50. The Composition of Permutable Functions and the Operational Calculus.

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1. It is well known that by the Pérès's transformation[1] the Volterra-Pérès's first composition of the functions which are permutable with a certain given function $f(x, y)$ of the first order[2] may be transformed into the calculus of the operators of closed cycle, i.e. of the linear translatable operators. The object of this note is to show that the theory may be also considered as a special case of the operational calculus under our formulation.[3] We will verify this by suitably defining the fundamental operator $\mathfrak{D}$.[4] In the following we may- and we shall assume that $f(x, y)$ is in the canonical form[5] in the domain $0 \leqq x \leqq y \leqq a$ $(a > 0)$.

2. The group of the fuctions $g(x, y)$[6] which are permutable with a given $f(x, y)$ can be generated by a Pérès's transformation,[7] i.e., by one of the transformations that maintain the property of composition: let us put

$$(1) \qquad h(x, y) \equiv \frac{\partial^2 f(x, y)}{\partial x \partial y}$$

1) We use the terminologies adopted in Volterra's book: Theory of functionals and of integral and integro-differential equations, (1930). See specially Chapter IV, Theory of composition and of permutable and of permutable functions. We quote this by [V]. See also Volterra-Pérès, Leçons sur la composition et les fonctions permutables, (1924). We quote this by [$V.$-$P.$].

2) For its definition, see [V] Chapter IV, § 7, Order of a function, (p. 110), and [$V.$-$P.$] Chapter I, § 10 (p. 10).

3) See T. Kitagawa: (1) A Formulation of Operational Calculus, These Proceedings, **13** (1937). (2) A Theorem on Orerational Equation, These Proceedings, **13** (1937). We quote these by [F] and [E] respectively.

4) $\mathfrak{D}$ prossesses the certain special properties. See [F] § 2.

5) That is to say, $f(x, y)$ is defined in the domain $0 \leqq x \leqq y \leqq a$, and

$$f(x, x)=1, \quad \left(\frac{\partial f(x, y)}{\partial x}\right)_{x=y} = \left(\frac{\partial f(x, y)}{\partial y}\right)_{x=y} = 0 \qquad (0 \leqq x \leqq a)$$

Further we assume that $\dfrac{\partial^2 f(x, y)}{\partial x \partial y}$ exists and is continuous with respect to (x, y). See [V] Chapter IV § 8, Group of the functions Permutable with a given Function (p. 111).

6) Each element $g(x, y)$ is defined and continuous in the domain $0 \leqq x \leqq y \leqq a$ and there

$$\int_x^y g(x, t) f(t, y)\, dt = \int_x^y f(x, t) g(t, y)\, dt,$$

that is to say,

$$\overset{**}{gf} = \overset{**}{fg}.$$

7) See [V] Chapter IV, § 9 Transformations which maintain the Law of Composition or Pérès's Transformations (p. 117). See also [$V.$-$P.$] Chapter IV, Les Transformations qui conservent la composition (pp. 56–67).

$$(2) \qquad l(x, y) \equiv -h - \overset{*}{h}\overset{*}{1}\overset{*}{h} - \overset{*}{h}\overset{*}{1}\overset{*}{h}\overset{*}{1}\overset{*}{h} - \cdots\cdots$$

$$(3) \qquad l_\eta(x, y) \equiv l(x+\eta, y+\eta)$$

and

$$(4) \qquad \varphi(\xi\,;\, x, y) \equiv \sum_{n=1}^{\infty} \int_0^{\xi} d\eta_n \int_0^{\eta_n} d\eta_{n-1} \int_0^{\eta_{n-1}} d\eta_{n-2} \cdots\cdots$$

$$\int_0^{\eta_2} d\eta_1 \overset{*}{l}\overset{*}{l}_{\eta_1} \overset{*}{l}_{\eta_2} \cdots\cdots \overset{*}{l}_{\eta_n}(x, y-\xi)\,,$$

then, by a suitable choice of a function $\mu(\xi)$ which is defined on the interval $0 \leqq \xi \leqq a$, $g(x, y)$ may be written as follows:

$$(5) \qquad g(x, y) = \mu(y-x) + \int_0^y \mu(\xi)\, \varphi(\xi\,;\, x, y)\, d\xi \equiv \Omega_x^y(\mu)\,,$$

for (x, y) belonging to the domain D defined by $0 \leqq x \leqq y \leqq a$. Such a function $\mu(\xi)$ $(0 \leqq \xi \leqq a)$ is uniquely determined for any given $g(x, y)$ defined on the domain D,[1] and conversely, for arbitrarily given function $\mu(\xi)$ which is defined and continuous on the interval $0 \leqq \xi \leqq a$, the righthand-side of (5) defines a function which is defined on D and which is permutable with $f(x, y)$ on the set D. Thus, by means of the transformation (5), we are able to define the one-to-one correspondence among the elements of $\{g(x, y)\}$ and those of the functions-set $\{\mu(\xi)\}$. Now we shall introduce:

Definition I. *If $g(x, y)$ is permutable with $f(x, y)$ and the function $\mu(\xi)$ defined by the Pérès transformation* (4) *is continuously differentiable on the interval $0 \leqq \xi \leqq a$, then $\mathfrak{D}_y\{g(x, y)\}$ is defined by*

$$(6) \qquad \mathfrak{D}_y\{g(x, y)\} \equiv \Omega_x^y\Big(\frac{d\mu(\xi)}{d\xi}\Big) \equiv \mu'(y-x) + \int_0^y \mu'(\xi)\, \varphi(\xi\,;\, x, y)\, d\xi\,.$$

In virtue of the facts that[2]

$$(7) \qquad \overset{*}{f}{}^{n+1}(x, y) = \Omega_x^y(\overset{*}{1}{}^n) = \Omega_x^y\Big(\frac{\xi^n}{n\,!}\Big)$$

and that

$$(8) \qquad |\overset{*}{f}{}^{n+1}(x, y)| \leqq \frac{|y-x|^n}{n\,!}\Big(1 + M\theta \frac{|y-x|}{n}\Big)$$

where $M = \underset{0 \leq x \leq y \leq a}{\mathrm{Max}} |f(x, y)|$ and $0 \leqq \theta \leqq 1$,
we obtain:

Theorem I. *For any x and y such that $0 \leqq x \leqq y \leqq a$, we have*

$$(9) \qquad \mathfrak{D}_y\{\overset{*}{f}{}^{n+1}(x, y)\} = \overset{*}{f}{}^{n}(x, y)\,, \qquad (n = 0, 1, 2, \ldots\ldots)$$

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1) This follows from the fact that there exists one and only one function $g(x, y)$ which is permutable with $f(x, y)$ and takes assigned values for $x=0$. See [V] Chapter IV, § 7 Order of a function, (p. 113).

2) See [V] Chapter IV § 9 (p. 115) and [V.-P.] Chapter IV, § 13 (p. 67).

Theorem II. *The function $j_\lambda(x, y)$ defined by the series*

$$j_\lambda(x, y) \equiv \sum_{n=0}^{\infty} \lambda^n \overset{*}{f}{}^{n+1}(x, y) \tag{10}$$

exists for any finite real or complex λ, and is continuous with respect to (x, y) in the domain $0 \leqq x \leqq y \leqq a$.

Owing to these facts we shall introduce

Definition II. *Functions $\{\overset{*}{f}{}^{n+1}(x, y)\}$ $(n=0, 1, 2, \ldots\ldots)$ are called as the Taylor-Bourlet series of y adjointed at the initial point x with respect to the operator $\mathfrak{D}$.*

3. We shall now consider three function-sets $(A)_t$, $(B)_t$ and $(C)_t$ which are associated with each t in $0 \leqq t \leqq a$ and which are defined as follows:[1]

1° $(A)_t$ is a class of all the functions of y which derive from elements of $\{g(x, y)\}$ by putting $x=t$; that is to say, $(A)_t$ is composed with all the functions of y $\{g(t, y)\}$.

2° $(B)_t$ is a class of all functions $\{g(t, y)\}$ which belong to $(A)_t$ and for which $\mathfrak{D}_y\{g(t, y)\}$ are defined by (5)

3° $(C)_t$ is a class of all functions $\{g(t, y)\}$ which belong to $(B)_t$ and such that $g(t, t)=0$.

Then we obtain[2]

Theorem III. *For any given $g(t, y) \in (A)_t$, the functional equation*

$$\mathfrak{D}_y\{h(t, y)\} = \lambda h(t, y) + g(t, y) \tag{11}$$

where y is any number subject to $t \leqq y \leqq a$, has one and only one solution in the function-set $(C)_t$.

Theorem IV. *If $g(t, y) \in (B)_t$ and $\mathfrak{D}_y\{g(t, y)\} = \lambda\, g(t, y)$ for every y subject to $t \leqq y \leqq a$, then there is a constant k such that $g(t, y) = k\, j_\lambda(t, y)$.*

Proof of Theorem III. First we notice that a function belonging $(B)_t$ is also an element of $(C)_t$ when and only when the corresponding $\mu(y)$ vanishes at $y=0$.

Then it is evident that the function

$$\sum_{n=0}^{\infty} \lambda^n \overset{*}{f}{}^{n+1} \overset{*}{g}(t, y) = \Omega_t^y\left(\int_0^{\xi} e^{\lambda(\xi-s)} \mu(s) ds\right) \tag{12}$$

belongs to $(C)_t$ and solves the functional equation.

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1) P. Flamant: Le dévelopement d'une transmutation linéaire par rapport à la différentiation finie. Rendic. d. Circ. Math. Palermo, **54** (1930).

See specially pp. 372–374. The name "Taylor-Bourlet series" originates from this paper.

The following papers are also intimately connected with this series.

G. D. Birkhoff: Sur une généralisation de la série de Taylor, Comptes Rendus, **164** (1917) pp. 942–945.

D. V. Widder: On the expansion of analytic functions of the complex variable in generalised Taylor's series, Trans. Am. Math. Soc., **31** (1929) pp. 43–52.

I. M. Sheffer: Concerning some methods of best approximation, and a theorem of Birkhoff, American Journ. Math., **57** (1935) (pp. 587–614). See specially p. 591.

2) For the meaning of the function-sets $(A)_t$, $(B)_t$ and $(C)_t$, see [F] § 2.

On the other hand if there is any solution in $(C)_t$, then the one-to-one correspondence between $\{g(x, y)\}$ and $\{\mu(\xi)\}$ leads us, in view of (12)[1] that the function $\nu(y)$ which corresponds to such $h(x, y)$ should solve the functional equation

$$\frac{d\nu(y)}{dy}=\lambda\nu(y)+\mu(y)\,, \tag{13}$$

and the $\nu(0)=0$. These two facts completes our proof. Similarly we may prove Theorem IV.

In virtue of these two facts, we know that the composition of permutable functions and allied functional equation may be reduced to the operational calculus and to allied functional equations which we have formulated previously. Consequently we are able to construct the Cauchy-Delsarte's series, with respect to the group of functions which are permutable with $f(x, y)$ which will, in some points, contribute to make *directly* clear the formal calculus of the operators. A corresponding problem for the case when $f(x, y)$ is of higher order remains unsolved.

1) Cf. The uniquencess-principles of $\mathfrak{D}$. See [F], § 2.