104. A Note on Zeros of Riemann Zeta-function.

By Fu Traing WANG.
Mathematical Institute, Tohoku Imperial University, Sendai.
(Comm. by M. FUJIWARA, M.I.A., Dec. 12, 1936.)

1. Let $N_0(T)$ be the number of zeros of $\zeta\left(\frac{1}{2}+it\right)$ for $0<t<T$, then

$$N_0(T) \geqslant \frac{T}{\pi e}+o(T),$$

which is a little better than that obtained by R. Kuzmin.[1)]

2. Proof. C. L. Siegel[2)] proved that the number of zeros of $\zeta\left(\frac{1}{2}+it\right)$ depends on those of $f(\sigma+it)$ for $\sigma<\frac{1}{2}$, where

$$f(s)=\int\limits_{0\swarrow 1} \frac{x^{-s}e^{\pi i x^2}}{e^{\pi i x}-e^{-\pi i x}}dx \qquad (s=\sigma+it),$$

the path of integration is the line parallel to the line bisecting the first and third quadrants and cutting the real axis in a point lying in (0, 1).

Put

$$g(s)=\pi^{-\frac{s+1}{2}}e^{-\frac{\pi i s}{4}}\Gamma\left(\frac{1+s}{2}\right)f(s),$$

and $U=T^a\left(\frac{13}{14}<a<1\right)$ and $N(T)$ be the number of zeros of $f(s)$ for s lying in the rectangle $-T^{\frac{3}{7}}<\sigma<\frac{1}{2}$, $T<t<T+U$. By making a detailed calculation as in Siegel's paper,[3)] we have

$$\text{(2.1)} \qquad N_0(T+U)-N_0(T)\geqslant 2N(T)+O(T^{\frac{13}{14}}).$$

By a similar method for calculating the mean value formula of zeta-function we have

$$\int_T^{T+U}|g(\sigma+it)|^2dt=\frac{1}{2}\frac{1}{\frac{1}{2}-\sigma}\sqrt{\frac{2}{\pi}}T^{\frac{1}{2}}U+O(U^2T^{-\frac{1}{2}})+O(T^{\frac{5}{4}})$$

for $\sigma<\frac{1}{4}$.

Using a known inequality[4)] we have

---

1) R. Kuzmin, C. R. Acad. de URSS. **2** (1934).

2) C. L. Siegel, Quellen und Studien zur Geschichte der Math. Astr. und Physik, **2** (1932), pp. 45–80.

3) C. L. Siegel. Loc. cit.

4) Hardy-Littlewood-Polya, Inequality.

$$\int_T^{T+U} \log |g(\sigma+it)|\, dt \leqslant \frac{U}{4}\log T + \frac{U}{2}\log \frac{1}{2\left(\frac{1}{2}-\sigma\right)}\sqrt{\frac{2}{\pi}} + O(U^2T^{-1}) + O(T^{\frac{3}{4}}).$$

On the other hand

$$\int_T^{T+U} \log \left| g\left(\frac{1}{2}+it\right)\right| dt = \frac{U}{4}\log T + \frac{U}{2}\log\sqrt{\frac{2}{\pi}} + 2\pi\sum_{\alpha>\frac{1}{2}}\left(\alpha-\frac{1}{2}\right) + O(U^2T^{-1}),$$

where α's run over all real parts of zeros of $f(s)$ for $\frac{1}{2}<\sigma<2$ and $T<t<T+U$. Hence

$$\left(\frac{1}{2}-\sigma\right)2\pi N(T) \geqslant \frac{U}{2}\log 2\left(\frac{1}{2}-\sigma\right) + O(U^2T^{-1}) + O(T^{\frac{13}{14}}).$$[1]

If we take $\sigma=\frac{1}{2}-\frac{e}{2}$, then

$$2\pi N(T) \geqslant \frac{U}{e} + o(U). \tag{2.2}$$

By (2.1) and (2.2) we have

$$N_0(T+U) - N_0(T) \geqslant \frac{1}{\pi e}U + o(U),$$[2]

for $T>t_0$.

Hence

$$N_0(T) - N_0(T-U) \geqslant \frac{1}{\pi e}U + o(U),$$

....

$$N_0(T-\overline{x-1}U) - N_0(T-xU) \geqslant \frac{1}{\pi e}U + o(U).$$

If we take $x=\left[\frac{T-t_0}{U}\right]$, then $T-xU \geqslant t_0$,

and

$$x \geqslant \frac{T-t_0}{U} - 1.$$

Hence

$$N_0(T) - N_0(T-xU) \geqslant \frac{T}{\pi e} + o(T).$$

Thus we have the required result.

---

1) Siegel, p. 68.

2) G. H. Hardy and J. E. Littlewood, Math. Zeit., **10** (1921).