

PAPERS COMMUNICATED

48. Some Remarks on the Uniqueness of Solution of a Differential Equation.

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I. Taking a function $f(x, y)$, continuous and limited ($\leq M$) in the domain R

$$0 \leq x \leq l, \quad -\infty < y < +\infty, \quad (1)$$

we consider the differential equation

$$y' = f(x, y) \quad (2)$$

and its solution passing through the origin $(0, 0)$, the existence of which being well known.

Many criterions were found for the uniqueness of such solution. Almost all of those criterions are established by means of an auxiliary differential equation, whose non-negative solution satisfying the initial condition

$$y(+0) = 0 \quad y'(+0) = 0 \quad (3)$$

is known to be $y \equiv 0$ uniquely.

In the place of such auxiliary differential equation, we may take a system S of curves, which is not necessarily a system of integral curves of a simple differential equation. We only require that the system S satisfies the following conditions.

(I.) Every curve of S lies within the domain R'

$$0 < x < l, \quad 0 < y \quad (4)$$

Every curve is simple and has continuous tangent. Its end points either approach indefinitely to the boundary $x=0$ or l of R' or tend to infinity.

(II.) Every curve of S is so oriented that the boundary $y=0$ of R' lies on the right hand side of the curve. Passing through any point (x, y) of R' , there goes at least a curve of S . The amplitude of the oriented tangent of the curve at (x, y) is determinate. We denote it by $\theta(x, y)$.

(III.) There is no curve of S , which approaches indefinitely to the origin, touching ultimately the axis of x .

Under such circumstances, we consider a non-negative differentiable function

$$y = \varphi(x), \quad 0 \leq x \leq l \quad (5)$$

If $\varphi(x)$ is not identically zero, and if the curve (5) always cross the curves of S from left to right, as x increases, then the portion of the

curve (5) for $0 \leq x < c$, where $\varphi(c) > 0$, lies wholly on the left side of a curve of S , passing through the point $(c, \varphi(c))$, so that $\varphi(0)$ may not be zero or at least the condition

$$\varphi(0) = 0, \quad \varphi'(0) = 0 \quad (6)$$

can not be satisfied. This means analytically that if the inequality

$$0 < \theta\{x, \varphi(x)\} - \tan^{-1} \varphi'(x) < \pi \pmod{2\pi} \quad (7)$$

holds good for the function (5) at every point x for which $\varphi(x) > 0$, then the condition (6) can not be satisfied. Here *the value of* \tan^{-1} is taken in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

From this fact, we can introduce, in a usual manner, the following criterion of uniqueness.

If we have

$$0 < \theta(x, y_2 - y_1) - \tan^{-1} \{f(x, y_2) - f(x, y_1)\} < \pi \pmod{2\pi} \quad (8)$$

for all $y_2 > y_1$, then the equation (2) can not have more than one solutions passing through the origin.

For, if there are many solutions, we must have the minimum and the maximum solutions $y_1(x)$ and $y_2(x)$. Putting them in the places of y_1 and y_2 of (8) respectively, we get the inequality (7) for the function $\varphi(x) = y_2(x) - y_1(x)$ which is non-negative and is not identically zero. Hence the condition

$$y_2(0) - y_1(0) = 0, \quad y_2'(0) - y_1'(0) = 0 \quad (9)$$

can not be satisfied. This is an absurd. Q. E. D.

Many other forms of criterions can be deduced, in a usual manner, by taking the system S instead of the auxiliary differential equation. In all cases, we can not replace $<$ of (8) by \leq . This is a defect of the theory.

II. The following criterion comes out from the entirely different point of view.

Let $A(y_1, y_2)$ be totally differentiable for all $y_1 \leq y_2$, and let

$$A(y, y) = y, \quad A(y_1, y_2) > y_2 \quad \text{for } y_1 < y_2 \quad (10)$$

If we have

$$A_{y_1}(y_1, y_2) f(x, y_1) + A_{y_2}(y_1, y_2) f(x, y_2) \leq f\{x, A(y_1, y_2)\} \quad (11)$$

for all $0 \leq x \leq l$, $y_1 \leq y_2$, then the equation (2) can not have more than one solutions.

For if we get the minimum and the maximum solutions $y_1(x)$ and $y_2(x)$, then, in virtue of (11), the function $\psi(x) = A\{y_1(x), y_2(x)\}$ must satisfy

$$\psi'(x) \leq f\{x, \psi(x)\}, \quad \psi(0) = 0 \quad (12)$$

Hence

$$\psi(x) = A\{y_1(x), y_2(x)\} \leq y_2(x) \quad (13)$$

which contradicts (10) at the point x for which $y_1(x) < y_2(x)$. Q. E. D.

If we take

$$A(y, y) = y, \quad A(y_1, y_2) < y_1 \quad \text{for } y_1 < y_2 \quad (14)$$

instead of (10) and invert the inequality sign of (11), then we get another criterion, parallel to the preceding.

III. Evidently, it is sufficient, for the same purpose, that the relations (8), (10) and (11) should be satisfied only for all y_1 and y_2 in the triangular domain

$$0 \leq x \leq l, \quad -Mx \leq y_1 \leq y_2 \leq Mx \quad (15)$$

(or the like domains) within which all solutions should lie.

Putting $A(y_1, y_2) = y_1 + t(y_2 - y_1)$, we can easily see that the solution of (2) is unique, when $f(x, y)$ is either convex or concave with respect to y in a wider triangular domain containing (15), as Mr. T. Kitagawa proved it more precisely in Japanese Journ. of Math., 9 (1932).
