# 101. Differential Geometry of Conics in the Projective Space of Three Dimensions.

## III. Differential invariant forms in the theory of a two-parameter family of conics (second report).

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**4.** *Normalization of* $\mathfrak{l}$. A two-parameter family of conics in the projective space of three dimentions can be represented by the equations in the parametric form

$$\mathfrak{a}=\mathfrak{a}(u^1, u^2), \quad \mathfrak{l}=\mathfrak{l}(u^1, u^2), \tag{19}$$

when we adopt the coordinate system of a conic in space, introduced in one of my previous papers[1]. For the system $\mathfrak{a}$ we have already completely discussed in the first report, and we may use the differential forms and the results in that report, because the present theory can be got by a proper combination of those of a conic-family in a plane (theory of $\mathfrak{a}$) and of a surface in space (theory for $\mathfrak{l}$). We must, therefore, introduce other differential invariant forms connected with the family, besides those introduced in the first report.

Put

$$H=h_{ij}du^idu^j=\frac{1}{\sqrt{G}}\,|\,\mathfrak{l}\;\mathfrak{l}_1\;\mathfrak{l}_2\;\mathfrak{l}_{ij}\,|\,du^idu^j, \tag{20}$$

which is an invariant differential form, where

$$G=g_{11}g_{22}=g_{12}^2$$

and $\mathfrak{l}_i$, $\mathfrak{l}_{ij}$ are the first and the second covariant derivatives of $\mathfrak{l}$ with respect to the form $g_{ij}du^idu^j$. Moreover we introduce the quantities $h_{ij}$ such that

$$h^{ij}\overline{h}_{ik}=\delta_k^j \tag{21}$$

and normalize the coordinates $\mathfrak{l}$ so that they satisfy the relation

$$h^{ij}g_{ij}=1, \tag{22}$$

since $h^{ij}$ is multiplied by $\rho^{-4}$ corresponding to a change of proportional facter: $\rho\mathfrak{l}$.

**5.** *Another differential form.* Consider the differential form of the third order

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1) Differential geometry of conics in the projective space of three dimensions, I. Fundamental theorem in the theory of a one-parameter family of conics, these Proceedings **4**, 255–258.

(23) $$c_{ijk}du^idu^jdu^k=\frac{1}{\sqrt{G}}|\mathfrak{l}\,\mathfrak{l}_i\,\mathfrak{l}_2\,d^3\mathfrak{l}|-\frac{3}{2}dH,$$

then it follows

(24) $$c_{ijk}=\frac{1}{2\sqrt{G}}\left\{3\,|\mathfrak{l},\mathfrak{l}_{1i},\mathfrak{l}_{2j},\mathfrak{l}_k|-|\mathfrak{l},\mathfrak{l}_1,\mathfrak{l}_2,\mathfrak{l}_{ijk}|\right\}du^idu^jdu^k$$

and $c_{ijk}$ is a symmetrical tensor. We can see that between $h^{ij}$ and $c_{ijk}$ the relations hold good

(25) $$h^{ij}c_{ijk}=0.$$[1]

Let us consider a surface enveloped by the planes $\mathfrak{l}(u^1, u^2)$ and its point-coordinates be $\mathfrak{y}(u^1, u^2)$, then

(26) $$\mathfrak{y}=\lambda\,||\mathfrak{l},\mathfrak{l}_1,\mathfrak{l}_2||,\quad \mathfrak{l}=\mu\,||\mathfrak{y},\mathfrak{y}_1,\mathfrak{y}_2||,$$

putting

$$\lambda=\varepsilon\mu=\frac{1}{\sqrt{G}}\qquad(\varepsilon=-\operatorname{sgn} G)$$

**6.** *New vectors* $\mathfrak{m}$, $\mathfrak{z}$. I will now denote in the following the covariant derivatives of a quantity $\mathfrak{p}$ with respect to the form $\overline{h_{ij}du^idu^j}$ by $\hat{\mathfrak{p}}_i$, then we get very easily

(27) $$c_{ijk}=\frac{1}{\sqrt{G}}|\mathfrak{l},\mathfrak{l}_1,\mathfrak{l}_2,\overline{\mathfrak{l}}_{ijk}|.$$

It is not difficult to find out the relations

(28) $$\begin{cases} h_{ij}=\mathfrak{y}\mathfrak{l}_{ij}=\mathfrak{l}\mathfrak{y}_{ij}=\mathfrak{y}\mathfrak{l}_{ij}=\mathfrak{l}\mathfrak{y}_{ij}, \\ c_{ijk}=\mathfrak{y}\mathfrak{l}_{ijk}=-\mathfrak{y}_{ijk}\mathfrak{l}. \end{cases}$$

For a new vector

(29) $$\mathfrak{m}=\frac{1}{2}h^{ij}\mathfrak{l}_{ij},$$

the relations subsist:

$(30)_1$ $$\mathfrak{y}_i\mathfrak{m}=\frac{1}{2}h^{jk}\mathfrak{l}_{jk}\mathfrak{y}_i=-\frac{1}{2}h^{jk}c_{ijk}=0,$$

$$\mathfrak{y}\mathfrak{m}=\frac{1}{2}h^{ij}\mathfrak{l}_{jk}\mathfrak{y}=\frac{1}{2}h^{ij}\overline{h}_{ij}=1,$$

hence

$(30)_2$ $$\mathfrak{m}_i\mathfrak{y}=0;$$

dually for a vector

(31) $$\mathfrak{z}=\frac{1}{2}h^{ij}\mathfrak{y}_{ij}$$

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1) See G. Fubini–E. Čech, Geometria proiettiva differenziale, vol. I, Bologna, 1926, pp. 64–67.

we have

$$\mathfrak{l}\mathfrak{z}=1,\quad \mathfrak{l}_i\mathfrak{z}=0,\quad \mathfrak{l}\mathfrak{z}_i=0. \tag{32}$$

By aid of these relations we can put

$$\left\{\begin{aligned} \hat{\mathfrak{l}}_{ij}&=c_{ijk}h^{km}\mathfrak{l}_m+h_{ij}\mathfrak{m}+p_{ij}\mathfrak{l},\\ \hat{\mathfrak{y}}_{ij}&=-c_{ijk}h^{km}\mathfrak{y}_m+h_{ij}\mathfrak{z}+\pi_{ij}\mathfrak{y}, \end{aligned}\right. \tag{33}$$

from which the two new differential forms

$$p_{ij}du^idu^j,\quad \pi_{ij}du^idu^j \tag{34}$$

appear. These forms are both apolar to $h^{ij}$, i.e.

$$h^{ij}p_{ij}=0,\quad h^{ij}\pi_{ij}=0, \tag{35}$$

for

$$2\mathfrak{m}=h^{ij}\hat{\mathfrak{l}}_{ij}=2\mathfrak{m}+h^{ij}p_{ij}\mathfrak{l},\quad \text{etc.}$$

From (30) and (32)

$$\left\{\begin{aligned} \mathfrak{m}_i&=k_i\mathfrak{l}+n_{ip}h^{pq}\mathfrak{l}_q,\\ \mathfrak{z}_i&=\lambda_i\mathfrak{y}+\mu_{ip}h^{pq}\mathfrak{y}^q, \end{aligned}\right. \tag{36}$$

where

$$\left\{\begin{aligned} &k_i+\lambda_i=\mathfrak{m}_i\mathfrak{z}+\mathfrak{m}\mathfrak{z}_i=\frac{\partial}{\partial u^i}(\mathfrak{m}\mathfrak{z})=\Omega_i,\\ &n_{ij}=\pi_{ij}+\Omega\overline{h}_{ij},\quad \mu_{ij}=p_{ij}+\Omega\overline{h}_{ij}, \end{aligned}\right. \tag{37}$$

since

$$\mathfrak{m}\mathfrak{y}_{ij}=\Omega\overline{h}_{ij}+\pi_{ij},\quad \mathfrak{m}_i\mathfrak{y}_j=-n_{ip}\overline{h}^{pq}h_{qj}=-n_{ij},\quad \text{etc.}$$

**7.** *Representability of $h^{ij}$ by other quantities.* In the general case we may now assume that the three differential forms

$$g_{ij}du^idu^j,\quad c_{ijl}a^{lk}_{\cdot\cdot k}du^idu^j,\quad q_{ij}du^idu^j=(p_{ij}+\pi_{ij})du^idu^j$$

are mutually linearly independent; then the quantities $h^{ij}$ must be linearly represented by $g^{ij}$, $c^{ij}_{\cdot\cdot l}a^{lk}_{\cdot\cdot k}$, $q^{ij}$, i.e.

$$h^{ij}=\alpha g^{ij}+\beta c^{ij}_{\cdot\cdot l}\,a^{lk}_{\cdot\cdot k}+\gamma q^{ij}, \tag{38}$$

but from (22), (25) and (35) $\alpha$, $\beta$ and $\gamma$ are determined by

$$\left\{\begin{aligned} 1&=\alpha+\beta c^{i}_{\cdot il}a^{lk}_{\cdot\cdot k}+\gamma q^{i}_{i},\\ 0&=\alpha c^{i}_{\cdot il}a^{lk}_{\cdot\cdot k}+\beta c_{ijl}c^{ij}_{\cdot\cdot m}a^{lk}_{\cdot\cdot k}a^{mn}_{\cdot\cdot n}+\gamma c^{ij}_{\cdot\cdot l}q_{ij}a^{lk}_{\cdot\cdot k},\\ 0&=\alpha q^{i}_{i}+\beta c_{ijl}a^{lk}_{\cdot\cdot k}q^{ij}+\gamma q_{ij}q^{ij}. \end{aligned}\right. \tag{39}$$

**8.** *Equations of integrability* (continued). Equations of integrability for the differential equations (33) and (36) are

$$\left\{\begin{aligned} c_{ij[k}p_{l]m}h^{jm}+\hat{p}_{i[k,l]}+\overline{h}_{i[k}k_{l]}&=0,\\ -c_{ij[k}\pi_{l]m}h^{jm}+\hat{\pi}_{i[k,l]}+\overline{h}_{i[k}\lambda_{l]}&=0, \end{aligned}\right. \tag{40}$$

$$(41)\quad \begin{cases} \hat{c}_{ij(k,l)}h^{jm}+p_{i(k,}\delta_{l)}{}^{m}+\overline{h}_{i(k}n_{l)j}h^{jm}+c_{ij(k}c_{l)np}h^{jp}h^{nm}=\dfrac{1}{2}\mathfrak{R}_{kli}{}^{\cdot\cdot\cdot m}, \\ -\hat{c}_{ij(k,l)}h^{jm}+\pi_{i(k,}\delta_{l)}{}^{m}+\overline{h}_{i(k}\mu_{l)j}h^{jm}+c_{ij(k}c_{l)np}h^{jp}h^{nm}=\dfrac{1}{2}\mathfrak{R}_{kli}{}^{\cdot m}, \end{cases}$$

$$(42)\quad \begin{cases} \hat{k}_{(i,j)}+n_{p(i}p_{j)k}h^{pk}=0, \\ \lambda_{(i,j)}+\mu_{p(i}\pi_{j)k}h^{pk}=0, \end{cases}$$

$$(43)\quad \begin{cases} k_{(i}\delta_{j)}{}^{m}+\hat{n}_{l(i,j)}h^{lm}+n_{l(i}c_{j)np}h^{lp}h^{nm}=0, \\ \lambda_{(i}\delta_{j)}{}^{m}+\hat{\mu}_{p(i,j)}h^{lm}-\pi_{l(i}c_{j)np}h^{lp}h^{nm}=0. \end{cases}$$

From (40) and (41) we get

$$(44)\quad \begin{cases} \delta^{k}{}_{(i}k_{j)}=c_{rp(j}p_{i)q}h^{pq}h^{rk}+\hat{p}_{r(j,i)}h^{rk}, \\ \delta^{k}{}_{(i}\lambda_{j)}=-c_{rp(j}\pi_{i)q}h^{pq}h^{rk}+\hat{\pi}_{r(j,i)}h^{rk}, \\ \delta^{k}{}_{(i}n_{j)p}=\dfrac{1}{2}\mathfrak{R}_{ijr}{}^{\cdot\cdot\cdot q}\overline{h}_{qp}h^{rk}+\hat{c}_{rp(j,i)}h^{rk}+p_{r(j}\overline{h}_{i)p}h^{rk}+c_{rq(j}c_{i)pl}h^{ql}h^{rk}, \\ \delta^{k}{}_{(i}\mu_{j)p}=\dfrac{1}{2}\mathfrak{R}_{ijr}{}^{\cdot\cdot\cdot p}\overline{h}_{qp}h^{rk}-\hat{c}_{rp(j,i)}h^{rk}+\pi_{r(j}\overline{h}_{i)p}h^{rk}+c_{rq(j}c_{i)pl}h^{ql}h^{rk}, \end{cases}$$

that is the quantities $k_i$, $\lambda_i$, $n_{jp}$, $\mu_{jp}$ are all represented by other quantities. It follows moreover from (41)

$$2\hat{c}_{rp(i,j)}+(p_{r(i}-\pi_{r)i})h_{j)p}+h_{r(i}(n_{j(p}-\mu_{j)p})=0,$$

for $r=p$ this relation becomes

$$2\hat{c}_{rp(i,j)}+(p_{r(i}-\pi_{r(i})h_{j)p}+h_{r(i}(\pi_{j)p}-p_{j)p})=0.$$

From this relation and

$$(45)\qquad h^{pq}\hat{c}_{qpi,j}=0$$

we have

$$(46)\qquad (p_{ij}-\pi_{ij})du^{i}du^{j}=\frac{1}{H}\begin{vmatrix} \hat{c}_{1i(1,2)}du^{i} & \overline{h}_{1j}du^{j} \\ \hat{c}_{2i(1,2)}du^{i} & \overline{h}_{2j}du^{j} \end{vmatrix},\quad H=h_{11}h_{22}-h_{12}{}^{2},$$

therefore $p_{ij}$ and $\pi_{ij}$ can de represented by $g_{ij}$, $q_{ij}$, $a_{ijk}$, $c_{ijk}$.

**9.** *The fundamental theorem.* By the above results we can prove the fundamental theorem:

*When the four differential forms $g_{ij}du^{i}du^{j}$, $q_{ij}du^{i}du^{j}$, $a_{ijk}du^{i}du^{j}du^{k}$ and $c_{ijk}du^{i}du^{j}du^{k}$ are given, between which the conditions for integrability above mentioned hold good, then the two-parameter family of conics having those forms as the fundamental forms in the projective space of three dimensions is uniquely determined, expect for projective transformations.*