162. *On the Summability of Fourier Series by Riesz's Logarithmic Means.*

By Fu Traing WANG.

Mathematical Institute, Tohoku Imperial University, Sendai.

(Comm. by M. FUJIWARA, M.I.A., Dec. 12, 1933.)

1. Let $f(t)$ be a summable and periodic function with period 2π, and let

$$f(t) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \qquad (1\cdot 1)$$

The Fourier series $(1\cdot 1)$ is said to be summable (R, k), for $t=x$, to sum s, provided that

$$R_\omega^k = \frac{a_0}{2} + \frac{1}{(\log \omega)^k} \sum_{n<\omega} \left(\log \frac{\omega}{n}\right)^k (a_n \cos nx + b_n \sin nx)$$

tends to a limit s, as $\omega \to \infty$.[1]

Let $$\phi(u) = \frac{1}{2}\{f(x+u) + f(x-u) - 2s\};$$

we write $$\phi(t) \to 0 \quad (R, \alpha)$$

as $t \to 0$, provided that

$$\phi_\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_t^\pi \left(\log \frac{u}{t}\right)^{\alpha-1} \frac{\phi(u)}{u} du = o\left[\left(\log \frac{1}{t}\right)^\alpha\right],$$

when $t \to 0$.

Concerning the summability of Fourier series by Riesz's logarithmic means, Prof. Hardy has given a theorem on $(R, 1)$ summability.[2] Now we can extend this theorem and obtain some other theorems. The proof of them will appear in Tohoku Mathematical Journal.

2. Suppose that k is a positive integer and $\phi_0(t) = \phi(t)$, then we have

Theorem A. *If*

$$\int_0^t |\phi_{k-1}(u)|\, du = O\left[t\left(\log \frac{1}{t}\right)^k\right],$$

then the necessary and sufficient condition that the series $(1\cdot 1)$ *should be summable* (R, k), *for* $t=x$, *to sum* s, *is that*

---

1) Hardy and Riesz: Theory of general Dirichlet's series.

2) Hardy: Quarterly Journal, **2** (1931).

$$\psi_k(t)=O\left[\left(\log\frac{1}{t}\right)^k\right],$$

and

$$\int_0^t \psi_k(u)du=o\left[t\left(\log\frac{1}{t}\right)^k\right],$$

when $t\to 0$.

Theorem B. *If*

$$\int_0^t |\psi_{k-1}(u)|\,du=o\left[t\left(\log\frac{1}{t}\right)^k\right],$$

then the necessary and sufficient condition that the series (1·1) *should be summable* (R, k) *for* $t=x$, *to sum* s, *is that*

$$\phi(t)\to 0 \quad (R, k),$$

when $t\to 0$.

Theorem C. *The necessary and sufficient condition that the series* (1·1) *should be summable by Riesz's logarithmic means, for* $t=x$, *to sum* s, *is that* $\phi(t)\to 0$ (R, k), *for some* k.

Theorem D. *If*

$$\phi_k(t)=\int_0^t (t-u)^{k-1}\phi(u)du=o(t^k),$$

when $t\to 0$, *then the series* (1·1) *is summable* (R, k), *for* $t=x$, *to sum* s.

Theorem E. *If* $\alpha>0$, *and*

$$\phi(t)\to 0 \quad (R, \alpha),$$

when $t\to 0$, *then the series* (1.1) *is summable* $(R, \alpha+\delta)$ $(\delta>0)$, *for* $t=x$, *to sum* s.

Theorem F. *If the Fourier series* (1·1) *is summable* (R, α), *for* $t=x$, *to sum* s, *then*

$$\phi(t)\to 0 \quad (R, \alpha+1+\delta),$$

when $t\to 0$.

Theorem G. *If*

$$\int_0^t |\phi(u)|\,du=O(t),$$

then the necessary and sufficient condition that the series (1·1) *should be summable by Riesz's logarithmic means of any positive order, for* $t=x$, *to sum* s, *is that*

$$\int_t^\pi \frac{\phi(u)}{u}du=o\left(\log\frac{1}{t}\right),$$

when $t\to 0$.