Congruences for coefficients of Drinfeld modular forms for $\Gamma_0(T)$

By SoYoung Choi

Department of Mathematics Education, Dongguk University,
Gyeongju 780-714, Republic of Korea

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Abstract: We find congruences for the t-expansion coefficients of Drinfeld modular forms for $\Gamma_0(T)$. We determine all the linear relations among the initial t-expansion coefficients of Drinfeld modular forms for $\Gamma_0(T)$.

Key words: Drinfeld modular forms; congruences.

1. Introduction. By studying the action of the Hecke operators, López [11] proved the existence of congruences for the coefficients of a Drinfeld modular form, the discriminant function Δ. Gallardo and López [6] showed that there exist congruences for the s-expansion coefficients of the Eisenstein series of weight $q^k - 1$ for any positive integer k. By using the Residue Theorem, we [1] found divisibility properties for the t-expansion coefficients of Drinfeld modular forms for $GL_2(\mathbf{F}_q[T])$. As a consequence we obtained congruence relations of t-expansion coefficients of Drinfeld modular forms for $GL_2(\mathbf{F}_q[T])$.

In the classical case, the study of the arithmetic properties of modular forms with algebraic integral coefficients is a rich and interesting branch in the theory of modular forms. (see [12] for many results and applications in this direction). In [3] Choie, Kohnen and Ono showed p-divisibility properties for Fourier coefficients of a modular form on $SL_2(\mathbf{Z})$ and determined all the linear relations among the initial Fourier coefficients. Here, the classical discriminant function played an important role in the study of congruence properties among the Fourier coefficients of modular forms on $SL_2(Z)$. El-Guindy [4] generalized methods in [3] by finding an analogy of the classical discriminant function and obtained similar results for cusp forms of level $N \in \{2, 3, 5, 7, 13\}$ (These are all the primes for which $\Gamma_0(N)$ is a genus zero group). Precisely, he showed that $a_f(p^e) \equiv 0 \bmod p$ under a certain condition when $N = 2$. For other cases, that is, $N \in \{3, 5, 7, 13\}$, he found congruence properties of linear combination of Fourier coefficients of a cusp form on $\Gamma_0(N)$. Here we note that $a_f(p^e)$ is the p^e-th coefficient of a modular form f and p is a prime.

These results motivate the research of divisibility properties for the t-expansion coefficients of Drinfeld modular forms. In this paper when q is not 2 we generalize the results in [1] to the Hecke congruence subgroup $\Gamma_0(T)$ of $GL_2(\mathbf{F}_q[T])$ (see Theorem 3.4, Corollary 3.5 and Theorem 4.1) by producing an analogy Δ_T of the discriminant function Δ. Specially, we find $(T^q - T)$-divisibility properties for the t-expansion coefficients of Drinfeld modular forms for $\Gamma_0(T)$ (Corollary 3.5). Moreover, since there is an automorphism of A which maps the polynomial T to $T - a$ and the congruences derived in this paper are invariant under this group action, we can obtain the same congruences for Hecke congruence subgroup $\Gamma_0(T - a)$ where $a \in \mathbf{F}_q$.

2. Preliminaries. Let K be the rational function field $\mathbf{F}_q(T)$ over the finite field $\mathbf{F}_q$ of characteristic p and $A = \mathbf{F}_q[T]$. Let K_∞ be the completion of K at $\infty = (1/T)$ and C be the completion of an algebraic closure of K_∞. Let deg be the unique valuation of K such that $\deg(f)$ is -1 times the usual degree of every polynomial $f \in A$. Normalize the absolute value $|\cdot|$ on K corresponding to deg so that $|T| = q$. There is a unique extension of $|\cdot|$ to C, which will be labelled by the same symbol.

Let $\Omega = C - K_\infty$ be Drinfeld's upper half plane. Then the group $GL_2(A)$ acts on Ω in the following way: if $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in GL_2(A)$ and $z \in \Omega$, then

$$\gamma z = \frac{az + b}{cz + d}.$$

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Let Q be a monic element of A. Consider the following Hecke congruence subgroup of $GL_2(A)$:

$$\Gamma_0(Q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(A) \mid c \equiv 0 \mod Q \right\}.$$

For each group $\Gamma_0(Q)$, the rigid analytic space $\Gamma_0(Q)\backslash\Omega$ is uniquely endowed with the structure of a smooth affine algebraic curve over C. We let $\overline{\Gamma_0(Q)\backslash\Omega}$ denote its smooth projective compactification. For a 1-form ω on $\overline{\Gamma_0(Q)\backslash\Omega}$, the Residue Theorem [10, Theorem 7.14.2] says the following

$$\sum_{p \in \overline{\Gamma_0(Q)\backslash\Omega}} \mathrm{Res}_p \omega = 0.$$

A cusp of $\overline{\Gamma_0(Q)\backslash\Omega}$ is a point of $\overline{\Gamma_0(Q)\backslash\Omega} - \Gamma_0(Q)\backslash\Omega$. At the level of C-valued points, we have $\overline{\Gamma_0(Q)\backslash\Omega} = \Gamma_0(Q)\backslash(\Omega \cup \mathbf{P}^1(K))$.

Let $L = \widetilde{\pi}A$ be the rank one A-lattice in C corresponding to the Carlitz module,

$$\rho_T = TX + X^q.$$

Let e_L be the exponential function associated to L (see [8, p. 672]). We define

$$t = t(z) := \frac{1}{e_L(\widetilde{\pi}z)} \text{ and } s = t^{q-1}.$$

A Drinfeld modular form (respectively, a meromorphic Drinfeld modular form) for $\Gamma_0(Q)$ of weight k and type m (where $k \geq 0$ is an integer and m is a class in $\mathbf{Z}/(q-1)$) is a holomorphic (respectively, meromorphic) function $f : \Omega \to C$ that satisfies:

(i) $f(\gamma z) = (\det \gamma)^{-m}(cz+d)^k f(z)$ for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(Q)$,

(ii) f is holomorphic (respectively, meromorphic) at the cusps of $\Gamma_0(Q)$.

If f is a meromorphic Drinfeld modular form for $\Gamma_0(Q)$ of weight k and type m, then the t-expansion of f is of the form

$$f = \sum_i a_f((q-1)i+m)t^{(q-1)i+m}.$$

Here and in what follows, we denote the unique nonnegative integer less than $q-1$ representing the congruence m by the same symbol. Let $M_k(\Gamma_0(Q))$ be the C-vector space of Drinfeld modular forms for $\Gamma_0(Q)$ of weight k and type 0. For convenience we call elements of $M_k(\Gamma_0(Q))$ simply Drinfeld modular forms for $\Gamma_0(Q)$ of weight k.

For $z \in \Omega$, we let $\Lambda_z = Az + A$ be the associated rank 2 A-lattice in C. Then it induces a Drinfeld module ϕ^z of rank 2 determined by

$$\phi^z_T = TX + g(z)X^q + \Delta(z)X^{q^2}.$$

The functions g and Δ in z are Drinfeld modular forms for $GL_2(A)$ of weights $q-1$ and q^2-1, respectively. We normalize $g(z)$ and $\Delta(z)$ as follows:

$$g_{\mathrm{new}}(z) = \widetilde{\pi}^{1-q}g(z) \quad \text{and} \quad \Delta_{\mathrm{new}}(z) = \widetilde{\pi}^{1-q^2}\Delta(z).$$

Hereafter we write $g(z)$ and $\Delta(z)$ for $g_{\mathrm{new}}(z)$ and $\Delta_{\mathrm{new}}(z)$, respectively (see [8, sect. 6]).

Let $A_+ = \{a \in A \mid a \text{ is monic}\}$, and $E = E(z) := \sum_{a \in A_+} at(az)$ (see [8, p. 686]). Then we have $(d\Delta/dz)/(\widetilde{\pi}\Delta) = E$.

Lemma 2.1. *The first few coefficients of g and E are given as follows*:

(i) $g(z) = 1 - (T^q - T)s - (T^q - T)s^{q^2-q+1} + \cdots$.

(ii) $E = t + t^{(q-1)^2+1} + \cdots$.

Proof. Notice that

$$E/t = 1 + s^{q-1}U_1^{-1} + s^{q^3-q^2+q-1}U_1^{-2} + \cdots$$
(see [8, p. 692, (10.5) Corollary])

and

$$g(z) = 1 - [1](s + s^{q^2-q+1}U_1^{1-q}) + \cdots$$
(see [8, p. 694, (10.11) Corollary])

where $U_1 = 1 - s^{q-1} + [1]s^q$ (see [8, p. 691]) and $[1] = T^q - T$ (see [8, p. 677, (4.2)]). These imply the assertion. □

Let G be any meromorphic Drinfeld modular form of weight 2 and type 1 for $\Gamma_0(T)$. Take $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and let

$$(G|_2^1\gamma)(z) := (\det \gamma)^m(cz+d)^{-k}G(\gamma z) = \sum_{i=i_\gamma}^{\infty} a_0((q-1)i+1)t(z/T)^{(q-1)i+1},$$

which is the $t(z/T)$-expansion of $G(z)$ at the cusp 0. We will call $(q-1)i_\gamma + 1$ the order of the zero of G at the cusp 0 if $a_0((q-1)i_\gamma + 1) \neq 0$. We can consider $\omega := G(z)dz$ as a 1-form on $\overline{\Gamma_0(T)\backslash\Omega}$.

Let

$$\pi : \Omega \to \Gamma_0(T)\backslash\Omega$$

be the quotient map and e_τ be the cardinality of $\Gamma_0(T)_\tau/(\Gamma_0(T)_\tau \cap Z(K))$ for each $\tau \in \Omega$. Here $\Gamma_0(T)_\tau$ is the stabilizer of τ in $\Gamma_0(T)$ and $Z(K)$ is the group of scalar matrices. It is known [9, p. 50] that e_τ is prime to p. We then have the following lemma.

Lemma 2.2. *With the same assumptions and notations as above we have*

$$Res_\infty \omega = \frac{a_G(1)}{\widetilde{\pi}}.$$

Proof. Since $dt^{q-1} = \widetilde{\pi} t^q dz$, we obtain $Res_\infty \omega = a_G(1)/\widetilde{\pi}$. □

3. Congruences of coefficients of Drinfeld modular forms for $\Gamma_0(T)$. Given a prime $P \in A$ and a Drinfeld modular form $f = \sum_{n\geq 0} a_n t^n$ with P-integral coefficients, we define the filtration of f modulo P to be the least weight k for which there exists a Drinfeld modular form f_0 of weight k with t-expansion $\sum_{n\geq 0} b_n t^n$ such that $a_n \equiv b_n \pmod{P}$ for all $n \geq 0$.

For a Drinfeld modular form to have filtration lower than its weight, two a priori unrelated infinite power series must line up when taken modulo a prime -a seemingly unlikely occurrence. In view of this, one might expect such events to have interesting consequences. Indeed, in the classical case, work of Deligne, Swinnerton-Dyer, and Serre suggests a relationship between the supersingular locus and modular forms with lower filtration than weight. Recently, Dobi and *et al.* [5] presented an analogous relationship for Drinfeld modular forms. Indeed, each non-zero Drinfeld modular form f for $GL_2(A)$ with coefficients in A has a unique polynomial $F(f,x) \in A[x]$ such that $f = g^a h^b F(f,j)$ where a, b are integers and j is the Drinfeld modular invariant. Let P be a prime and f have filtration l. Then they showed that $x^c F(f,x) - S_q(P:x)^d \equiv 0 \pmod{P}$ for some integer c, d depending on l. Here $S_q(P:x)$ is the Drinfeld supersingular locus (see [5, p. 2] for the definition).

Thus, congruences for the t-expansion coefficients of Drinfeld modular forms require a research in some sense. The author [1] found certain divisibility properties for t-expansion coefficients of Drinfeld modular forms for $GL_2(\mathbf{F}_q[T])$. In this section we generalize this result for the Hecke congruence subgroup $\Gamma_0(T)$.

From now on we assume that q is not 2.

Proposition 3.1. $dim_C M_{k(q-1)}(\Gamma_0(T)) = k+1$ $(k \geq 1)$.

Proof. See [7, p. 93, 6.5 Table]. □

Let $\Delta_T := (g(Tz) - g(z))/(T^q - T) \in M_{q-1}(\Gamma_0(T))$. Then Δ_T has a s-expansion at the cusp ∞ of the form

$$\Delta_T = s - s^q + \cdots$$

because

$$t(Tz) = t^q + \cdots \in A[[t]]$$

(see [8, p. 678 and p. 682, (6.2)])

and

$$g(Tz) - g(z) = (1 - (T^q - T)s^q + \cdots) - (1 - (T^q - T)s - (T^q - T)s^{q^2-q+1} + \cdots).$$

Let h be the Poincaré series $P_{q+1,1}$ of weight $q+1$ and type 1 (see [8, p. 681] for the definition). Let $j_{(T)} = j_{(T)}(z) := h(z)/h(Tz) = 1/s + \cdots$ be the Drinfeld modular function for $\Gamma_0(T)$ (see [8, p. 692, (10.4) Corollary]). Then the latter is holomorphic on $\Omega \cup \{0\}$ and has a simple pole at ∞ (see [2, the proof of Proposition 2.1.]).

Proposition 3.2. *The set* $\{ j_{(T)}^i \Delta_T^k \mid 0 \leq i \leq k\}$ *is a basis of* $M_{k(q-1)}(\Gamma_0(T))$.

Proof. Since $j_{(T)}^i \Delta_T^k = s^{k-i} + \cdots$ for every $0 \leq i \leq k$, these forms are linearly independent over C. Since the $j_{(T)}^i \Delta_T^k$ $(0 \leq i \leq k)$ are holomorphic on $\Omega \cup \{\infty, 0\}$ and hence contained in $M_{k(q-1)}(\Gamma_0(T))$, they form a basis of $M_{k(q-1)}(\Gamma_0(T))$. □

Since $\{ j_{(T)} \Delta_T, \Delta_T\}$ is a basis of $M_{q-1}(\Gamma_0(T))$, there are $a, b \in C$ such that $g = a\Delta_T + b j_{(T)} \Delta_T$. This means that g/Δ_T is holomorphic on $\Omega \cup \{0\}$ and has a simple pole at ∞. Hereafter we write j_T for g/Δ_T.

Proposition 3.3. *The modular from* Δ_T *has no zero on* $\Omega \cup \{0\}$.

Proof. Since g has no zero at the cusp 0 and g/Δ_T is holomorphic at the cusp 0, Δ_T has no zero at the cusp 0.

Since $\Delta(Tz) = j_{(T)} \Delta_T(z)^{q+1} + a\Delta_T(z)^{q+1}$ for some $a \in C$ and $\Delta(Tz)$ has no zero on Ω, Δ_T has no zero on Ω. □

The following theorem is motivated by the classical results of p-divisibility properties for Fourier coefficients of modular forms on $SL_2(\mathbf{Z})$. These classical results play an important role in the p-adic theory of modular forms. Unfortunately the author could not find a similar role for Theorem 3.4 in the function field case. This requires further research.

Theorem 3.4 gives mysterious congruence properties of Drinfeld modular forms.

Theorem 3.4. *Let* f *be a Drinfeld modular form for* $\Gamma_0(T)$ *of weight* $k(q-1)$*. Let* fE *have the following* t*-expansion at* ∞

$$fE = \sum_{n=0}^{\infty} a_{fE}((q-1)n+1)t^{(q-1)n+1} \in A[[t]].$$

For every integer $b > 0$ such that $k < p^b$ we have that

$$a_{fE}((q-1)p^b+1) \equiv 0 \ mod \ (T^q - T).$$

Proof. For any non-negative integer a the function $j_T^a hf/\Delta_T^{k+1}$ is a meromorphic Drinfeld modular form for $\Gamma_0(T)$ of weight 2 and type 1, and holomorphic on Ω. Take $\gamma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Then the $t(z/T)$-expansion of $h(z)$ at the cusp 0 is of the form

$$h(z) = -t\left(\frac{z}{T}\right)^q + \cdots.$$

Hence

$$((j_T^a hf/\Delta_T^{k+1})|_2^1\gamma)(z) = at\left(\frac{z}{T}\right)^q + \cdots \quad \text{for some } a \in C$$

which means it has a zero of order at least q at 0.

By the Residue Theorem on $\overline{\Gamma_0(T)\backslash\Omega}$ and Lemma 2.2, the coefficient of t in

$$\frac{j_T^a hf}{\Delta_T^{k+1}}$$

vanishes.

Let $a \geq 0$, $b > 0$ be integers such that $a + k + 1 = p^b$. The facts that $g \equiv 1 \bmod (T^q - T)$ (see [8, p. 684, claim *(ii)* of (6.9) and (6.12)]) and $E \equiv -h \bmod (T^q - T)$ (see [8, p. 687, (8.5) and (9.1) Theorem]) show that the coefficient of t in

$$\frac{j_T^a hf}{\Delta_T^{k+1}} \equiv \frac{-fE}{\Delta_T^{p^b}} \equiv -(s^{-p^b} + s^{p^b(q-2)} + \cdots)fE \mod (T^q - T)$$

is zero mod $(T^q - T)$, where $\cdots$ means "higher terms in t". This implies the assertion. □

Corollary 3.5. *Let $f \in M_{k(q-1)}(\Gamma_0(T))$ have the following t-expansion at ∞*

$$f = \sum_{n=0}^{\infty} a_f((q-1)n)t^{(q-1)n} \in A[[t]].$$

If $q > p+1$ and $p > k$ then we have that $a_f((q-1)p) \equiv 0$ mod $(T^q - T)$.

Proof. Take $b = 1$ in Theorem 3.4. Noticing that the condition $q > p+1$ implies $a_{fE}((q-1)p+1) = a_f((q-1)p)$ we obtain the assertion. □

4. Linear relations among Drinfeld modular form coefficients. In this section, we give all the linear relations among the initial t-expansion coefficients of Drinfeld modular forms in $M_{k(q-1)}(\Gamma_0(T))$.

For any integer $N \geq 0$, we let $L_{k(q-1),N}(\Gamma_0(T)) :=$

$$\left\{(c_0, c_1, \ldots, c_{k+1+N}) \in C^{k+N+2} \;\middle|\; \sum_{i=0}^{k+N+1} c_i a_f((q-1)i) = 0 \quad \forall f = \sum_{n=0}^{\infty} a_f((q-1)n)t^{(q-1)n} \in M_{k(q-1)}(\Gamma_0(T))\right\}$$

be the space of linear relations satisfied by the first $k+N+2$ t-expansion coefficients of all the forms $f \in M_{k(q-1)}(\Gamma_0(T))$.

To state our result, for each Drinfeld modular form $u \in M_{N(q-1)}(\Gamma_0(T))$ define the elements $b(k, N, u; i)$ of C by

$$\frac{hu}{\Delta_T^{k+N+1}} = \sum_{i=0}^{k+N+1} b(k,N,u;i)t^{-(q-1)i+1} + \sum_{i=1}^{\infty} c(k,N,u;i)t^{(q-1)i+1}.$$

In this notation, we have the following

Theorem 4.1. *The map $\phi_{k,N}$: $M_{N(q-1)}(\Gamma_0(T)) \to L_{k(q-1),N}(\Gamma_0(T))$ defined by*

$$\phi_{k,N}(u) = (b(k,N,u;0), b(k,N,u;1), \cdots, b(k,N,u;k+N+1))$$

provides a linear isomorphism from $M_{N(q-1)}(\Gamma_0(T))$ onto $L_{k(q-1),N}(\Gamma_0(T))$.

Proof. Let

$$u \in M_{N(q-1)} \quad \text{and} \quad f = \sum_{i=0}^{\infty} a_f((q-1)i)t^{(q-1)i} \in M_{k(q-1)}.$$

The meromorphic Drinfeld modular form hfu/Δ_T^{k+N+1} is holomorphic on Ω and has a zero of order at least q at 0. By the Residue Theorem on $\overline{\Gamma_0(T)\backslash\Omega}$ and Lemma 2.2, the coefficient $\sum_{i=0}^{k+N+1} b(k, N, u; i)a_f((q-1)i)$ of t in

$$\frac{hfu}{\Delta_T^{k+N+1}}$$

is zero. Therefore the map $\phi_{k,N}$ is well-defined. Clearly, $\phi_{k,N}$ is linear. Suppose that $\phi_{k,N}(u) = 0$. This assumption implies that $\frac{hfu}{\Delta_T^{k+N+1}}$ is double cuspidal. Since the genus of $\overline{\Gamma_0(T)\backslash\Omega}$ is zero, $\frac{hfu}{\Delta_T^{k+N+1}}$ is the zero function which says that u is the zero function. Thus $\phi_{k,N}$ is injective.

Since the $k+1$ functionals $\{a_f(0), a_f(1), \cdots,$

$a_f(k)\}$ form a basis for the dual space $(M_{k(q-1)})^*$ (see the proof of Proposition 3.2), we conclude that $\dim_C L_{k(q-1),N}(\Gamma_0(T)) = N+1 = \dim M_{N(q-1)}$ so $\phi_{k,N}$ is an isomorphism. $\square$

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