42. *A Theorem on Operational Equation.*

By Tosio KITAGAWA.

Mathematical Institute, Faculty of Science, Osaka Imperial University.

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1. In addition to the convections of our previous note[1] we shall here make some assumptions, and we shall prove a theorem on operational equation which corresponds to that of Schürer on the solution of linear differential equation of infinite order with constant coefficients.[2] The solutions now in our consideration correspond to those of finite grade[3] on the theory of differential equation of infinite order.

2. The assumptions which we will add to those[4] of our previous note are the following:

1°. The function-set $(C)_t$ is now defined as consisted of all elements $g(x)$ of $(B)_t$ which satisfy the boundary condition at t

$$L_t\Big\{g(x)\Big\}=0\,, \tag{1}$$

where L_t is a linear functional of $g(x)$. Here we assume that, for any λ of $\mathfrak{M}$, $j_\lambda(x, t_0)$ does not satisfy the boundary condition at the point t_0 that is,

$$L_{t_0}\Big\{j_\lambda(x, t_0)\Big\}=a_0 \neq 0\,, \tag{2}$$

where a_0 is independent of λ.

2°. In the following t_0 is fixed, and therefore we may and we shall write $j_\lambda(x)$ in stead of $j_\lambda(x, t_0)$.

3°. Let $\mathfrak{X}$ be a system of subset[5] of Y_{t_0} which constitutes a corpus.[6] For any $Y \in \mathfrak{X}$ and for any function $f(x) \in (A)_{t_0}$, we shall define a function $f_Y(x)$ which is only defined on Y and which there equals to $f(x)$. We assume that $(A)_{t_0}$ possesses the property that, for any fixed Y of $\mathfrak{X}$, the set of all $f_Y(x)$ constitutes a normalised Banach space, whose norm will be designated by $\|f_Y(x)\|_Y$ or simply by $\|f(x)\|_Y$.[7]

4°. A sequence of functions $\Big\{f_n(x)\Big\}$ in $(A)_{t_0}$ is said *to be a Cauchy-sequence in the generalised sense*, if, however we may choose Y from $\mathfrak{X}$,

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1) T. Kitagawa: *A* Formulation of Operational Calculus, This Proceeding, **13**. We quote this by [F]. See specially § 2.

2) F. Schürer: Eine gemeinsame Methode zur Behandlung gewisser Funktional gleichungsprobleme. Leipziger Berichte, vol. **70** (1918).
See specially C.L-Gleichungen hoher Ordnung p. 210.

3) See, for example, Davis: The theory of linear operators, (1936) Chapter V, Grades defined by Special Operators.

4) See [F] § 2 and § 3.

5) Under a subsat of X, we understand "echte" subset.

6) Under a corpus, we understand a system of stets for which if $Y \in \mathfrak{X}$ and $Z \in \mathfrak{X}$, then $Y.\ Z$, $Y-Z$ and $Y+Z$ also belong to $\mathfrak{X}$.

7) For example, let (A) be consisted of all functions which are quarely integrable in any bounded measurable set Y of real-axis, and let

$$\|f(x)\|_Y \equiv \|f_Y(x)\|_Y = \sqrt[2]{\int_Y |f(t)|^2 dt}$$

the sequence $\{f_{n,Y}(x)\}$ is a Cauchy-sequence in the ordinary sense with respect to the norm $\|f_{n,Y}(x)\|_Y$. The space $(A)_{t_0}$ is called *to be complete in the generalised sense,* if, to any given Cauchy-sequence in the generalised sense, there corresponds a function (uniquely determined) $f(x)$ in (A) such that, for any $Y \in \mathfrak{X}$, we have

$$\lim_{n\to\infty} \| f_n(x) - f(x) \|_Y = 0 . \tag{3}$$

In this case we say that $\{f_n(x)\}$ *converges to* $f(x)$ *in the generalised sense.*[1]

5.° Let us designate by (B^∞)[2] the set of $(A)_{t_0}$ which is constituted by of all functions $f(x)$ in $(A)_{t_0}$ for which $\mathfrak{D}^n f(x)$ exists and belongs to $(A)_{t_0}$ for every non-negative integer n. (We mean that $\mathfrak{D}^0 f(x) \equiv f(x)$.) The function-set $\big(E(\lambda)\big)$[3] is constituted by all functions which belong to (B^∞) and which satisfy the condition that, for any $Y \in \mathfrak{X}$, we have

$$\overline{\lim_{n\to\infty}} \left\| \frac{\mathfrak{D}^n f(x)}{\big(\varphi(\lambda)\big)^n} \right\|_Y \equiv \rho(Y) < +\infty . \tag{4}$$

6°. Let $f(x) \in (B^\infty)$ and also let $f_n(x) \in (B^\infty)$. Let λ be a given complex number. We say that a sequence $\{f_n(x)\}$ *converges in the set* $\big(E(\lambda)\big)$ *to* $f(x)$ if, for any given $Y \in \mathfrak{X}$ and for any given positive number ε, there corresponds a number $n(\varepsilon, Y)$ such that, for any $m \geqq n(\varepsilon, Y)$, we have

$$\left\| \frac{\mathfrak{D}^p \big\{ f_n(x) - f(x) \big\}}{\big(\varphi(\lambda)\big)^p} \right\|_Y < \varepsilon , \qquad (p = 0, 1, 2 \ldots) . \tag{5}$$

7°. A linear operation $\Lambda g(x)$ whose domain and range belong to $(A)_{t_0}$ is said *to be continuous in the set* $\big(E(\lambda)\big)$ if, whenever every $g_n(x)$ and $g(x)$ belong to $\big(E(\lambda)\big)$ and $\{g_n(x)\}$ converges in the set $\big(E(\lambda)\big)$ to $g(x)$, $\Lambda g_n(x)$ converges to $\Lambda g(x)$ in the generalised sense. A linear functional $L\{g(x)\}$ whose domain belongs to $(A)_{t_0}$ is said to be continuous in the set $\big(E(\lambda)\big)$ if, whenever every $g_n(x)$ and $g(x)$ belong to $\big(E(\lambda)\big)$ and $g_n(x)$ converges in the set $\big(E(\lambda)\big)$ to $g(x)$, $L\{g_n(x)\}$ tends to $L\{g(x)\}$ as $n \to \infty$.

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1) The function space given on footnote (7) on p. 147 is complete in this generalised sense.

2) The class (B^∞) is that which is consisted of infinitely many time differentiable function, if specially we put $\mathfrak{D}f(x) = f'(x)$.

3) cf B. A. Lengyel and M. H. Stone: Elementary Proof of the Spectral Theorem, Annals Math., **37** (1936). Specially Theorem 4, § 4. Our method is much suggested by their relevant method.

Let it be remarked that

(i) If $f(x) \in (E(\lambda))$, then, for every non-negative integer p, $\mathfrak{D}^p f(x) \in (E(\lambda))$.

(ii) If $f(x)$ and $g(x)$ belong to $(E(\lambda))$, then $f(x) + g(x) \in (E(\lambda))$.

(iii) If $f(x) \in (E(\lambda))$ and $|\varphi(\lambda_1)| > |\varphi(\lambda)|$, then $f(x) \in (E(\lambda_1))$.

8°. In the following we assume that if, for any non-negative integer p, $\{\mathfrak{D}^p g_n(x)\}$ converges to $h_p(x)$ in the generalised sense, then $h_p(x) \in (B^\infty)$ and we have $h_p(x) = \mathfrak{D}^p h_0(x)$, for every such p.[1]

2. After these preparations, the theorem which we will communicate may be stated in the following form:

Theorem. *Let $(A)_{t_0}$ be complete in the generalised sense, and let Λ be a linear operator which is permutable with $\mathfrak{D}$ and which is continuous in the set $\big(E(\lambda_1)\big)$. Let the linear functionals L_0 be continuous in the set $\big(E(\lambda_1)\big)$. Let $f(x)$ belong to $\big(E(\lambda_0)\big)$, and let it be a solution of the functional equation*

$$\Lambda f(x) = 0\,, \qquad (x \in Y_{t_0})\,. \tag{6}$$

Then a $\mathfrak{C}$-section of Cauchy-Delsarte's series of $f(x)$ with respect to Λ is independent of $\mathfrak{C}$, provided $\mathfrak{C}$ contains in its interior the part of $\mathfrak{M}$ for which $|\varphi(\lambda)| \leqq |\varphi(\lambda_0)|$ and $\mathfrak{C}$ is contained in the part of $\mathfrak{M}$ for which $|\varphi(\lambda)| \leqq |\varphi(\lambda_1)|$.[2]

Proof. We shall prove this theorem by showing that the residue in S which correspond to any point λ of $\mathfrak{M}$ for which $|\varphi(\lambda_1)| > |\varphi(\lambda)| > |\varphi(\lambda_0)|$ is equal to zero.

Here we may assume without loss of generality that $|\varphi(\lambda_0)| \neq 0$.

We shall put

$$g_n(x) = -\sum_{\nu=0}^{n} \frac{\mathfrak{D}^\nu f(x)}{\big(\varphi(\lambda)\big)^{\nu+1}}\,, \qquad (n = 1, 2, 3, \ldots)\,. \tag{7}$$

Then evidently every $g_n(x)$ belongs to $\big(E(\lambda_0)\big)$, and further, for any $Y \in Y_{t_0}$, we have, for $m > n$,

$$\leqq \frac{\rho(Y)}{|\varphi(\lambda)|} \sum_{p=n+1}^{m} \left| \frac{\varphi(\lambda_0)}{\varphi(\lambda)} \right|^p, \tag{8}$$

which, yields us, in virtue of the completeness of $(A)_{t_0}$ in the generalised sense, that $\{\mathfrak{D}^p g_n(x)\}$ converges to $h_p(x)$ in the generalised sense, for every non-negative integer p.

Therefore, by Assumption 8° of this note, $h_p(x) \in (B^\infty)$ and $\mathfrak{D}^p h_0(x) = h_p(x)$. The estimation (8) now gives us that

$$\mathfrak{D}^p h_0(x) = -\sum_{\nu=0}^{\infty} \frac{\mathfrak{D}^{p+\nu} f(x)}{\varphi(\lambda)^{\nu+1}}\,, \tag{9}$$

for every non-negative integer p, and further that $h_0(x) \in \big(E(\lambda_0)\big)$.

The expression (9) shows that $h_0(x)$ solves the functional equation

$$\mathfrak{D} h_0(x) = \varphi(\lambda) h_0(x) + f(x)\,. \tag{10}$$

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1) In the special case considered in footenote 2) in the previous page, $(E(\lambda))$ concides with the class which is consisted of all indefinitely many time differentiable functions with the grade not greater than $|\lambda|$. (In this case $\varphi(\lambda) = \lambda$).

2) If $\{f_n(x)\}$ converges in the set $(E(\lambda))$ to $f(x)$, then it does so in any set $(E(\lambda_1))$, whenever $\lambda_1 \in \mathfrak{M}$ and $|\varphi(\lambda_1)| > |\varphi(\lambda)|$.

Noticing (2), we shall now put

(11)
$$k_n(x)=g_n(x)-\frac{L_{t_0}\left\{g_n(x)\right\}}{L_{t_0}\left\{j_\lambda(x)\right\}}j_\lambda(x)$$

and

(12)
$$k(x)=h_0(x)-\frac{L_{t_0}\left\{h_0(x)\right\}}{L_{t_0}\left\{j_\lambda(x)\right\}}j_\lambda(x)\,.$$

Then $k(x)$ also statisfy the functional equation (10), and moreover $L_{t_0}\left\{k(x)\right\}=0$.

Consequently it follows that[1)]

(13)
$$k(x)=\mathfrak{L}_\lambda^{t_0}\left[f(x)\right].$$

Here let it be noticed that, since $j_\lambda(x)\in\left(E(\lambda)\right)$ and $|\varphi(\lambda)|>|\varphi(\lambda_0)|$ by Hypothesis, it follows that $k_n(x)$ and $k(x)$ belong to $\left(E(\lambda)\right)$.

In view of (11) and (12), we have

(14)
$$\left\|\frac{\mathfrak{D}^p\left\{k_n(x)-k(x)\right\}}{\left(\varphi(x)\right)^p}\right\|_Y\leqq\left\|\frac{\mathfrak{D}^p\left\{g_n(x)-h_0(x)\right\}}{\left(\varphi(\lambda)\right)^p}\right\|_Y$$
$$+\left|\frac{L_{t_0}\left\{g_n(x)-h_0(x)\right\}}{a_0}\right|\,\|j_\lambda(x)\|_Y$$

By the continuity of the linear functional L_{t_0}, the second term of the right-hand side converges to zero as $n\to\infty$.

Therefore, in combination with the estimation (8), it follows that $\left\{k_n(x)\right\}$ converges in the set $\left(E(\lambda)\right)$ to $k(x)$, and consequently, in virtue of the continuity of $\varLambda$ in $\left(E(\lambda)\right)$, $\left\{\varLambda k_n(x)\right\}$ converges to $\varLambda k(x)$ in the generallised sense.

On the other hand, since $\varLambda$ and $\mathfrak{D}$ are permutable with each other and $g(x)\in(B^\infty)$, it follows that

(15)
$$\varLambda\left\{g_n(x)\right\}=\varLambda\left\{-\sum_{\nu=0}^{n}\frac{\mathfrak{D}^\nu f(x)}{\left(\varphi(\lambda)\right)^{\nu+1}}\right\}=-\sum_{\nu=0}^{n}\frac{\varLambda\mathfrak{D}^\nu f(x)}{\left(\varphi(\lambda)\right)^{\nu+1}}$$
$$=-\sum_{\nu=0}^{n}\frac{\mathfrak{D}^\nu\varLambda f(x)}{\left(\varphi(\lambda)\right)^{\nu+1}}=0\,,$$

and consequently that $\varLambda h_0(x)=0$.

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1) By the assumption 1° and by the second principle of uniqueness setteled in the previous note. See [F] § 2.

Therefore we now know that

$$(16)\qquad \Lambda\left(\mathfrak{L}_\lambda^{t_0}\left[f(x)\right]\right)=\Lambda\left(k(x)\right)=-\frac{L_{t_0}\left\{h(x)\right\}}{a_0}\Lambda j_\lambda(x)$$

and that, if $\mathfrak{C}_\lambda$ is a sufficiently small circle about λ,

$$(17)\qquad S_{\mathfrak{C}_\lambda}(x, t_0; f)\equiv\frac{-1}{2\pi i}\oint_{\mathfrak{C}_\lambda}\frac{j_\lambda(x, t_0)}{G\left(\varphi(\lambda)\right)j_\lambda(t_0, t_0)}\Lambda\left(\mathfrak{L}_\lambda^{t_0}\left[f(t_0)\right]\right)d\varphi(\lambda)$$

$$=-\frac{1}{2\pi i a_0}\oint_{\mathfrak{C}_\lambda} j_\lambda(x, t_0)d\varphi(\lambda)=0\,.$$

which we were to prove.[1]

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1) The theorem we have proved corresponds to Schürer, loc. cit., C. 2. Alle Lösungen der L-gleichung unendlich hoher Ordnung im Bereiche $[L^\infty, q]$ sind Lösungen einer L-gleichung endlicher Ordnung. (p. 210).