PAPERS COMMUNICATED

99. *Theory of Connections in the Generalized Finsler Manifold, III.*

By Akitsugu KAWAGUCHI.
Mathematical Institute, Hokkaido Imperial University, Sapporo.

(Comm. by M. FUJIWARA, M.I.A., Oct. 12, 1933.)

I propose, in this paper, to extend the results of my investigation,[1] so that a theory on the Finsler space by E. Cartan[2] can be deduced from mine by specialization.

1. In a generalized Finsler manifold, whose every element $(x^\nu, \overset{i}{p}{}^\nu)$ is a point x^ν (in an n-dimensional manifold) associated to a system of the line elements $\overset{1}{p}{}^\nu, \overset{2}{p}{}^\nu, \ldots\ldots, \overset{r}{p}{}^\nu$, we define a connection from an element $(x^\nu, \overset{i}{p}{}^\nu)$ to its consecutive element $(x^\nu + dx^\nu, \overset{i}{p}{}^\nu + d\overset{i}{p}{}^\nu)$ for an ordinary vector v^ν as follows:

$$\delta v^\nu = dv^\nu + v^\lambda \Big(\Gamma^\nu_{\lambda\mu} dx^\mu + \sum_i^{1,r} \overset{i}{\Lambda}{}^\nu_{\lambda\mu} d\overset{i}{p}{}^\mu\Big) \tag{1. 1}$$

and for a pseudo-vector of the class k: $\mathfrak{r}^\nu$

$$\delta \mathfrak{r}^\nu = d\mathfrak{r}^\nu + \mathfrak{r}^\lambda \Big(\Gamma^\nu_{\lambda\mu} dx^\mu + \sum_i \overset{i}{\Lambda}{}^\nu_{\lambda\mu} d\overset{i}{p}{}^\mu\Big) - k\mathfrak{r}^\nu \Gamma_\mu dx^\mu , \tag{1. 2}$$

where $\Gamma^\nu_{\lambda\mu}$, Γ_μ and $\overset{i}{\Lambda}{}^\nu_{\lambda\mu}$ depend upon $x^\nu, \overset{1}{p}{}^\nu, \ldots\ldots, \overset{r}{p}{}^\nu$. The variations $d\overset{i}{p}{}^\nu$ are independent not only of one another but also of dx^ν. For a transformation of coordinate systems $\bar{x}^\nu = \bar{x}^\nu(x^\lambda)$ the new parameters $'\Gamma^\nu_{\lambda\mu}$ and $'\Gamma_\mu$ of the connection have the forms

(1. 3)

$$'\Gamma^\nu_{\lambda\mu} = P^\nu_\alpha Q^\beta_\lambda Q^\gamma_\mu \Gamma^\alpha_{\beta\gamma} + P^\nu_\alpha \frac{\partial Q^\alpha_\lambda}{\partial \bar{x}^\mu} + P^\nu_\alpha Q^\beta_\lambda \sum_i \overset{i}{\Lambda}{}^\alpha_{\beta\gamma} \Big(\overset{i}{p}{}^\delta P^\pi_\delta \frac{\partial Q^\gamma_\pi}{\partial \bar{x}^\mu} + i\overset{i}{p}{}^\gamma \frac{\partial \log a}{\partial \bar{x}^\mu} \Big),$$

$$'\Gamma_\mu = Q^\beta_\mu \Gamma_\beta + \frac{\partial \log a}{\partial \bar{x}^\mu} \tag{1. 4}$$

respectively and the other parameters $'\overset{i}{\Lambda}{}^\nu_{\lambda\mu}$

---

1) A. Kawaguchi, Theory of connections in the generalized Finsler manifold, Proc. **7** (1931), 211–214; and Die Differentialgeometrie in der verallgemeinerten Mannigfaltigkeit, Rendiconti del Circolo Matematico di Palermo, **56** (1932), 245–276.

2) E. Cartan, Sur les espaces de Finsler, Comptes Rendus, Paris, **196** (1933), 582–586.

(1. 5) $$'\overset{i}{\Lambda}{}^{\nu}_{\lambda\mu}=\alpha^{-i}P^{\nu}_{\alpha}Q^{\beta}_{\lambda}Q^{\tau}_{\mu}\overset{i}{\Lambda}{}^{\alpha}_{\beta\tau},$$

where $$P^{\nu}_{\alpha}=\frac{\partial\bar{x}^{\nu}}{\partial x^{\alpha}},\qquad Q^{\alpha}_{\nu}=\frac{\partial x^{\alpha}}{\partial\bar{x}^{\nu}},$$

and α is a function of position which appears in the formula of transformation of pseudo-vectors.[1] (1. 3) shows that $\Gamma^{\nu}_{\lambda\mu}$ are not transformed as the parameters of an affine connection and (1. 5) that $\overset{i}{\Lambda}{}^{\nu}_{\lambda\mu}$ are affinors of the class $-i$.

In accordance with the rules of transformation we can derive r affinors $\overset{i}{P}{}^{\nu}_{\lambda\mu}$ from the parameters $\Gamma^{\nu}_{\lambda\mu}$, Γ_{μ}, $\overset{i}{\Lambda}{}^{\nu}_{\lambda\mu}$ and that of the base-connections:[2]

(1. 6) $$\overset{i}{P}{}^{\nu}_{\lambda\mu}=\Gamma^{\nu}_{\lambda\mu}-i\delta^{\nu}_{\lambda}\Gamma_{\mu}-\overset{i}{\Gamma}{}^{\nu}_{\lambda\mu}+i\delta^{\nu}_{\lambda}\overset{i}{\Gamma}_{\mu}-\sum_{j}\overset{j}{\Lambda}{}^{\nu}_{\lambda\tau}(\overset{j}{\Gamma}{}^{\tau}_{\pi\mu}-j\delta^{\tau}_{\pi}\overset{j}{\Gamma}_{\mu})\overset{j}{p}{}^{\pi}.$$

2. There are in this connection not only the covariant derivatives $\nabla_{\mu}v^{\nu}$, which are the coefficients of dx^{ν} in δv^{ν}, but also many others $\overset{i}{\nabla}_{\mu}v^{\nu}$, the coefficients of the covariant differential $\overset{i}{\delta}\overset{i}{p}{}^{\nu}$ with respect to the base-connections:

(2. 1) $$\nabla_{\mu}v^{\nu}=\frac{\partial v^{\nu}}{\partial x^{\mu}}+\Gamma^{\nu}_{\lambda\mu}v^{\lambda}-\sum_{i}\overset{i}{\nabla}_{\lambda}v^{\nu}(\overset{i}{\Gamma}{}^{\lambda}_{\pi\mu}-i\overset{i}{\Gamma}_{\mu}\delta^{\lambda}_{\pi})\overset{i}{p}{}^{\pi},$$

(2. 2) $$\overset{i}{\nabla}_{\mu}v^{\nu}=\frac{\partial v^{\nu}}{\partial\overset{i}{p}{}^{\mu}}+\overset{i}{\Lambda}{}^{\nu}_{\lambda\mu}v^{\lambda}.$$

We haxe namely

(2. 3) $$\delta v^{\nu}=dx^{\mu}\nabla_{\mu}v^{\nu}+\sum_{i}\overset{i}{\delta}\overset{i}{p}{}^{\mu}\overset{i}{\nabla}_{\mu}v^{\nu}.$$

These covariant derivatives $\nabla_{\mu}v^{\nu}$ as well as $\overset{i}{\nabla}_{\mu}v^{\nu}$ are all affinors, being contravariant for the upper suffix and covariant for the lower. The affinor $\overset{i}{\nabla}_{\mu}v^{\nu}$ is of the class $-i$, when v^{ν} is an ordinary vector.

3. There exist many curvature tensors of our connection:

(3. 1) $$R^{\cdot\nu}_{\cdot\lambda\mu\omega}=K^{\cdot\nu}_{\cdot\lambda\mu\omega}-\sum_{i}\overset{i}{R}{}^{\cdot\nu}_{\cdot\lambda\mu\rho}(\overset{i}{\Gamma}{}^{\rho}_{\tau\omega}-i\delta^{\rho}_{\tau}\overset{i}{\Gamma}_{\omega})\overset{i}{p}{}^{\tau}+\sum_{i}\overset{i}{R}{}^{\cdot\nu}_{\cdot\lambda\omega\rho}(\overset{i}{\Gamma}{}^{\rho}_{\tau\mu}-i\delta^{\rho}_{\tau}\overset{i}{\Gamma}_{\mu})\overset{i}{p}{}^{\tau}$$
$$-\sum_{i,j}\overset{i,j}{R}{}^{\cdot\nu}_{\cdot\lambda\rho\tau}(\overset{i}{\Gamma}{}^{\rho}_{\alpha\mu}-i\delta^{\rho}_{\alpha}\overset{i}{\Gamma}_{\mu})(\overset{j}{\Gamma}{}^{\tau}_{\beta\omega}-j\delta^{\tau}_{\beta}\overset{j}{\Gamma}_{\omega})\overset{i}{p}{}^{\alpha}\overset{j}{p}{}^{\beta},$$

---

1) For a pseudo-vector $\mathfrak{r}^{\nu}$ of the class k

$$'\mathfrak{r}^{\nu}=\alpha^{k}P^{\nu}_{\beta}\mathfrak{r}^{\beta}.$$

2) A. Kawaguchi, loc. cit.

$$(3.\ 2)\qquad \overset{i}{R}{}^{\nu}_{\cdot\lambda\mu\omega}=\overset{i}{K}{}^{\nu}_{\cdot\lambda\mu\omega}-\sum_{j}\overset{j,i}{R}{}^{\nu}_{\cdot\lambda\rho\omega}(\overset{j}{\Gamma}{}^{\rho}_{\tau\mu}-j\delta^{\rho}_{\tau}\overset{i}{\Gamma}{}_{\mu})\overset{j}{p}{}^{\tau},$$

$$(3.\ 3)\qquad \overset{i,j}{R}{}^{\nu}_{\cdot\lambda\mu\omega}=\frac{\partial\overset{i}{\Lambda}{}^{\nu}_{\lambda\mu}}{\partial\overset{j}{p}{}^{\omega}}-\frac{\partial\overset{j}{\Lambda}{}^{\nu}_{\lambda\omega}}{\partial\overset{i}{p}{}^{\mu}}+\overset{j}{\Lambda}{}^{\nu}_{\rho\omega}\overset{i}{\Lambda}{}^{\rho}_{\lambda\mu}-\overset{i}{\Lambda}{}^{\nu}_{\rho\mu}\overset{j}{\Lambda}{}^{\rho}_{\lambda\omega},$$

which are all affinors, where

$$(3.\ 4)\qquad K^{\nu}_{\cdot\lambda\mu\omega}=\frac{\partial\Gamma^{\nu}_{\lambda\mu}}{\partial x^{\omega}}-\frac{\partial\Gamma^{\nu}_{\lambda\omega}}{\partial x^{\mu}}+\Gamma^{\nu}_{\rho\omega}\Gamma^{\rho}_{\lambda\mu}-\Gamma^{\nu}_{\rho\mu}\Gamma^{\rho}_{\lambda\omega},$$

$$(3.\ 5)\qquad \overset{i}{K}{}^{\nu}_{\cdot\lambda\mu\omega}=\frac{\partial\Gamma^{\nu}_{\lambda\mu}}{\partial\overset{i}{p}{}^{\omega}}-\frac{\partial\overset{i}{\Lambda}{}^{\nu}_{\lambda\omega}}{\partial x^{\mu}}+\overset{i}{\Lambda}{}^{\nu}_{\rho\omega}\Gamma^{\rho}_{\lambda\mu}-\Gamma^{\nu}_{\rho\mu}\overset{i}{\Lambda}{}^{\rho}_{\lambda\omega},$$

which are not affinors. $\overset{i}{R}{}^{\nu}_{\cdot\lambda\mu\omega}$ is of the class $-i$ and $\overset{i,j}{R}{}^{\nu}_{\cdot\lambda\mu\omega}$ of the class $-(i+j)$. It follows after some calculation

$$(3.\ 6)\qquad \begin{aligned}(\delta_2\delta_1-\delta_1\delta_2)v^{\nu}&=\{K^{\nu}_{\cdot\lambda\mu\omega}d_1x^{\mu}d_2x^{\omega}+\sum_{i}\overset{i}{K}{}^{\nu}_{\cdot\lambda\mu\omega}(d_1x^{\mu}d_2\overset{i}{p}{}^{\omega}-d_1\overset{i}{p}{}^{\omega}d_2x^{\mu})\\&\qquad+\sum_{i,j}\overset{i,j}{R}{}^{\nu}_{\cdot\lambda\mu\omega}d_1\overset{i}{p}{}^{\mu}d_2\overset{j}{p}{}^{\omega}\}v^{\lambda}\\&=\{R^{\nu}_{\cdot\lambda\mu\omega}d_1x^{\mu}d_2x^{\omega}+\sum_{i}\overset{i}{R}{}^{\nu}_{\cdot\lambda\mu\omega}(d_1x^{\mu}\overset{i}{\delta}_2\overset{i}{p}{}^{\omega}-\overset{i}{\delta}_1\overset{i}{p}{}^{\omega}d_2x^{\mu})\\&\qquad+\sum_{i,j}\overset{i,j}{R}{}^{\nu}_{\cdot\lambda\mu\omega}\overset{i}{\delta}_1\overset{i}{p}{}^{\mu}\overset{j}{\delta}_2\overset{j}{p}{}^{\omega}\}v^{\lambda}\end{aligned}$$

for the surrounding of the vector v^{ν} about an infinitesimal circuit.

4. On the other hand, we have

$$(4.\ 1)\qquad \begin{aligned}(\overset{j}{\nabla}_{\omega}\overset{i}{\nabla}_{\mu}-\overset{i}{\nabla}_{\mu}\overset{j}{\nabla}_{\omega})v^{\nu}&=\overset{i,j}{R}{}^{\nu}_{\cdot\lambda\mu\omega}v^{\lambda}-\overset{j}{\Lambda}{}^{\pi}_{\mu\omega}\overset{i}{\nabla}_{\pi}v^{\nu}+\overset{i}{\Lambda}{}^{\pi}_{\omega\mu}\overset{j}{\nabla}_{\pi}v^{\nu}\\&=(\overset{i,j}{R}{}^{\nu}_{\cdot\lambda\mu\omega}-\overset{j}{\Lambda}{}^{\pi}_{\mu\omega}\overset{i}{\Lambda}{}^{\nu}_{\lambda\pi}+\overset{i}{\Lambda}{}^{\pi}_{\omega\mu}\overset{j}{\Lambda}{}^{\nu}_{\lambda\pi})v^{\lambda}-\left(\overset{j}{\Lambda}{}^{\pi}_{\mu\omega}\frac{\partial v^{\nu}}{\partial\overset{i}{p}{}^{\pi}}-\overset{i}{\Lambda}{}^{\pi}_{\mu\omega}\frac{\partial v^{\nu}}{\partial\overset{j}{p}{}^{\pi}}\right),\end{aligned}$$

$$(4.\ 2)\qquad \begin{aligned}(\overset{i}{\nabla}_{\rho}\nabla_{\mu}-\nabla_{\mu}\overset{i}{\nabla}_{\rho})v^{\nu}&=R^{\nu}_{\cdot\lambda\mu\rho}v^{\lambda}+\overset{i}{P}{}^{\lambda}_{\rho\mu}\overset{i}{\nabla}_{\lambda}v^{\nu}-\overset{i}{\Lambda}{}^{\lambda}_{\mu\rho}\nabla_{\lambda}v^{\nu}\\&\qquad-\sum_{j}\overset{j}{\nabla}_{\lambda}v^{\nu}\left(\frac{\partial\overset{j}{\Gamma}{}^{\lambda}_{\pi\mu}}{\partial\overset{i}{p}{}^{\rho}}-j\delta^{\lambda}_{\pi}\frac{\partial\overset{j}{\Gamma}{}_{\mu}}{\partial\overset{i}{p}{}^{\rho}}\right)\overset{j}{p}{}^{\pi},\end{aligned}$$

$$(4.\ 3)\qquad (\nabla_{\omega}\nabla_{\mu}-\nabla_{\mu}\nabla_{\omega})v^{\nu}=R^{\nu}_{\cdot\lambda\mu\omega}v^{\lambda}+2T^{\lambda}_{\cdot\omega\mu}\nabla_{\lambda}v^{\nu}+\sum_{i}\overset{i}{Q}{}^{\lambda}_{\cdot\tau\omega\mu}\overset{i}{p}{}^{\tau}\overset{i}{\nabla}_{\lambda}v^{\nu},$$

where

$$(4.\ 4)\qquad \begin{aligned}T^{\lambda}_{\cdot\omega\mu}=\Gamma^{\lambda}_{[\omega\mu]}-\sum_{j}(\overset{j}{\Gamma}{}^{\lambda}_{\pi[\mu}-j\delta^{\tau}_{\pi}\overset{j}{\Gamma}{}_{[\mu})\overset{j}{p}{}^{\pi}\overset{j}{\Lambda}{}^{\lambda}_{\omega]\tau}&=\overset{i}{P}{}^{\lambda}_{[\omega\mu]}-\overset{i}{\Gamma}{}^{\lambda}_{[\omega\mu]}\\&\qquad+i\delta^{\lambda}_{[\omega}(\Gamma_{\mu]}-\overset{i}{\Gamma}{}_{\mu]}),\end{aligned}$$

$$(4.\ 5)\qquad \frac{1}{2}\overset{i}{Q}{}^{\lambda}_{\cdot\tau\mu\omega}=\frac{\partial \overset{i}{\Gamma}{}^{\lambda}_{\tau[\mu}}{\partial x^{\omega]}}-i\delta^{\lambda}_{\tau}\frac{\partial \overset{i}{\Gamma}_{[\mu}}{\partial x^{\omega]}}-\overset{i}{\Gamma}{}'^{\lambda}_{\pi[\mu}\overset{i}{\Gamma}{}^{\pi}_{|\tau|\omega]}+\sum_{j}(\overset{j}{\Gamma}{}^{\pi}_{\rho[\mu}-j\delta^{\pi}_{\rho}\overset{j}{\Gamma}_{[\mu})\overset{j}{p}{}^{\omega}\times\frac{\partial}{\partial \overset{j}{p}{}^{\pi}}(\overset{i}{\Gamma}{}'^{\lambda}_{|\tau|\mu]}-i\delta^{\lambda}_{|\tau|}\overset{i}{\Gamma}_{\omega]})\,.$$

$T^{\lambda}_{\cdot\omega\mu}$ and $\overset{i}{Q}{}^{\lambda}_{\cdot\tau\mu\omega}$ are also affinors, of which the former is the *torsion tensor* of the connection and the latter are the *curvature tensors* of the base-connections.

5. In Cartan's case we may assume $\overset{1}{\Gamma}_{\mu}=0$, as there exists an integral invariant $s=\int\sqrt{F(x, x')}$, which will be taken as the parameter for the line element $\overset{1}{p}{}^{\nu}=\frac{dx^{\nu}}{ds}$. Let $\overset{1}{\Lambda}_{\lambda\mu\nu}=g_{\lambda\sigma}\overset{i}{\Lambda}{}^{\sigma}_{\mu\nu}$[1] be symmetric and $\overset{1}{\Lambda}_{\lambda\mu\nu}\overset{1}{p}{}^{\nu}=0$, being $g_{\lambda\mu}=F_{x'^{\lambda}x'^{\mu}}$. For the parameters of the base-connection we take

$$(5.\ 1)\qquad \overset{1}{\Gamma}{}^{\nu}_{\lambda\mu}=\Gamma^{\nu}_{\lambda\mu}-\overset{1}{\Lambda}{}^{\nu}_{\lambda\tau}\overset{1}{\Gamma}{}^{\tau}_{\pi\mu}\overset{1}{p}{}^{\pi}=\Gamma^{\nu}_{\lambda\mu}-\overset{1}{\Lambda}{}^{\nu}_{\lambda\tau}\Gamma^{\tau}_{\pi\mu}\overset{1}{p}{}^{\pi}$$

and assume $\delta g_{\lambda\mu}=0$. Then the curvature tensors (3. 1)–(3. 3) are nothing but those introduced by Cartan. In this case we have

$$(5.\ 2)\qquad \overset{1}{P}{}^{\nu}_{\lambda\mu}=0\,,$$

$$(5.\ 3)\qquad T^{\nu}_{\cdot\lambda\mu}=0\,,$$

$$(5.\ 4)\qquad \overset{1}{Q}{}^{\lambda}_{\cdot\tau\mu\omega}\overset{1}{p}{}^{\tau}=R^{\nu}_{\cdot\tau\mu\omega}\overset{1}{p}{}^{\tau}\,.$$

From (5. 3) we can conclude that Cartan's connection is symmetric or has no torsion.

---

1) Cartan, loc. cit. Cartan uses the notation $C^{\nu}_{\lambda\mu}$ for our $\overset{1}{\Lambda}{}^{\nu}_{\lambda\mu}$.