49. On Differential Operators permutable with Lie Continuous Groups of Transformations.

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1. In the present paper, we shall generalize Casimir's theorem[1] on semi-simple continuous groups, which may be stated as follows:

Let $X_1, X_2, \ldots\ldots, X_r$ generate a semi-simple continuous group and satisfy the law of compositions such that

$$[X_i, X_k]=C^{\alpha}_{ik}X_{\alpha}, \qquad (i, k=1, 2, \ldots\ldots, r).$$

If (g^{ik}) denotes the inverse matrix of the coefficient matrix $(C^{\alpha}_{i\beta}C^{\beta}_{k\alpha})$ of Cartan's quadratic form

$$\varphi(\lambda, \lambda)=C^{\alpha}_{i\beta}C^{\beta}_{k\alpha}\lambda^i\lambda^k,$$

then the differential operator of the second order

$$P(X)=g^{ik}X_iX_k$$

is permutable with every element X_ω, that is,

$$X_\omega P(X)=P(X)X_\omega, \qquad (\omega=1, 2, \ldots\ldots, r).$$

By means of this theorem, Profs. B. L. van der Waerden,[2] H. Casimir and Richard Brauer[3] gave the algebraic proof of Weyl's theorem[4] that all reducible representations of semi-simple continuous group are completely reducible.

2. In general, we assume that an r-parametric continuous group G of transformation is generated by r infinitesimal transformations

$$X_\omega=\xi^k_\omega(x^1_, x^2_, \ldots\ldots, x^n)\frac{\partial}{\partial x^k}, \qquad (\omega=1, 2, \ldots\ldots, r),$$

where $\xi^k_\omega(x^3_, x^2_, \ldots\ldots, x^n)$ are analytic in a neighborhood of the origin. Then, we consider the symmetric differential operators of the ν-th order, defined as follows:

$$P_0(X)=g, \qquad P_1(X)=g^iX_i, \qquad P_2(X)=g^{ik}X_iX_k,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots, \qquad P_\nu(X)=g^{ikj\ldots l}X_iX_kX_j\ldots\ldots X_l,$$

where

$$g^{ik}=g^{ki}, \qquad \ldots\ldots\ldots, \qquad g^{ikj\ldots l}=g^{kij\ldots l},$$
$$g^{ikj\ldots l}=g^{ijk\ldots l} \quad \text{etc.},$$

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1) H. Casimir: Proc. Kon..Acad. Amsterdam, **34** (1931), 844.
K. Toyoda: Japanese Journal of Mathematics, **12** (1935), 17.
2) H. Casimir und B. L. van der Waerden: Math. Annalen, **111** (1935), 1.
3) R. Brauer: Math. Zeitschrift, **41** (1936), 330.
4) H. Weyl: Math. Zeitschrift, **24** (1926), 328.

and we form

$$P(X)=\sum P_\nu(X)=\sum g^{ik\cdots l}X_iX_k\cdots\cdots X_l\,.$$

First, we have

Theorem 1. *If a symmetric polynomial $P(\Lambda)$ be an absolute invariant of the contragredient adjoint group H^* generated by r infinitesimal transformations*

$$E^*_\omega=\Lambda_a C^a_{\omega k}\frac{\partial}{\partial\Lambda_k}\,,\qquad (\omega=1,2,\cdots\cdots,r)\,,$$

then the corresponding differential operator $P(X)$ is permutable with every element X_ω, that is

$$X_\omega P(X)=P(X)X_\omega\,,\qquad (\omega=1,2,\cdots\cdots,r)\,.$$

Proof. By the law of compositions

$$[X_\omega,X_k]=C^a_{\omega k}X_a=E^*_\omega X_k\,,\qquad (\omega,k=1,2,\cdots\cdots,r)\,,$$

we get

$$e^{X_\omega}X_k\,e^{-X_\omega}=e^{E^*_\omega}X_k\,.$$

Now, for the sake of simplicity, we consider a particular case where

$$P(X)=g^{ik}X_iX_k\,,$$

then, we have

$$e^{X_\omega}P(X)e^{-X_\omega}=g^{ik}e^{X_\omega}X_i\,e^{-X_\omega}\,e^{X_\omega}X_k\,e^{-X_\omega}$$

$$=g^{ik}e^{E^*_\omega}X_i\,e^{E^*_\omega}X_k\,.$$

Hence, supposing that a symmetric polynomial $P(\Lambda)$ is an absolute invariant of the contragredient adjoint group H^*, it follows that

$$e^{X_\omega}P(X)=P(X)\,e^{X_\omega},\qquad (\omega=1,2,\cdots\cdots,r)\,,$$

that is

$$X_\omega P(X)=P(X)X_\omega\,,\qquad (\omega=1,2,\cdots\cdots,r)\,.$$

Remark. In order that we exclude the condition of symmetry, we have to consider an invariant bilinear form $P(\Lambda,\Lambda^*)=g^{ik}\Lambda_i\Lambda^*_k$ instead of an invariant quadratic form $P(\Lambda)=g^{ik}\Lambda_i\Lambda_k$.

Corollary. *If a complete system of linear partial differential equations*

$$E^*_\omega F(\Lambda)=0\,,\qquad (\omega=1,2,\cdots\cdots,r)\,,$$

has s independent symmetric solutions $F_1(\Lambda)$, $F_2(\Lambda)$, ……, $F_s(\Lambda)$, then arbitrary function $\Omega\big(F_1(X),F_2(X),\cdots\cdots,F_s(X)\big)$ of $F_1(X)$, $F_2(X)$, ……, $F_s(X)$, is permutable with every element X_ω.

3. Let the parameter group[1] G_0 of G be generated by r infinitesimal transformations

1) K. Toyoda: Science Reports of the Tohoku Imperial University, **24** (1935), 269.

$$A_\omega = a_\omega^k(\lambda^1, \lambda^2, \ldots\ldots, \lambda^r)\frac{\partial}{\partial\lambda^k}, \qquad (\omega = 1, 2, \ldots\ldots, r),$$

where λ^k denote canonical parameters and $a_i^k(0)=\delta_i^k$ is Kronecker's delta.

Then, we have

Theorem 2. *In order that a symmetric differential operator*

$$P(A) = \sum_{\nu=0}^{p} P_\nu(A) = \sum g^{ik\ldots l} A_i A_k \ldots\ldots A_l$$

is permutable with every element A_ω, *it is necessary that each symmetric polynomial* $P_\nu(A)$, $(\nu=0, 1, \ldots\ldots, p)$ *is an absolute invariant of the contragredient adjoint group* H^*.

Proof. If we suppose that

$$P(A) = g^i A_i + g^{ik} A_i A_k = 0,$$

we get

$$g^i a_i^\alpha(\lambda)\frac{\partial}{\partial\lambda^\alpha} + g^{ik} a_i^\alpha(\lambda)\frac{\partial a_k^\beta(\lambda)}{\partial\lambda^\alpha}\frac{\partial}{\partial\lambda^\beta} + g^{ik} a_i^\alpha(\lambda) a_k^\beta(\lambda)\frac{\partial^2}{\partial\lambda^\alpha\partial\lambda^\beta} = 0,$$

whence we obtain $g^{ik}=0$ and consequently $g^i=0$.

Also, we have

Theorem 3. *Let* $P(x)$ *be a differential operator permutable with every element* X_ω *and* $g(X)$ *be an absolute invariant of the group* G. *If* $f(x)$ *be a solution of the differential equation*

$$P(X)f(x) = g(x),$$

then $f(e^{X_\omega}x)$ *is also a solution of the same differential equation.*

Proof. If $f(x)$ be a solution of the partial differential equation

$$P(X)f(x) = g(x),$$

then we get

$$P(X)f(e^{X_\omega}x) = P(X)e^{X_\omega}f(x) = e^{X_\omega}P(X)f(x)$$

$$= e^{X_\omega}g(x) = g(x).$$

4. Finally, we shall give another proof for Casimir's theorem, which runs as follows.

Theorem 4. *If* $X_1, X_2, \ldots\ldots, X_r$ *generate a semi-simple continuous group and* (g^{ik}) *be the inverse matrix of the coefficient matrix* $(C_{i\beta}^\alpha C_{k\alpha}^\beta)$ *of Cartan's quadratic form*

$$\varphi(\lambda, \lambda) = C_{i\beta}^\alpha C_{k\alpha}^\beta \lambda^i \lambda^k,$$

then the differential operator of the second order

$$P(X) = g^{ik} X_i X_k$$

is permutable with every element X_ω.

Proof. Since Cartan's quadratic form

$$\varphi(\lambda, \lambda) = C_{i\beta}^\alpha C_{k\alpha}^\beta \lambda^i \lambda^k = g_{ik}\lambda^i\lambda^k$$

is an absolute invariant of the adjoint group[1] H generated by r infinitesimal transformations

$$E_\omega = -\lambda^a C^k_{\omega a}\frac{\partial}{\partial \lambda^k}, \qquad (\omega=1, 2, \ldots\ldots, r),$$

we have

$$E_\omega \varphi(\lambda, \lambda) = \varphi(\lambda, E_\omega\lambda) + \varphi(E_\omega\lambda, \lambda)$$
$$= 2\varphi(\lambda, E_\omega\lambda) = 0,$$

whence

$$g_{ia}C^a_{\omega k} + g_{ka}C^a_{\omega i} = 0, \qquad (\omega, i, k = 1, 2, \ldots\ldots, r).$$

Therefore, we obtain

$$g^{ia}C^k_{\omega a} + g^{ka}C^i_{\omega a} = 0, \qquad (\omega, i, k = 1, 2, \ldots\ldots, r),$$

which shows that the symmetric quadratic form

$$P(\Lambda) = g^{ik}\Lambda_i\Lambda_k$$

is an absolute invariant of the contragredient adjoint group H^*, therefore by means of Theorem 1 we can prove the following

Corollary. *If $X_1, X_2, \ldots\ldots, X_r$ generate a semi-simple continuous group, then the determinants of all matrices (C^ω_{ik}), $(\omega=1, 2, \ldots\ldots, r)$, vanish simultaneously.*

Furthemore, we have

Theorem 5.[2] *In order that a continuous group G contains an element other than the identical element in the central, it is necessary and sufficient that there exists a differential operator $P(X)$ of the first order which is permutable with every element X_ω.*

Proof. If a differential operator of the first order

$$P(X) = g + g^i X_i$$

is permutable with every element X_ω, then we have

$$E^*_\omega P(\Lambda) = g^i \Lambda_a C^a_{\omega i} = 0, \qquad (\omega=1, 2, \ldots\ldots, r),$$

which shows that $g^i X_i$ is contained in the central.

Remark. But, if G is a soluble group generated by X_1, X_2 such that $[X_1, X_2] = X_1$, then G has no differential operator $P(X)$ permutable with every element X_ω.

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1) K. Toyoda: Science Reports of the Tohoku Imperial University, **25** (1936), 621.

2) This theorem was remarked by Prof. Kôsaku Yosida. (全國紙上數學談話會, 123 號).