On injectivity, vanishing and torsion-free theorems for algebraic varieties

Dedicated to Professor Yoichi Miyaoka on the occasion of his 60th birthday

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Abstract: We give a short and almost self-contained proof of generalizations of Kollár's vanishing and torsion-free theorems. Although they are contained in Ambro's much more general results on embedded normal crossing pairs, we give an alternate and direct reduction argument to the mixed Hodge theory. In this sense, this paper gives a more readable account of the application to the log minimal model program for log canonical pairs.

Key words: Vanishing theorem; torsion-freeness; injectivity theorem; Hodge theory.

1. Introduction. The main purpose of this paper is to give a short and almost self-contained proof of the following theorem.

Theorem 1.1 (Torsion-free and vanishing theorems)**.** *Let Y be a smooth projective variety and B a boundary $\mathbf{Q}$-divisor such that* Supp$B$ *is simple normal crossing. Let $f : Y \to X$ be a projective morphism and L a Cartier divisor on Y such that $H \sim_{\mathbf{Q}} L - (K_Y + B)$ is f-semi-ample.*

(i) *Let q be an arbitrary non-negative integer. Every non-zero local section of $R^q f_* \mathcal{O}_Y(L)$ contains in its support the f-image of some strata of (Y, B), where a* stratum *of (Y, B) denotes Y itself or an lc center of (Y, B).*
(ii) *Assume that $H \sim_{\mathbf{Q}} f^* H'$ for some ample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor H' on X. Then $H^p(X, R^q f_* \mathcal{O}_Y(L)) = 0$ for all $p > 0$ and $q \geq 0$.*

Although this theorem is a very special case of [A, Theorem 3.2], it will play important roles in the log minimal model program for log canonical pairs. In [A], Ambro proved the above theorem for *embedded normal crossing pairs*. His proof is rather difficult involving a highly technical notion of normal crossing pairs. For a systematic and thorough treatment, we refer the reader to [F1, Chapter 2].

The author has found a straightforward proof of the cone theorem for log canonical pairs, which does not use *quasi-log varieties*. The proof will be published in the forthcoming [F2]. The cone theorem for log canonical pairs is the starting point of the log minimal model program for log canonical pairs. Being free from resolution of singularities and perturbation of coefficients, the proof of the cone theorem in [F2] will be even easier than the original proof of the cone theorem for Kawamata log terminal pairs. Both [A, Chapter 3] and [F1, Chapter 2] were intended for the experts and rather involved. Although it was the feature of [A] to prove the cone theorem in the context of *quasi-log varieties*, the proof of the cone theorem without *quasi-log varieties* was not available. Thus Theorem 1.1 had to be proved for embedded normal crossing pairs, which was the most difficult part in [A]. Both of [A, Chapter 3] and [F1, Chapter 2] adopted Esnault–Viehweg's framework explained in [EV]. Here, we give a short proof of Theorem 1.1 after Kollár's philosophy explained in, for example, [KM, §2.4]. It is the first time that we use Kollár's philosophy to treat Theorem 1.1. in the literature. Hopefully, the approach adopted here will clarify the nature of Theorem 1.1.

We summarize the contents of this paper. Section 2 is a short review of the Hodge theoretic aspect of the injectivity theorem. We would like to emphasize that the E_1-degeneration in [D] is sufficient for our purposes. We do not know whether the E_1-degeneration discussed in [EV, (3.2, c)] follows from the one in [D] if $A \neq 0$ in [EV, (3.2, c)] (cf. [EV, 3.18. Remarks. a)]). In Section 3, we give a short proof of Theorem 1.1. It is a standard argument once the fundamental injectivity theorem is given in Section 2. In Section 4, we will explain two

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applications of Theorem 1.1. The first one contains the extension theorem from log canonical centers. It is very strong, seems inaccessible by the Kawamata–Viehweg–Nadel vanishing theorem (cf. Remark 4.2), and is intended for use in the log minimal model program for log canonical pairs. Although [A] proved it in the context of quasi-log varieties, this paper gives a more accessible account. The final theorem is the Kodaira vanishing theorem for log canonical pairs, which was not explicitly stated in [A].

Notation. Let X be a normal variety and B an effective $\mathbf{Q}$-divisor such that K_X+B is $\mathbf{Q}$-Cartier. Then we can define the discrepancy $a(E,X,B)\in\mathbf{Q}$ for every prime divisor E *over* X. If $a(E,X,B)\geq -1$ for every E, then (X,B) is called *log canonical*. We sometimes abbreviate log canonical to *lc*. Assume that (X,B) is log canonical. If E is a prime divisor over X such that $a(E,X,B)=-1$, then $c_X(E)$ is called a *log canonical center* (*lc center*, for short) of (X,B), where $c_X(E)$ is the closure of the image of E on X. A *stratum* of (X,B) denotes X itself or an lc center of (X,B).

Let r be a rational number. The integral part $\llcorner r\lrcorner$ is the largest integer $\leq r$ and the fractional part $\{r\}$ is defined by $r-\llcorner r\lrcorner$. We put $\ulcorner r\urcorner=-\llcorner -r\lrcorner$ and call it the round-up of r. For a $\mathbf{Q}$-divisor $D=\sum_{i=1}^r d_iD_i$, where D_i is a prime divisor for every i and $D_i\neq D_j$ for $i\neq j$, we call D a *boundary* $\mathbf{Q}$-divisor if $0\leq d_i\leq 1$ for every i. We note that $\sim_{\mathbf{Q}}$ denotes the $\mathbf{Q}$-linear equivalence of $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisors. We put $\llcorner D\lrcorner=\sum\llcorner d_i\lrcorner D_i$, $\ulcorner D\urcorner=\sum\ulcorner d_i\urcorner D_i$, $\{D\}=\sum\{d_i\}D_i$, $D^{<1}=\sum_{d_i<1}d_iD_i$, and $D^{=1}=\sum_{d_i=1}D_i$.

We will work over $\mathbf{C}$, the complex number field, throughout this paper.

2. Hodge theoretic aspect. In this section, we will prove the following injectivity theorem, which is the same as [EV, 5.1. b)]. The proof given here is more in the sprint of Kollár than in the sprit of Esnault–Viehweg.

We use the classical topology throughout this section.

Proposition 2.1 (Fundamental injectivity theorem)**.** *Let X be a smooth projective variety and $S+B$ a boundary $\mathbf{Q}$-divisor on X such that the support of $S+B$ is simple normal crossing and $\llcorner S+B\lrcorner=S$. Let L be a Cartier divisor on X and D an effective Cartier divisor whose support is contained in* Supp$B$. *Assume that $L\sim_{\mathbf{Q}}K_X+S+B$. Then the natural homomorphisms*

$$H^q(X,\mathcal{O}_X(L))\to H^q(X,\mathcal{O}_X(L+D)),$$

which are induced by the inclusion $\mathcal{O}_X\to\mathcal{O}_X(D)$, are injective for all q.

Before we prove Proposition 2.1, let us recall some results on the Hodge theory.

2.2. Let V be a smooth projective variety and Σ a simple normal crossing divisor on V. Let $\iota: V\setminus\Sigma\to V$ be the natural open immersion. Then $\iota_!\mathbf{C}_{V\setminus\Sigma}$ is quasi-isomorphic to the complex $\Omega^\bullet_V(\log\Sigma)\otimes\mathcal{O}_V(-\Sigma)$. By this quasi-isomorphism, we can construct the following spectral sequence

$$E_1^{pq}=H^q(V,\Omega^p_V(\log\Sigma)\otimes\mathcal{O}_V(-\Sigma))$$
$$\Rightarrow H_c^{p+q}(V\setminus\Sigma,\mathbf{C}).$$

By the Serre duality, the right hand side

$$H^q(V,\Omega^p_V(\log\Sigma)\otimes\mathcal{O}_V(-\Sigma))$$

is dual to $H^{n-q}(V,\Omega^{n-p}_V(\log\Sigma))$, where $n=\dim V$. By the Poincaré duality, $H_c^{p+q}(V\setminus\Sigma,\mathbf{C})$ is dual to $H^{2n-(p+q)}(V\setminus\Sigma,\mathbf{C})$. Therefore,

$$\dim H_c^k(V\setminus\Sigma,\mathbf{C})$$
$$=\sum_{p+q=k}\dim H^q(V,\Omega^p_V(\log\Sigma)\otimes\mathcal{O}_V(-\Sigma))$$

by Deligne (cf. [D, Corollaire (3.2.13) (ii)]). Thus, the above spectral sequence degenerates at E_1. We will use this E_1-degeneration in the proof of Proposition 2.1. By the above E_1-degeneration, we obtain

$$H_c^k(V\setminus\Sigma,\mathbf{C})\simeq\bigoplus_{p+q=k}H^q(V,\Omega^p_V(\log\Sigma)\otimes\mathcal{O}_V(-\Sigma)).$$

In particular, the natural inclusion $\iota_!\mathbf{C}_{V\setminus\Sigma}\subset\mathcal{O}_V(-\Sigma)$ induces surjections

$$H_c^p(V\setminus\Sigma,\mathbf{C})\simeq H^p(V,\iota_!\mathbf{C}_{V\setminus\Sigma})\to H^p(V,\mathcal{O}_V(-\Sigma))$$

for all p.

Proof of Proposition 2.1. We put $\mathcal{L}=\mathcal{O}_X(L-K_X-S)$. Let ν be the smallest positive integer such that $\nu L\sim\nu(K_X+S+B)$. In particular, νB is an integral Weil divisor. We take the ν-fold cyclic cover $\pi': Y'=\mathrm{Spec}_X\bigoplus_{i=0}^{\nu-1}\mathcal{L}^{-i}\to X$ associated to the section $\nu B\in|\mathcal{L}^\nu|$. More precisely, let $s\in H^0(X,\mathcal{L}^\nu)$ be a section whose zero divisor is νB. Then the dual of $s:\mathcal{O}_X\to\mathcal{L}^\nu$ defines an $\mathcal{O}_X$-algebra structure on $\bigoplus_{i=0}^{\nu-1}\mathcal{L}^{-i}$. Let $Y\to Y'$ be the normalization and $\pi: Y\to X$ the composition morphism. For details, see [EV, 3.5. Cyclic covers]. We can take a finite cover $\varphi: V\to Y$ such that V is smooth and T is a simple normal crossing divisor on V, where $\psi=\pi\circ\varphi$ and $T=\psi^*S$, by Kawamata's covering trick (cf. [EV, 3.17. Lemma]). Let $\iota' : Y \setminus \pi^*S \to Y$ be the natural open immersion and U the smooth locus of Y. We denote the natural open immersion $U \to Y$ by j. We put $\widetilde{\Omega}^p_Y(\log(\pi^*S)) = j_*\Omega^p_U(\log(\pi^*S))$ for any p. Then it can be checked easily that

$$\iota'_!\mathbf{C}_{Y\setminus\pi^*S} \xrightarrow{qis} \widetilde{\Omega}^\bullet_Y(\log(\pi^*S)) \otimes \mathcal{O}_Y(-\pi^*S)$$

is a direct summand of

$$\varphi_*(\iota_!\mathbf{C}_{V\setminus T}) \xrightarrow{qis} \varphi_*(\Omega^\bullet_V(\log T) \otimes \mathcal{O}_V(-T)),$$

where *qis* means a quasi-isomorphism. On the other hand, we can decompose $\pi_*(\widetilde{\Omega}^\bullet_Y(\log(\pi^*S)) \otimes \mathcal{O}_Y(-\pi^*S))$ and $\pi_*(\iota'_!\mathbf{C}_{Y\setminus\pi^*S})$ into eigen components of the Galois action of $\pi : Y \to X$. We write these decompositions as follows:

$$\begin{aligned}\pi_*(\iota'_!\mathbf{C}_{Y\setminus\pi^*S}) &= \bigoplus_{i=0}^{\nu-1}\mathcal{C}_i \\ &\subset \bigoplus_{i=0}^{\nu-1}\mathcal{L}^{-i}(\llcorner iB \lrcorner - S) = \pi_*\mathcal{O}_Y(-\pi^*S),\end{aligned}$$

where $\mathcal{C}_i \subset \mathcal{L}^{-i}(\llcorner iB \lrcorner - S)$ for every i. We put $\mathcal{C} = \mathcal{C}_1$. Then we have that

$$\mathcal{C} \xrightarrow{qis} \Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)$$

is a direct summand of

$$\psi_*(\iota_!\mathbf{C}_{V\setminus T}) \xrightarrow{qis} \psi_*(\Omega^\bullet_V(\log T) \otimes \mathcal{O}_V(-T)).$$

The E_1-degeneration of the spectral sequence

$$\begin{aligned}E_1^{pq} &= H^q(V, \Omega^p_V(\log T) \otimes \mathcal{O}_V(-T)) \\ &\Rightarrow \mathbf{H}^{p+q}(V, \Omega^\bullet_V(\log T) \otimes \mathcal{O}_V(-T)) \\ &\simeq H^{p+q}(V, \iota_!C_{V\setminus T})\end{aligned}$$

implies the E_1-degeneration of

$$\begin{aligned}E_1^{pq} &= H^q(X, \Omega^p_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) \\ &\Rightarrow \mathbf{H}^{p+q}(X, \Omega^\bullet_X(\log(S+B)) \otimes \mathcal{L}^{-1}(-S)) \\ &\simeq H^{p+q}(X, \mathcal{C}).\end{aligned}$$

Therefore, the inclusion $\mathcal{C} \subset \mathcal{L}^{-1}(-S)$ induces surjections

$$H^p(X, \mathcal{C}) \to H^p(X, \mathcal{L}^{-1}(-S)).$$

The following arguments are the same as those in [KM]. We describe them for readers' convenience.

We can check the following simple property by seeing the monodromy action of the Galois group of $\pi : Y \to X$ on $\mathcal{C}$ around $\mathrm{Supp}B$.

Corollary 2.3 (cf. [KM, Corollary 2.54]). *Let $U \subset X$ be a connected open set such that $U \cap \mathrm{Supp}B \neq \emptyset$. Then $H^0(U, \mathcal{C}|_U) = 0$.*

This property is utilized via the following fact. The proof is obvious.

Lemma 2.4 (cf. [KM, Lemma 2.55]). *Let F be a sheaf of Abelian groups on a topological space X and $F_1, F_2 \subset F$ subsheaves. Let $Z \subset X$ be a closed subset. Assume that*
(1) $F_2|_{X\setminus Z} = F|_{X\setminus Z}$, and
(2) if U is connected, open and $U \cap Z \neq \emptyset$, then $H^0(U, F_1|U) = 0$.
Then F_1 is a subsheaf of F_2.

As a corollary, we obtain:

Corollary 2.5 (cf. [KM, Corollary 2.56]). *Let $M \subset \mathcal{L}^{-1}(-S)$ be a subsheaf such that $M|_{X\setminus\mathrm{Supp}B} = \mathcal{L}^{-1}(-S)|_{X\setminus\mathrm{Supp}B}$. Then the injection*

$$\mathcal{C} \to \mathcal{L}^{-1}(-S)$$

factors as

$$\mathcal{C} \to M \to \mathcal{L}^{-1}(-S).$$

Therefore,

$$H^i(X, M) \to H^i(X, \mathcal{L}^{-1}(-S))$$

is surjective for every i.

Proof. The first part is clear from Corollary 2.3 and Lemma 2.4. This implies that we have maps

$$H^i(X, \mathcal{C}) \to H^i(X, M) \to H^i(X, \mathcal{L}^{-1}(-S)).$$

As we saw above, the composition is surjective. Hence so is the map on the right. □

Therefore, we obtain that

$$H^q(X, \mathcal{L}^{-1}(-S-D)) \to H^q(X, \mathcal{L}^{-1}(-S))$$

is surjective for every q. By the Serre duality, we obtain

$$\begin{aligned}&H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S)) \\ &\quad \to H^q(X, \mathcal{O}_X(K_X) \otimes \mathcal{L}(S+D))\end{aligned}$$

is injective for every q. This means that

$$H^q(X, \mathcal{O}_X(L)) \to H^q(X, \mathcal{O}_X(L+D))$$

is injective for every q.

3. Proof of the main theorem. In this section, we prove Theorem 1.1. First, we prove a generalization of Kollár's injectivity theorem (cf. [A, Theorem 3.1]). It is a straightforward consequence of Proposition 2.1 and will produce the desired torsion-free and vanishing theorems.

Theorem 3.1 (Injectivity theorem). *Let X be a smooth projective variety and Δ a boundary* **Q**-

divisor such that SuppΔ *is simple normal crossing. Let L be a Cartier divisor on X and D an effective Cartier divisor that contains no lc centers of (X,Δ). Assume the following conditions.*

(i) *$L \sim_{\mathbf{Q}} K_X + \Delta + H$,*

(ii) *H is a semi-ample $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor, and*

(iii) *$tH \sim_{\mathbf{Q}} D + D'$ for some positive rational number t, where D' is an effective $\mathbf{Q}$-Cartier $\mathbf{Q}$-divisor that contains no lc centers of (X,Δ).*

Then the homomorphisms

$$H^q(X,\mathcal{O}_X(L)) \to H^q(X,\mathcal{O}_X(L+D)),$$

which are induced by the natural inclusion $\mathcal{O}_X \to \mathcal{O}_X(D)$, are injective for all q.

Proof. We put $S = \llcorner \Delta \lrcorner$ and $B = \{\Delta\}$. We can take a resolution $f : Y \to X$ such that f is an isomorphism outside Supp$(D + D' + B)$, and that the union of the support of $f^*(S + B + D + D')$ and the exceptional locus of f has a simple normal crossing support on Y. Let B' be the strict transform of B on Y. We write $K_Y + S' + B' = f^*(K_X + S + B) + E$, where S' is the strict transform of S, and E is f-exceptional. It is easy to see that $E_+ = \ulcorner E \urcorner \geq 0$. We put $L' = f^*L + E_+$ and $E_- = E_+ - E \geq 0$. We note that E_+ is Cartier and E_- is an effective $\mathbf{Q}$-Cartier divisor with $\llcorner E_- \lrcorner = 0$. Since f^*H is semi-ample, we can write $f^*H \sim_{\mathbf{Q}} aH'$, where $0 < a < 1$ and H' is a general Cartier divisor on Y. We put $B'' = B' + E_- + \frac{\varepsilon}{t} f^*(D + D') + (1-\varepsilon)aH'$ for some $0 < \varepsilon \ll 1$. Then $L' \sim_{\mathbf{Q}} K_Y + S' + B''$. By the construction, it is easy to see that $\llcorner B'' \lrcorner = 0$, the support of $S' + B''$ is simple normal crossing on Y, and Supp$B'' \supset$ Suppf^*D. So, Proposition 2.1 implies that the homomorphisms $H^q(Y,\mathcal{O}_Y(L')) \to H^q(Y,\mathcal{O}_Y(L' + f^*D))$ are injective for all q. By Lemma 3.2 below, $R^q f_*\mathcal{O}_Y(L') = 0$ for all $q > 0$ and it is easy to see that $f_*\mathcal{O}_Y(L') \simeq \mathcal{O}_X(L)$. By the Leray spectral sequence, the homomorphisms $H^q(X,\mathcal{O}_X(L)) \to H^q(X,\mathcal{O}_X(L+D))$ are injective for all q. $\square$

Let us recall the following well-known easy lemma.

Lemma 3.2. *Let V be a smooth projective variety and B a boundary $\mathbf{Q}$-divisor on V such that* Supp$B$ *is simple normal crossing. Let $f : V \to W$ be a birational morphism onto a projective variety W. Assume that f is an isomorphism at the generic point of every lc center of (V,B) and that D is a Cartier divisor on V such that $D - (K_V + B)$ is nef. Then $R^i f_*\mathcal{O}_V(D) = 0$ for every $i > 0$.*

Proof. We use the induction on the number of irreducible components of $\llcorner B \lrcorner$ and on the dimension of V. If $\llcorner B \lrcorner = 0$, then the lemma follows from the Kawamata–Viehweg vanishing theorem (cf. [KM, Corollary 2.68]). Therefore, we can assume that there is an irreducible divisor $S \subset \llcorner B \lrcorner$. We consider the following short exact sequence

$$0 \to \mathcal{O}_V(D-S) \to \mathcal{O}_V(D) \to \mathcal{O}_S(D) \to 0.$$

By induction, we see that $R^i f_*\mathcal{O}_V(D-S) = 0$ and $R^i f_*\mathcal{O}_S(D) = 0$ for every $i > 0$. Thus, we have $R^i f_*\mathcal{O}_V(D) = 0$ for $i > 0$. $\square$

Let us start the proof of the main theorem: Theorem 1.1.

Proof of Theorem 1.1. We begin the proof of (i). We can assume that H is semi-ample by replacing L (resp. H) with $L + f^*A'$ (resp. $H + f^*A'$), where A' is a very ample Cartier divisor on X. Assume that $R^q f_*\mathcal{O}_Y(L)$ has a local section whose support does not contain the image of any (Y,B)-stratum. Then we can find a very ample Cartier divisor A with the following properties.

(a) f^*A contains no lc centers of (Y,B), and

(b) $R^q f_*\mathcal{O}_Y(L) \to R^q f_*\mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)$ is not injective.

We can assume that $H - f^*A$ is semi-ample by replacing L (resp. H) with $L + f^*A$ (resp. $H + f^*A$). If necessary, we replace L (resp. H) with $L + f^*A''$ (resp. $H + f^*A''$), where A'' is a very ample Cartier divisor on X. Then, we have

$$H^0(X, R^q f_*\mathcal{O}_Y(L)) \simeq H^q(Y,\mathcal{O}_Y(L))$$

and

$$H^0(X, R^q f_*\mathcal{O}_Y(L) \otimes \mathcal{O}_X(A)) \simeq H^q(Y,\mathcal{O}_Y(L + f^*A)).$$

We see that

$$H^0(X, R^q f_*\mathcal{O}_Y(L)) \to H^0(X, R^q f_*\mathcal{O}_Y(L) \otimes \mathcal{O}_X(A))$$

is not injective by (b) if A'' is sufficiently ample. So,

$$H^q(Y,\mathcal{O}_Y(L)) \to H^q(Y,\mathcal{O}_Y(L + f^*A))$$

is not injective. It contradicts Theorem 3.1. This completes the proof of (i).

Let us start the proof of (ii). We take a general member $A \in |mH'|$, where m is a sufficiently divisible positive integer, such that $A' = f^*A$ and $R^q f_*\mathcal{O}_Y(L + A')$ is Γ-acyclic for every q. By (i), we have the following short exact sequences,

$$0 \to R^q f_*\mathcal{O}_Y(L) \to R^q f_*\mathcal{O}_Y(L + A') \to R^q f_*\mathcal{O}_{A'}(L + A') \to 0.$$

for all q. Note that $R^q f_*\mathcal{O}_{A'}(L+A')$ is Γ-acyclic by the induction on $\dim X$ and $R^q f_*\mathcal{O}_Y(L+A')$ is also Γ-acyclic by the above assumption. We consider the spectral sequences

$$E_2^{pq} = H^p(X, R^q f_*\mathcal{O}_Y(L)) \to H^{p+q}(Y, \mathcal{O}_Y(L)),$$

and

$$\overline{E}_2^{pq} = H^p(X, R^q f_*\mathcal{O}_Y(L+A')) \to H^{p+q}(Y, \mathcal{O}_Y(L+A')).$$

Thus, $E_2^{pq} = 0$ for $p \geq 2$ in the following commutative diagram of spectral sequences.

$$\begin{array}{ccc} E_2^{pq} & \Longrightarrow & H^{p+q}(Y, \mathcal{O}_Y(L)) \\ \varphi^{pq}\big\downarrow & & \big\downarrow\varphi^{p+q} \\ \overline{E}_2^{pq} & \Longrightarrow & H^{p+q}(Y, \mathcal{O}_Y(L+A')). \end{array}$$

We note that φ^{1+q} is injective by Theorem 3.1. We have $E_2^{1q} \to H^{1+q}(Y, \mathcal{O}_Y(L))$ is injective by the fact that $E_2^{pq} = 0$ for $p \geq 2$. We also have that $\overline{E}_2^{1q} = 0$ by the above assumption. Therefore, we obtain $E_2^{1q} = 0$ since the injection $E_2^{1q} \to H^{1+q}(Y, \mathcal{O}_Y(L+A'))$ factors through $\overline{E}_2^{1q}$. This implies that $H^p(X, R^q f_*\mathcal{O}_Y(L)) = 0$ for every $p > 0$. □

4. Applications. In this final section, we give two applications of Theorem 1.1. The next theorem is powerful enough and seems inaccessible by the classical approaches (cf. Remark 4.2). We recommend the reader to see [F2] for some applications to the log minimal model program for log canonical pairs.

Theorem 4.1 (cf. [A, Theorem 4.4]). *Let X be a normal projective variety and B a boundary $\mathbf{Q}$-divisor on X such that (X, B) is log canonical. Let L be a Cartier divisor on X. Assume that $L-(K_X+B)$ is ample. Let $\{C_i\}$ be any set of lc centers of the pair (X, B). We put $W = \bigcup C_i$ with a reduced scheme structure. Then we have*

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(L)) = 0, \quad H^i(X, \mathcal{O}_X(L)) = 0,$$

and

$$H^i(W, \mathcal{O}_W(L)) = 0$$

for all $i > 0$, where $\mathcal{I}_W$ is the defining ideal sheaf of W on X. In particular, the restriction map

$$H^0(X, \mathcal{O}_X(L)) \to H^0(W, \mathcal{O}_W(L))$$

is surjective. Therefore, if (X, B) has a zero-dimensional lc center, then the linear system $|L|$ is not empty and the base locus of $|L|$ contains no zero-dimensional lc centers of (X, B).

Before the proof of Theorem 4.1, we give a very important remark.

Remark 4.2. In the last sentence in Theorem 4.1, we do not assume that the zero-dimensional lc center is isolated in the non-klt locus of the pair (X, B), neither do we assume that there exists another boundary $\mathbf{Q}$-divisor B' on X such that (X, B') is klt. Therefore, it can not be proved by the traditional arguments depending on the Kawamata–Viehweg–Nadel vanishing theorem. So, Theorem 4.1 is not a technical improvement of the known results.

We begin the proof of Theorem 4.1.

Proof of Theorem 4.1. Let $f: Y \to X$ be a resolution such that $\mathrm{Supp} f_*^{-1}B \cup \mathrm{Exc}(f)$ is a simple normal crossing divisor. We can further assume that $f^{-1}(W)$ is a simple normal crossing divisor on Y. We can write

$$K_Y + B_Y = f^*(K_X + B).$$

Let T be the union of the irreducible components of $B_Y^{=1}$ that are mapped into W by f. We put $A = \ulcorner -(B_Y^{<1}) \urcorner$. Then A is an effective f-exceptional divisor. Thus, we have $f_*\mathcal{O}_Y(A) \simeq \mathcal{O}_X$. On the other hand, it is easy to see that $f_*\mathcal{O}_Y(A-T) \simeq \mathcal{I}_W$, where $\mathcal{I}_W$ is the defining ideal sheaf of W. We note that $f(T) = W$. Since

$$f^*L + A - T - (K_Y + \{B_Y\} + B_Y^{=1} - T) \sim_{\mathbf{Q}} f^*(L - (K_X + B)),$$

and

$$f^*L + A - (K_Y + \{B_Y\} + B_Y^{=1}) \sim_{\mathbf{Q}} f^*(L - (K_X + B)),$$

we have

$$H^i(X, \mathcal{I}_W \otimes \mathcal{O}_X(L)) \simeq H^i(X, f_*\mathcal{O}_Y(A-T) \otimes \mathcal{O}_X(L)) = 0,$$

and

$$H^i(X, \mathcal{O}_X(L)) \simeq H^i(X, f_*\mathcal{O}_Y(A) \otimes \mathcal{O}_X(L)) = 0$$

for all $i > 0$ by Theorem 1.1 (ii). By the following long exact sequence

$$\cdots \to H^i(X, \mathcal{O}_X(L)) \to H^i(W, \mathcal{O}_W(L)) \to H^{i+1}(X, \mathcal{I}_W \otimes \mathcal{O}_X(L)) \to \cdots,$$

we obtain $H^i(W, \mathcal{O}_W(L)) = 0$ for all $i > 0$. □

Remark 4.3. We note that we do not need Ambro's vanishing theorem for embedded normal crossing pairs (cf. [A, Theorem 3.2]) to obtain $H^i(W, \mathcal{O}_W(L)) = 0$ for $i > 0$ in Theorem 4.1.

We close this paper with the Kodaira vanishing theorem for log canonical pairs, which was not explicitly stated in [A]. For a more general result containing the Kawamata–Viehweg vanishing theorem, see [F1, Theorem 2.48].

Theorem 4.4 (Kodaira vanishing theorem for lc pairs)**.** *Let X be a normal projective variety and B a boundary $\mathbf{Q}$-divisor on X such that (X, B) is log canonical. Let L be a $\mathbf{Q}$-Cartier Weil divisor on X such that $L-(K_X+B)$ is ample. Then $H^q(X, \mathcal{O}_X(L))=0$ for every $q>0$.*

Proof. Let $f: Y \to X$ be a resolution of (X, B) such that $K_Y = f^*(K_X+B)+\sum_i a_iE_i$ with $a_i \geq -1$ for every i and $\mathrm{Supp}\sum E_i$ is simple normal crossing. We can assume that $\sum_i E_i \cup \mathrm{Supp} f^*L$ is a simple normal crossing divisor on Y. We put $E=\sum_i a_iE_i$ and $F=\sum_{a_j=-1}(1-b_j)E_j$, where $b_j=\mathrm{mult}_{E_j}\{f^*L\}$. We note that $A=L-(K_X+B)$ is ample by the assumption. So, we have $f^*A = f^*L - f^*(K_X+B) = \ulcorner f^*L+E+F\urcorner - (K_Y+F+\{-(f^*L+E+F)\})$. We can easily check that $f_*\mathcal{O}_Y(\ulcorner f^*L+E+F\urcorner) \simeq \mathcal{O}_X(L)$ and that $F+\{-(f^*L+E+F)\}$ has a simple normal crossing support and is a boundary $\mathbf{Q}$-divisor on Y. By Theorem 1.1 (ii), we obtain that $\mathcal{O}_X(L)$ is Γ-acyclic. Thus, we have $H^q(X, \mathcal{O}_X(L))=0$ for every $q>0$. $\square$

The reader can find more advanced topics and many other applications in [F1–F8]. This paper is a gentle introduction to Chapter 2 in [F1]. We recommend the reader to see [F1].

Added in the proof: In the proof of Theorem 1.1(i), in order to take a very ample Cartier divisor A with the properties (a) and (b), we sometimes have to replace L (resp. B) with $L-B'$ (resp. $B-B'$) after taking suitable blow-ups of Y, where B' is the union of suitable components of $\llcorner B\lrcorner$. We will discuss the details in [F8].

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