Generic Torelli theorem for quintic-mirror family

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(Communicated by Masaki KASHIWARA, M.J.A., Sept. 12, 2008)

Abstract: This article is a geometric application of polarized logarithmic Hodge theory of Kazuya Kato and Sampei Usui. We prove generic Torelli theorem for the well-known quintic-mirror family in two ways by using different logarithmic points at the boundary of the fine moduli of polarized logarithmic Hodge structures.

Key words: Quintic-mirror family; logarithmic Hodge theory; moduli; Torelli theorem.

1. Review of quntic-mirror family. We review the construction of the mirror family of the pencil joining the quintic hypersurface of Fermat type and the union of the coordinate hyperplanes in a 4-dimensional complex projective space after [M1].

Let $\psi \in \mathbf{P}^1$, and let

(1) $$Q_\psi = \left\{\sum_{j=1}^5 x_j^5 - 5\psi \prod_{j=1}^5 x_j = 0\right\} \subset \mathbf{P}^4.$$

Let $\mu_5 = \{\alpha \in \mathbf{C} \mid \alpha^5 = 1\}$.

The singular members of the pencil (1) are listed as follows:

(2) Over each point $\psi \in \mu_5 \subset \mathbf{C} \subset \mathbf{P}^1$, Q_ψ contains $5^3 = 125$ ordinary double points $(\alpha_1, \dots, \alpha_5) \in (\mu_5)^5/\mu_5 \subset \mathbf{P}^4$ with $\psi\alpha_1 \cdots \alpha_5 = 1$.

(3) Over $\infty \in \mathbf{P}^1$, $Q_\infty = \bigcup_j \{x_j = 0\}$. The union of coordinate hyperplanes.

Let $G = \{\alpha \in (\mu_5)^5 \mid \alpha_1 \cdots \alpha_5 = 1\}/\mu_5$. By multiplication, this becomes a group which acts on $\mathbf{P}^4$ coordinate-wise. This action of G preserves Q_ψ. Taking the quotient Q_ψ/G, the following canonical singularities appear:

(4) For each pair of distinct indices j, k, a compound du Val singularity cA_4 appears as the quotient of the curve $Q_\psi \cap \{x_j = x_k = 0\} \setminus (\bigcup_{m \neq j,k} \{x_m = 0\})$.

(5) For each triple of distinct indices j, k, l, the point which is the quotient of the five points $Q_\psi \cap \{x_j = x_k = x_l = 0\}$ belongs to the closure of three curves in (4).

Moreover, we see that holomorphic 3-forms on Q_ψ are G-invariant for every $\psi \in \mathbf{C}$ by adjunction formula.

For $\psi \in \mathbf{C}$, it is known that there is a simultaneous minimal desingularization W_ψ of these quotient singularities, and that holomorphic 3-forms extend to nowhere vanishing forms on W_ψ.

We thus have a pencil $(W_\psi)_{\psi \in \mathbf{P}^1}$ whose singular fibers are listed as follows:

(6) Over each point $\psi \in \mu_5 \subset \mathbf{C} \subset \mathbf{P}^1$, W_ψ has one ordinary double point.

(7) Over $\psi = \infty \in \mathbf{P}^1$, W_ψ is a normal crossing divisor in the total space, whose components are all rational.

The other members W_ψ are smooth with Hodge numbers $h^{p,q} = 1$ for $p + q = 3$.

By the action of $\alpha \in \mu_5$, $(x_1, \dots, x_5) \mapsto (\alpha^{-1}x_1, x_2, \dots, x_5)$, we have $W_{\alpha\psi} \simeq W_\psi$. Let $\lambda = \psi^5$, and let

$$\begin{array}{ccc} (W_\lambda)_\lambda & = & ((W_\psi)_\psi)/\mu_5 \\ \downarrow & & \downarrow \\ (\lambda\text{-plan}) & = & (\psi\text{-plan})/\mu_5. \end{array}$$

This pencil $(W_\lambda)_{\lambda \in \mathbf{P}^1}$ is the mirror of the original pencil (1). (For more details of the above construction, see e.g. [M1].)

2. Review of fine moduli space $\Gamma \backslash D_\Xi$. We review some facts in [KU] that are necessary in the present article.

Let $w = 3$, and $h^{p,q} = 1$ $(p + q = 3, p, q \geq 0)$. Let $H_0 = \bigoplus_{j=1}^4 \mathbf{Z}e_j$, and $\langle e_3, e_1\rangle_0 = \langle e_4, e_2\rangle_0 = 1$. Let D be the corresponding classifying space of polarized Hodge structures, and $\check{D}$ the compact dual.

Let $S =$ (square free positive integers), and let $m \in S$. Define $N_\alpha, N_\beta, N_m \in \mathrm{End}(H_0, \langle\ ,\ \rangle_0)$ as follows:

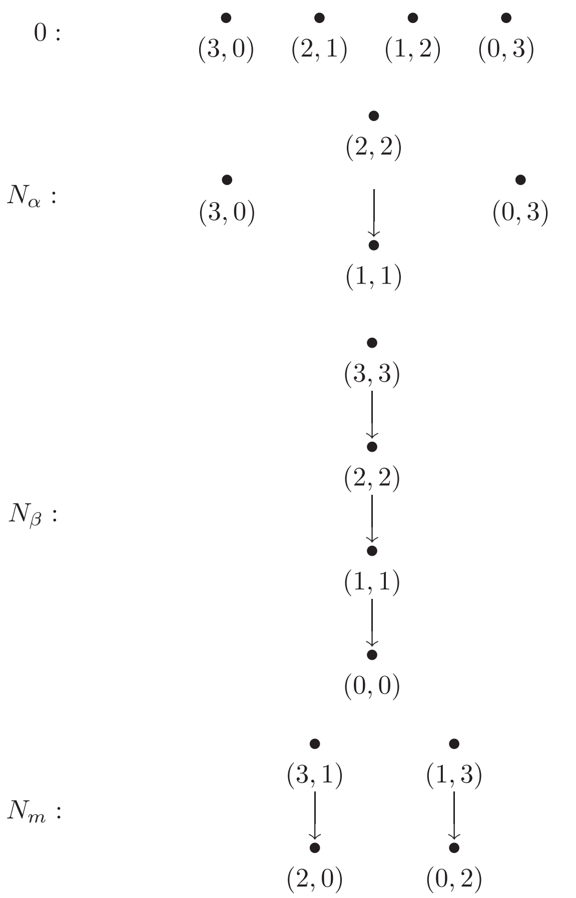
$$N_\alpha(e_3) = e_1, \quad N_\alpha(e_j) = 0 \quad (j \neq 3);$$

2000 Mathematics Subject Classification. Primary 14C30; Secondary 14D07, 32G20.

Dedicated to Professor Phillip A. Griffiths on his seventieth birthday.

$$N_\beta(e_4) = e_3, \quad N_\beta(e_3) = -e_1,$$
$$N_\beta(e_1) = -e_2, \quad N_\beta(e_2) = 0;$$
$$N_m(e_1) = e_3, \quad N_m(e_4) = -me_2,$$
$$N_m(e_2) = N_m(e_3) = 0.$$

Then the respective Hodge diamonds are



Let $\sigma_\alpha = \mathbf{R}_{\geq 0} N_\alpha$, $\sigma_\beta = \mathbf{R}_{\geq 0} N_\beta$, and $\sigma_m = \mathbf{R}_{\geq 0} N_m$ ($m \in S$).

Proposition [KU, §12.3]**.** *Let* $\Xi =$ (*rational nilpotent cones in* $\mathfrak{g}_\mathbf{R}$ *of rank* ≤ 1). *Then* $\Xi = \{\mathrm{Ad}(g)\sigma \mid \sigma = \{0\}, \sigma_\alpha, \sigma_\beta, \sigma_m\ (m \in S), g \in G_\mathbf{Q}\}$. *This is a complete fan, i.e., if there exists* $Z \subset \check{D}$ *such that* (σ, Z) *is a nilpotent orbit, then* $\sigma \in \Xi$.

Here a pair (σ, Z) is a *nilpotent orbit* if $Z = \exp(\mathbf{C}N)F$, $NF^p \subset F^{p-1}$ $(\forall p)$ and $\exp(iyN)F \in D$ $(y \gg 0)$ hold for $\mathbf{R}_{\geq 0}N = \sigma$ and $F \in Z$.

In [KU], the fine moduli space $\Gamma \backslash D_\Xi$ of polarized logarithmic Hodge structures is constructed. We briefly explain this according to the present case. Let Γ be a neat subgroup of $G_\mathbf{Z} := \mathrm{Aut}(H_0, \langle\, , \,\rangle_0)$ of finite index. As a set, $D_\Xi = \{(\sigma, Z) : \text{nilpotent orbit} \mid \sigma \in \Xi,\ Z \subset \check{D}\}$. Let $\sigma \in \Xi$, and let $\Gamma(\sigma) = \Gamma \cap \exp(\sigma)$. If $\sigma = \{0\}$, then $D \simeq \{(\{0\}, F) \mid F \in D\} \subset D_\Xi$. If $\sigma \neq \{0\}$, then $\Gamma(\sigma) \simeq \mathbf{N}$. Let γ be its generator and let $N = \log \gamma$. Define

$$E_\sigma = \left\{ (q, F) \in \mathbf{C} \times \check{D} \;\middle|\; \begin{array}{l} \exp((2\pi i)^{-1} \log(q) N) F \in D \text{ if } q \neq 0, \text{ and} \\ \exp(\mathbf{C}N)F \text{ is a } \sigma\text{-nilpotent orbit if } q = 0 \end{array} \right\},$$

and the map

$$(1)\quad E_\sigma \to \Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma, \quad (q, F) \mapsto \begin{cases} \exp((2\pi i)^{-1}\log(q)N)F \ \bmod\ \Gamma(\sigma)^{\mathrm{gp}} & \text{if } q \neq 0, \\ (\sigma, \exp(\mathbf{C}N)F) \ \bmod\ \Gamma(\sigma)^{\mathrm{gp}} & \text{if } q = 0. \end{cases}$$

Here $\Gamma(\sigma)^{\mathrm{gp}}$ is the subgroup of Γ generated by $\Gamma(\sigma)$. $\mathbf{C} \times \check{D}$ is obviously an analytic manifold. We endow this with the logarithmic structure M associated to the divisor $\{0\} \times \check{D}$. The strong topology of E_σ in $\mathbf{C} \times \check{D}$ is defined as follows: A subset U of E_σ is open if, for any analytic space Y and any analytic morphsm $f : Y \to \mathbf{C} \times \check{D}$ such that $f(Y) \subset E_\sigma$, $f^{-1}(U)$ is open in Y. By (1), the quotient topology, the sheaf $\mathcal{O}$ of local rings over $\mathbf{C}$, and the logarithmic structure M are introduced on $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma$. Introduce the corresponding structures $\mathcal{O}$, M on $\Gamma \backslash D_\Xi$ so that $\Gamma(\sigma)^{\mathrm{gp}} \backslash D_\sigma \to \Gamma \backslash D_\Xi$ $(\sigma \in \Xi)$ are local isomorphisms and form an open covering. Then, the resulting $\Gamma \backslash D_\Xi$ is a "logarithmic manifold", which is nearly a logarithmic analytic space but has "slits" at the boundaries. In fact, in the present case, we have in [KU, §12.3] that $\dim_\mathbf{C} D = 4$, but $\dim_\mathbf{C} \Gamma(\sigma_\alpha)^{\mathrm{gp}} \backslash (D_{\sigma_\alpha} - D) = 2$, $\dim_\mathbf{C} \Gamma(\sigma_\beta)^{\mathrm{gp}} \backslash (D_{\sigma_\beta} - D) = 1$ and $\dim_\mathbf{C} \Gamma(\sigma_m)^{\mathrm{gp}} \backslash (D_{\sigma_m} - D) = 1$.

3. Period map. Let Δ be a unit disc, $\Delta^* = \Delta - \{0\}$, and $\mathfrak{h} \to \Delta^*$, $z \mapsto q = e^{2\pi i z}$, the universal covering. Let $\varphi : \Delta^* \to \langle\gamma\rangle \backslash D$ be a period map, where $\langle\gamma\rangle$ is the monodromy group generated by a unipotent element γ, and $\tilde{\varphi} : \mathfrak{h} \to D$ a lifting. Let $N = \log \gamma$. The map $\exp(-zN)\tilde{\varphi}(z)$ from $\mathfrak{h}$ to D drops down to the map $\psi : \Delta^* \to D$.

Then the nilpotent orbit theorem of Schmid asserts that there exists the limit $\psi(0)$, denoted by F, and that $\tilde{\varphi}(z)$ and $\exp(zN)F$ are closing as $\mathrm{Im}\, z \to \infty$.

Let $\sigma = \mathbf{R}_{\geq 0}N$. In our space $\langle\gamma\rangle \backslash D_\sigma$ we have, moreover, $\exp(zN)F \to ((\sigma, \exp(\sigma_\mathbf{C})F) \bmod \langle\gamma\rangle)$ as $\mathrm{Im}\, z \to \infty$. Hence $\varphi(q) \to ((\sigma, \exp(\sigma_\mathbf{C})F) \bmod \langle\gamma\rangle)$ as $q \to 0$ in $\langle\gamma\rangle \backslash D_\sigma$. (For details, see [KU].)

Fix a point $b \in \mathbf{P}^1 - \{0,1,\infty\}$ on the λ-plane, identify $H^3(W_b, \mathbf{Z}) = H_0$, and let

(1) $$\Gamma = \mathrm{Image}(\pi_1(\mathbf{P}^1 - \{0,1,\infty\}) \to G_{\mathbf{Z}}).$$

This Γ is not neat. In fact, the local monodromy around 0 is of order 5. Let

(2) $$\mathbf{P}^1 - \{0,1,\infty\} \to \Gamma\backslash D$$

be the period map. Since K_{W_λ} is trivial, the differential of (2) is injective everywhere. Endow $\mathbf{P}^1$ with the logarithmic structure associated to the divisor $\{1,\infty\}$. Then, by §1, §2 and [KU], (2) extends to a morphism

(3) $$\begin{aligned} \varphi : \mathbf{P}^1 &\to \Gamma\backslash D_\Xi, \\ 0 &\mapsto (\text{point}) \bmod \Gamma \in \Gamma\backslash D, \\ 1 &\mapsto (\mathrm{Ad}(g)(\sigma_\alpha)\text{-nilpotent orbit} \\ &\qquad \text{for some } g \in G_{\mathbf{Q}}) \bmod \Gamma, \\ \infty &\mapsto (\mathrm{Ad}(g)(\sigma_\beta)\text{-nilpotent orbit} \\ &\qquad \text{for some } g \in G_{\mathbf{Q}}) \bmod \Gamma. \end{aligned}$$

of logarithmic ringed spaces (cf. [KU, 4.3.1 (i)]). The image of the extended period map φ is an analytic curve, which is not affected by the slits of the space $\Gamma\backslash D_\Xi$.

Let $X = \Gamma\backslash D_\Xi$. Let $P_1 = 1, P_\infty = \infty \in \mathbf{P}^1$, and let $Q_1 = \varphi(P_1), Q_\infty = \varphi(P_\infty) \in X$. Then, by the observation of local monodromy and holomorphic 3-form basing on the descriptions in §1 and §2, we have

(4) $$\varphi^{-1}(Q_\lambda) = \{P_\lambda\} \quad \text{for } \lambda = 1, \infty.$$

4. Generic Torelli theorem. We use the notation in the previous sections.

Theorem. *The period map* φ *in* §3 (3) *is the normalization of analytic spaces over its image.*

Proof. We use the fs logarithmic point P_1 and its image Q_1 at the boundaries (§3). Since $\varphi^{-1}(Q_1) = \{P_1\}$ (§3 (4)), it is enough to show that the local ramification index at Q_1 is one, i.e.,

Claim. $(M_X/\mathcal{O}_X^\times)_{Q_1} \to (M_{\mathbf{P}^1}/\mathcal{O}_{\mathbf{P}^1}^\times)_{P_1}$ is surjective.

Let $N := N_\alpha$ be the nilpotent endomorphism introduced in §2.

Let $\tilde{q}$ be a local coordinate on a neighborhood U of $P_1 = 1$ in $\mathbf{P}^1$, and let $z = (2\pi i)^{-1} \log \tilde{q}$ be a branch over $U - \{P_1\}$. Then $\exp(-zN)e_3 = e_3 - ze_1$ is single-valued. Let $\omega(\tilde{q})$ be a local frame of the locally free $\mathcal{O}_{\mathbf{P}^1}$-module F^3. Write $\omega(\tilde{q}) = \sum_{j=1}^4 a_j(\tilde{q})e_j$, and define $t = -a_1(\tilde{q})/a_3(\tilde{q})$. Then

$$\begin{aligned} t &= \frac{\langle e_3, \omega(\tilde{q})\rangle_0}{\langle e_1, \omega(\tilde{q})\rangle_0} = \frac{\langle \exp(-zN)e_3, \omega(\tilde{q})\rangle_0 + z\langle e_1, \omega(\tilde{q})\rangle_0}{\langle e_1, \omega(\tilde{q})\rangle_0} \\ &= z + (\text{single-valued holomorphic function in } \tilde{q}). \end{aligned}$$

Let $q = e^{2\pi i t}$. Then $q = u\tilde{q}$ for some $u \in \mathcal{O}_{\mathbf{P}^1, P_1}^\times$. Let V be a neighborhood of Q_1 in $X = \Gamma\backslash D_\Xi$. We have a composite morphism of fs logarithmic local ringed spaces

$$U \to V \to \mathbf{C}, \quad \tilde{q} \mapsto q = e^{2\pi i(-a_1/a_3)} \ (= u\tilde{q}).$$

Hence the composite morphism $(M_{\mathbf{P}^1}/\mathcal{O}_{\mathbf{P}^1}^\times)_{P_1} \leftarrow (M_X/\mathcal{O}_X^\times)_{Q_1} \leftarrow (M_{\mathbf{C}}/\mathcal{O}_{\mathbf{C}}^\times)_0$ of reduced logarithmic structures is an isomorphism. The claim follows. In fact, that is an isomorphism since the rank of $(M_X/\mathcal{O}_X^\times)_{Q_1}$ is one in the present case. □

5. The second proof, and logarithmic generic Torelli theorem. In this section, we give another proof of the generic Torelli theorem in §4 by using the fs logarithmic points P_∞ and Q_∞ at the boundaries.

Since $\varphi^{-1}(Q_\infty) = \{P_\infty\}$ (§3 (4)), it is enough to show the following

Claim. $(M_X/\mathcal{O}_X^\times)_{Q_\infty} \to (M_{\mathbf{P}^1}/\mathcal{O}_{\mathbf{P}^1}^\times)_{P_\infty}$ is surjective.

Proof. Let N be the logarithm of the local monodromy at $\lambda = \infty$. Let β^1, β^2, α_1, α_2 be the integral symplectic basis of H_0 given in [M1, Appendix C], and let

$$g_0 = \alpha_2, \quad g_1 = 2\alpha_1 + \beta^1, \quad g_2 = \beta^2, \quad g_3 = \alpha_1 + \beta^1.$$

Then g_3, g_2, g_1, g_0 is another integral symplectic basis such that g_0, g_1 is a good integral basis of $\mathrm{Image}(N^2)$ in the sense of [M1, 2, Appendix C]. Using the result there, we have

$$\begin{aligned} &(N(g_3), N(g_2), N(g_1), N(g_0)) \\ &\quad = (g_3, g_2, g_1, g_0)\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -5 & -9/2 & 0 & 0 \\ -9/2 & 25/6 & 1 & 0 \end{pmatrix}. \end{aligned}$$

Let $\tilde{q}$ be a local coordinate on a neighborhood U of $P_\infty = \infty$ in $\mathbf{P}^1$, and let $z = (2\pi i)^{-1} \log \tilde{q}$ be a branch over $U - \{P_\infty\}$. Then $\exp(-zN)g_1 = g_1 - zg_0$ is single-valued. Let $\omega(\tilde{q})$ be a local frame of the locally free $\mathcal{O}_{\mathbf{P}^1}$-module F^3. Write $\omega(\tilde{q}) = \sum_{j=0}^3 b_j(\tilde{q})g_j$, and define $t = b_3(\tilde{q})/b_2(\tilde{q})$ and $q = e^{2\pi i t}$. Then, as in §4, we see $q = u\tilde{q}$ for some $u \in \mathcal{O}_{\mathbf{P}^1, P_\infty}^\times$, and the claim follows. □

Combining the results in §4 and §5, we have the following refinement.

Theorem. *The period map* φ *in* §3 (3) *is the normalization of fs logarithmic analytic spaces over its image. Here the normalization of the image of* φ *is endowed with the pull-back logarithmic structure.*

Note. Canonical coordinates in [M1,M2], like q in the proofs of Claims in §4 and §5, can be understood more naturally in the context of PLH.

Problem 1. Eliminate the possibility that the image $\varphi(\mathbf{P}^1)$ would have singularities in Theorems in §4 and §5.

Problem 2. Describe the global monodromy group Γ explicitly. Generators of Γ are computed [COGP], [M1, Appendix C], and Γ is known to be Zariski dense in $G_{\mathbf{Z}} = \mathrm{Sp}(4, \mathbf{Z})$ [COGP], [D, 13].

Acknowledgements. The author is grateful to Kazuya Kato and Chikara Nakayama. The author is also grateful to the referee for the useful comments together with the references [S], [V].

Partly supported by the Grant-in-Aid for Scientific Research (B) No. 19340008, Japan Society for the Promotion of Science.

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