper and bornologous.

Let $f, g: X \to Y$ be maps. We define that f is *close* to g, denoted $f \simeq g$, if there exists $R > 0$ such that $d(f(x), g(x)) < R$, for all $x \in X$. We define that X and Y be *coarsely equivalent* if there exist coarse maps $f: X \to Y$ and $g: Y \to X$ such that $g \circ f$ and $f \circ g$ are close to the identity maps of X and Y, respectively. A *coarse space* is a coarsely equivalent class of proper metric spaces. The category of coarse spaces consists of coarse spaces and coarse maps.

Let $\varphi: X \to \mathbf{C}$ be a bounded continuous map. For each $r > 0$, we define a map $V_r\varphi: X \to \mathbf{R}$ by

$$V_r\varphi(x) := \sup\{|\varphi(y) - \varphi(x)| : d(x, y) < r\}.$$

We define that φ is a *Higson function* if for each $r > 0$, $V_r\varphi$ vanishes at infinity. The Higson functions on a proper metric space X form a unital C^*-algebra, denoted by $C_h(X)$. It follows from the Gelfand-Naimark theorem that there exists a unique compact Hausdorff space hX such that $C(hX) = C_h(X)$. The compactification hX of X is called the Higson compactification. Its boundary $hX \setminus X$ is denoted by νX and called the Higson corona. The Higson corona is a functor from the category of coarse spaces into the category of compact Hausdorff spaces. Namely, a coarse map $f: X \to Y$ induces the unique continuous map $\nu f: \nu X \to \nu Y$ and moreover if coarse maps $f, g: X \to Y$ are close then $\nu f = \nu g$. We remark that the Higson corona of a proper metric space is never second countable. We refer to [3] for a general reference of coarse geometry and the Higson compactification.

Let X be a proper metric space and let G be a finitely generated semi-group acting on X. A coarse action, defined below, of G on X induces a continuous action of G on the Higson corona νX. The main subject of this article is to study these actions from a dynamical point of view. Details of proofs of our main results will be published elsewhere.

2. Coarse action. Let X and G be as above.

Definition 2.1. An action of G on X is *coarse* if for each element g of G, the map $\Psi_g: X \to X$ defined by $x \mapsto g \cdot x$ is a coarse map.

Definition 2.2. For a point x_0 of X, the orbit map $\Phi_{x_0}: G \to X$ is defined by $g \mapsto g \cdot x_0$. We define:

(a) The orbit of x_0 is *proper* if so is Φ_{x_0}.

(b) The orbit of x_0 is *bornologous* if so is Φ_{x_0}.

(c) The orbit of x_0 is *coarse* if so is Φ_{x_0}.

A typical example of the coarse action with coarse orbits is the action of G on itself.

Lemma 2.3. *Let G be a finitely generated group or $G = \mathbf{N}^k$ with a left-invariant word metric for some generating set. The action of G on G by the left-translation $(g, h) \mapsto gh$ is a coarse action. Furthermore, any orbit of $h \in G$ is coarse.*

Since a coarse map induces a continuous map between the Higson coronae, a coarse action induces

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a continuous action on the Higson corona. Our main theorem is the following

Theorem 2.4. *Assume that $G = \mathbf{N}^k$ or $\mathbf{Z}^k$ and G acts on X as a coarse action. Suppose that there exists a point of X whose orbit is coarse. Then the induced action of G has a fixed point in the Higson corona νX. That is, there exists a point x of νX such that $g \cdot x = x$ for any element g of G.*

Example 2.5. Let G be a finitely generated group with an element h of infinite order. Then a group action of $\mathbf{Z}$ on G given by $(n, g) \mapsto h^n g$ is a coarse action and any orbit is coarse. Thus the action of $\mathbf{Z}$ has a fixed point in the Higson corona νG. Moreover, if G is a hyperbolic group, this action extends to the Gromov boundary $\partial_g G$. Then this action of $\mathbf{Z}$ has a fixed point in $\partial_g G$, since there exists a G-equivariant map $\nu G \to \partial_g G$. This is a well-known fact on the boundary of hyperbolic groups (c.f. Proposition 10 and Theorem 30 in [2, Chapter 8]).

Example 2.6. The wreath product $\mathbf{Z} \wr \mathbf{Z}$ contains $\mathbf{Z}^n$ as a subgroup for any positive integer n (see page 135 of [3]). Thus the action of $\mathbf{Z}^n$ on $\mathbf{Z} \wr \mathbf{Z}$ is coarse and the induced action of $\mathbf{Z}^n$ has a fixed point in $\nu(\mathbf{Z} \wr \mathbf{Z})$.

We cannot generalize Theorem 2.4 to a free group action as follows:

Proposition 2.7. *The action of the free group F_2 on νF_2 induced by the left-translation $F_2 \times F_2 \to F_2$ has no fixed point. That is, there is no point x of νF_2 such that $g \cdot x = x$ for all elements g of F_2.*

Proof. If the induced action of F_2 on νF_2 has a fixed point, the induced action of F_2 on the Gromov boundary $\partial_g F_2$ also has a fixed point. However, we can show that for any point z of $\partial_g F_2$, there exists an element g of F_2 such that $g \cdot z \neq z$. □

3. Coarse fixed point. Let G be a finitely generated semi-group acting on X. We call a point x of X, a *coarse fixed point* if its orbit

$$G \cdot x = \{g \cdot x : g \in G\} \subset X$$

is a bounded set. If G is an infinite group and x is a coarse fixed point, then the orbit of x is not proper. In the following two cases, the converse holds.

Proposition 3.1. *Let X be a metric space such that any bounded subset $D \subset X$ is a finite set. Suppose that $\mathbf{N}$ acts on X. Then a point of X whose orbit is not proper is a coarse fixed point.*

Proof. Suppose that the orbit of x_0 is not proper. Then there exists a bounded set $D \subset X$ such that

$$\{n \in \mathbf{N} : n \cdot x_0 \in D\}$$

is an infinite set. Because D is a finite set, there exist positive integers $m > n$ such that $m \cdot x_0 = n \cdot x_0$. For any integer $l > m$, there exist integers $k > 0$ and $r = 0, \cdots, m - n - 1$ satisfying $l - n = k(m - n) + r$. Thus we have $l \cdot x_0 = (n + r) \cdot x_0$. It follows that $\mathbf{N} \cdot x_0 \subset \{x_0, 1 \cdot x_0, 2 \cdot x_0, \cdots, (m-1) \cdot x_0\}$. □

Proposition 3.2. *Let X be a proper metric space. Suppose that $\mathbf{N}$ acts on X as an isometry. Then each point of X whose orbit is not proper is a coarse fixed point.*

Proof. We only give a sketch of a proof. Suppose that the orbit of x_0 is not proper. Then there exists a bounded subset $D \subset X$ such that $\{n \in \mathbf{N} : n \cdot x_0 \in D\}$ is an infinite set. Put $K = B(D, 1) \cap \mathbf{N} \cdot x_0$. Here $B(D, 1)$ is the 1-neighborhood of D. Since the action is an isometry, there exist points $x_1, \cdots, x_N$ of K and positive integers $T_1, \ldots, T_N$ such that

$$\overline{K} \subset \bigcup_{i=1}^{N} B(x_i, 1),$$

and $T_j \cdot x$ lies in $\bigcup_{i=1}^{N} B(x_i, 1)$ for any point x of $B(x_j, 1)$ where j runs from 1 to N.

Using this decomposition of K, we can show that $\mathbf{N} \cdot x_0 \subset B(x_0, R)$ for some $R > 0$. □

If the orbit is *not coarse*, there are two possibilities; that is, the orbit is *not proper*, or, the orbit is *not bornologous*. However, if the action is an isometry, any orbit is bornologous.

Proposition 3.3. *Let X be a proper metric space with an isometric action of $\mathbf{N}$. Then the action is a coarse action and any orbit is bornologous.*

Proof. An isometric action is coarse. For any given point x of X, put $L = d(1 \cdot x, x)$. Then we have $d((i+1) \cdot x, i \cdot x) = L$ for all integers $i > 0$. Hence for any pair of integers $m \geq n \geq 0$, we have

$$d(\Phi_x(m), \Phi_x(n)) = d(m \cdot x, n \cdot x) \leq \sum_{i=n}^{m-1} d((i+1) \cdot x, i \cdot x) = L|m - n|.$$

Thus Φ_x is bornologous. □

Corollary 3.4 (Coarse version of Brouwer's fixed point theorem). *Let X be a proper metric space and $f : X \to X$ be an isometry. Then at least one of the following holds*:

(a) *The map f has a coarse fixed point in X.*

(b) *The map νf has a fixed point in νX.*

Example 3.5. The Gromov boundary of the hyperbolic plane $\mathbf{H}^2$ is S^1. Let $f : \mathbf{H}^2 \to \mathbf{H}^2$ be a continuous map such that f extends to the Gromov boundary. Then Brouwer's fixed point theorem says that $f\colon \mathbf{H}^2 \cup S^1 \to \mathbf{H}^2 \cup S^1$ has a fixed point. Let Γ be a discrete group of isometries acting freely on $\mathbf{H}^2$ with quotient a compact surface. Γ is coarsely equivalent to $\mathbf{H}^2$ and its Gromov boundary is also S^1. Let $f\colon \Gamma \to \Gamma$ be an isometry. Then Corollary 3.4 says that $f\colon \Gamma \cup S^1 \to \Gamma \cup S^1$ has a coarse fixed point on Γ, or, a fixed point on S^1.

In Corollary 3.4, the assumption that the map f is an isometry is essential.

Remark 3.6. In [1, Section 4] we give an example of a proper coarse space X and a coarse map $f\colon X \to X$ such that the following hold:

(a) The map f has no coarse fixed point in X.

(b) The map νf has no fixed point in νX.

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