152. Note on the System of the Orthogonal Functions.

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Let $\{\varphi_i(s)\}$ be an orthogonal system of functions which are defined and squarely integrable in (0, 1), and $x(s)$ be a function defined in the same interval. The formal series

$$(1) \qquad \sum_{i=1}^{\infty} \varphi_i(s) \int_0^1 x(s)\varphi_i(s)ds$$

is called the expansion of $x(s)$ by the system $\{\varphi_i(s)\}$.

Concerning the expansion (1), Haar had, in his Dissertation,[1] proved the following theorems.

I. *Let s_0 be a point in* (0, 1), *and put*

$$K_n(s_0, s) = \sum_{i=1}^{n} \varphi_i(s_0)\varphi_i(s)$$

and

$$\omega_n = \int_0^1 |K_n(s_0, s)|\, ds.$$

If $\{\omega_n\}$ is not bounded, then there exists a continuous function whose expansion diverges at $s=s_0$.

II. *If every continuous function is uniformly approximable by the system $\{\varphi_i(s)\}$ in* (0, 1) *and $\{\omega_n\}$ is bounded, then the expansion of every continuous function converges at $s=s_0$.*

In the present paper, we prove these theorems by using theorems in the theory of linear operations, and at the same time, prove the following theorem.

III. *If the hypothesis in* I *is satisfied, then the set of functions*

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1) A. Haar: Zur Theorie der orthogonalen Funktionensysteme, Math. Ann. **69** (1912).

Cf. H. Steinhaus: Sur les développments orthogonaux, Bull. de Acad. de Cracovie, Série A (1926).

Banach-Steinhaus: Sur les principes de la condénsation de singularités, Fund. Math., **9** (1927).

W. Orlicz: Einige Bemerkungen über die Divergenzpunktmengen von Orthogonalentwicklungen, Studia Math. **2** (1930).

W. Orlicz: Eine Bemerkungen über Divergenzphänomene von Orthogonalentwicklungen, ibid.

2) The theorem in Haar's paper is a little more precise than this, but essencially equivalent.

whose expansions are divergent (indeed the partial sums are not bounded) at $s=s_0$, is of the second category in the space of all continuous functions.

We use, here, the terminology in Banach's work.[1]

Lemma α.[2] *Let $\{u_n(x)\}$ be a sequence of linear functionals defined in a space of type (B). If $\overline{\lim}\,\|u_n(x)\|$ is finite in a set of second category in the space in question, then there exists a constant M independent of n, such that $\|u_n\|<M$.*

Lemma β.[3] *Let $\{u_n(x)\}$ be a sequence of linear functionals defined in a space of type (B). If $u_n(x)$ converges in a set which is dense in a sphere in the space and there exists a constant M independent of n such that $\|u_n\|<M$, then $u_n(x)$ converges at every point in the space.*

Lemma γ.[4] *Every linear functional $u(x)$ defined in the space (C) (which consists of all continuous functions) has the form*

$$u(x)=\int_0^1 x(s)dg(s)\,,$$

where $g(s)$ is a function of bounded variation independent of $x(s)$ and

$$\|u_n\|=\int_0^1 |dg(s)|\,.$$

Now we will prove II. Put

$$u_n(x)=\sum_{i=1}^{n}\varphi_i(s_0)\int_0^1 x(s)\varphi_i(s)ds=\int_0^1 x(s)K_n(s_0, s)ds\,.$$

Then the functional $u_n(x)$ is defined in the space (C) and is linear. And since every continuous function is uniformly approximable by the system $\{\varphi_i(s)\}$ from the assumption, the set of linear combinations of finite number of $\varphi_i(s)$ is dense in (C). And the expansion of such a linear combination is plainly convergent at $s=s_0$. On the other hand, by Lemma γ, there exist the functions $g_n(s)$ of bounded variation such that

$$\int_0^1 x(s)dg_n(s)=\int_0^1 x(s)K_n(s_0, s)ds=\int_0^1 x(s)d\Big(\int_0^s K_n(s_0, v)dv\Big)\,. \tag{2}$$

Since the relation (2) is valid for every continuous function $x(s)$, we have

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1) S. Banach: Théorie des operation linéaires, 1932, Warszawa.

2) Banach-Steinhaus: loc. cit., *See* Lemma 2.

3) Banach-Steinhaus: loc. cit., *See* Lemma 3.

4) F. Riesz: Sur les opérations fonctionelles linéaires, Comptes Rendus, **149** (1909). Banach: loc. cit., p. 60.

$$g_n(s) = \text{constant} + \int_0^s K_n(s_0, v)dv$$

(the constant may depend on n).

Thus if there exists a constant M independent of n such that $\int_0^1 |K_n(s_0, s)|ds < M$, then $||u_n|| = \int_0^1 |dg_n(s)| < M$. Hence by Lemma ($\beta$), $u_n(x)$ is convergent at every point $x(s) \subset (C)$.

Next we will prove II and III. It is sufficient to prove III.

Here we use the same notations as in the proof of I. If we suppose that the set A of continuous functions whose expansions converge at $s = s_0$, is of the second category in the space (C), then by Lemma α we have $||u_n|| < M$, M being independent of n, for $\overline{\lim}|u_n(x)|$ is finite for every $x \subset A$. Thus as in the proof of I, we have $\omega_n = \int_0^1 |K_n(s_0, s)|ds < M$. This contradicts the hypothesis that $\{\omega_n\}$ is not bounded.