53. *On the Fundamental Theorem of the Tensor Calculus.*

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Let $\mathfrak{G}$ be a connected semi-simple Lie group with s real parameters (r. p.). Consider the

Problem of the construction of all matrix groups continuously homomorphic to the universal covering group $\tilde{\mathfrak{G}}$ of $\mathfrak{G}$.
This Problem was treated by H. Weyl in his famous memoirs.[1] His method consists in the combined use of the "unitarian trick" and the theory of "Gewichts."

In the present note I intend to show that the Problem can be solved, in principle, without the theory of "Gewichts." We make use of the *theory of almost periodic functions* in place of the theory of "Gewichts." Then we easily obtain the fundamental theorem of the tensor calculus, as was developed in Weyl's paper loc. cit.

The Lemma below will make clear a critical point which, so far as I know, has never been considered in the literature. It asserts that any matrix Lie ring $\mathfrak{D}$ homomorphic to the Lie ring of $\mathfrak{G}$ coincides with the Lie ring of the Lie group *generated* by $\mathfrak{D}$.

I. *A Lemma.* Let a matrix group $\mathfrak{D}$ be continuously homomorphic to $\tilde{\mathfrak{G}}$. $\tilde{\mathfrak{G}}$ being semi-simple with $\mathfrak{G}$, the homomorphic mapping $\tilde{\mathfrak{G}} \to \mathfrak{D}$ is *open*[2] (*Gebietstreu*). Thus $\mathfrak{D}$ is locally compact with $\tilde{\mathfrak{G}}$ and hence $\mathfrak{D}$ is a Lie group.[3] The Lie ring with r. p. of $\mathfrak{D}$ and $\mathfrak{G}$ be denoted by $\mathfrak{R}_{\mathfrak{D}}$ and $\mathfrak{R}_{\mathfrak{G}}$ respectively. As $\mathfrak{D}$ is *open* homomorphic to $\tilde{\mathfrak{G}}$, $\mathfrak{D}$ is locally *open* homomorphic to $\mathfrak{G}$. Thus $\mathfrak{R}_{\mathfrak{D}}$ is homomorphic to $\mathfrak{R}_{\mathfrak{G}}$ with real numbers as *Operatorenbereich.*

Next let a matrix Lie ring $\mathfrak{R}_{\mathfrak{D}'}$ with r. p. be homomorphic to $\mathfrak{R}_{\mathfrak{G}}$ with real numbers as Operatorenbereich. We denote by $\bar{\mathfrak{D}}'$ the Lie *group germ* whose Lie ring is $\mathfrak{R}_{\mathfrak{D}'}$. Let $\mathfrak{D}'$ be the set of all the products of the form $A_1A_2 \cdots A_k$, $A_i \in \bar{\mathfrak{D}}'$, and the limit matrices of such products, so long as they are non-singular. $\mathfrak{D}'$ is called the group *generated* by $\mathfrak{R}_{\mathfrak{D}'}$. It is a connected locally compact group, and hence it is a Lie group.[3] The Lie ring with r. p. of $\mathfrak{D}'$ coincides with $\mathfrak{R}_{\mathfrak{D}'}$, as $\mathfrak{R}_{\mathfrak{D}'}$ is semi-simple with $\mathfrak{R}_{\mathfrak{G}}$.[4] Therefore the connected matrix group $\mathfrak{D}'$ is continuously (in reality, *open*) homomorphic to $\tilde{\mathfrak{G}}$.

Summing up we have the

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1) H. Weyl: Math. Zeitsch. **23** (1925), **24** (1926).

2) K. Yosida: Tohoku Math. J. **43**, 2 (1937). For the general properties of the continuous (not necessarily open) representation of the topological group Cf. K. Yosida: Jap. J. of Math. **13** (1937) and K. Yosida: Proc. **12** (1936).

3) J. von Neumann: Math. Zeitsch. **30** (1929). See also K. Yosida: Jap. J. of Math. **13** (1936).

4) K. Yosida: Tôhoku Math. J. loc. cit.

Lemma. A connected matrix group $\mathfrak{D}$ *continuously homomorphic to* $\tilde{\mathfrak{G}}$ *is a Lie group whose Lie ring* $\mathfrak{R}_{\mathfrak{D}}$ *with r. p. is homomorphic to the Lie ring* $\mathfrak{R}_{\mathfrak{G}}$ *with r. p. of* $\mathfrak{G}$. *Conversely the group* $\mathfrak{D}$ *generated by a matrix Lie ring* $\mathfrak{R}_{\mathfrak{D}}$ *with r. p. homomorphic to* $\mathfrak{R}_{\mathfrak{G}}$ *with r. p. is a connected Lie group continuously homomorphic to* $\tilde{\mathfrak{G}}$, *such that the Lie ring with r. p. of* $\mathfrak{D}$ *coincides with* $\mathfrak{R}_{\mathfrak{D}}$. *In these statements the term "continuously" can be replaced by the term "open."*

Remark. The above lemma is not always true if $\mathfrak{G}$ is not semi-simple, as the following example shows us:

$$\mathfrak{G}=\text{additive group of real numbers,}$$
$$\mathfrak{R}_{\mathfrak{D}}=\left\{\begin{Vmatrix}\sqrt{-1} & 0\\ 0 & \sqrt{-1}\,a\end{Vmatrix}\right\}\text{where}\quad a/\pi \neq \text{rational.}$$

II. *Reduction of the Problem.* $\mathfrak{G}$ being semi-simple, the adjoint representation $\mathfrak{A}$ of $\mathfrak{G}$ is locally topologically isomorphic to $\mathfrak{G}$. Thus we may replace $\tilde{\mathfrak{G}}$ by the universal covering group $\tilde{\mathfrak{A}}$ of $\mathfrak{A}$ in the Problem.

Let the Lie ring with r. p. of $\mathfrak{A}$ be denoted by $\mathfrak{R}_{\mathfrak{A}}$. For any Lie ring $\mathfrak{R}$ with r. p. we denote by $\{\mathfrak{R}\}$ the Lie ring obtained from $\mathfrak{R}$ by regarding these r. p. as complex parameters (c. p.). Surely[1] $\{\mathfrak{R}_{\mathfrak{A}}\}$ is semi-simple with $s(2s)$ c. p. (r. p.).

If a matrix Lie ring $\mathfrak{R}_{\mathfrak{D}}$ with r. p. is homomorphic to $\mathfrak{R}_{\mathfrak{A}}$ with r. p., $\{\mathfrak{R}_{\mathfrak{D}}\}$ is homomorphic to $\{\mathfrak{R}_{\mathfrak{A}}\}$ with complex numbers as Operatorenbereich. Conversely let a matrix Lie ring $\mathfrak{R}'_{\mathfrak{D}'}$ with c. p. be homomorphic to $\{\mathfrak{R}_{\mathfrak{A}}\}$ with c. p., then the homomorphic image $\mathfrak{R}_{\mathfrak{D}'}$ of $\mathfrak{R}_{\mathfrak{A}}$ is a Lie ring with r. p.

Hence, by the Lemma, the Problem is reduced to the

Problem I of the construction of all matrix Lie rings with c. p. homomorphic to $\{\mathfrak{R}_{\mathfrak{A}}\}$ *with complex numbers as Operatorenbereich.*

The Lie ring $\{\mathfrak{R}_{\mathfrak{A}}\}$ with $2s$ r. p. has a base $(x_1, x_2, \dots, x_s, \sqrt{-1}\,x_1, \sqrt{-1}\,x_2, \dots, \sqrt{-1}\,x_s)$ such that $(x_1, x_2, \dots, x_s)$ with s r. p. constitutes the Lie ring of a semi-simple Lie group germ $\bar{\mathfrak{A}}_u$ of unitary matrices with s r. p. This results from the "unitarian trick" of H. Weyl.[2] By the Lemma, the semi-simple $\bar{\mathfrak{A}}_u$ is a vicinity of the connected compact semi-simple Lie group $\mathfrak{A}_u$ generated by $(x_1, x_2, \dots, x_s)$ with s r. p.

Thus the Problem I is reduced to the

Problem II of the construction of all matrix Lie rings with r. p. homomorphic to the Lie ring $(x_1, x_2, \dots, x_s)$ *with s r. p. of* $\mathfrak{A}_u$.

As $\mathfrak{A}_u$ is compact semi-simple, the universal covering group $\tilde{\mathfrak{A}}_u$ of $\mathfrak{A}_u$ is compact by Weyl's theorem.[3] By Schreier's theorem,[4] any connected matrix group locally *open* homomorphic to $\mathfrak{A}_u$ is open homomorphic to $\tilde{\mathfrak{A}}_u$ and vice versa.

1) E. Cartan: Ann. Ecol. Norm. Sup. **31** (1914). The elements of $\mathfrak{R}_{\mathfrak{A}}$ are real matrices.

2) H. Weyl: loc. cit.

3) H. Weyl: loc. cit.

4) O. Schreier: Abh. Math. Seminar Hamburg, **4** (1926).

$\tilde{\mathfrak{A}}_u$ being semi-simple we see, by the Lemma, that the Problem II is equivalent to the

Problem III of the construction of all continuous representations of $\tilde{\mathfrak{A}}_u$.

III. *Solution of the Problem III.* $\tilde{\mathfrak{A}}_u$ is a connected compact Lie group. Hence, by von Neumann-Pontrjagin-Freudenthal's theorem,[1] there exists a matrix group $\mathfrak{M}$ topologically isomorphic to $\tilde{\mathfrak{A}}_u$.

As $\mathfrak{M}$ is a compact matrix group, all continuous representations of $\mathfrak{M}$ can be obtained by taking complex conjugates and decomposing Kronecker's products from $\mathfrak{M}$, by van Kampen's theorem.[2]

Therefore the Problem III and hence the original Problem is completely solved.

Remark. A compact matrix group is *completely reducible.*[3] Thus the above $\mathfrak{M}$ is obtained as the *sum representation* of a finite number of suitably chosen *irreducible* continuous representations of $\tilde{\mathfrak{A}}_u$. Hence, by the Lemma and the arguments in II, we see that the Lie ring with r. p. of such matrix group $\mathfrak{M}$ with r. p. can be obtained in the following way. There exists a finite number of *irreducible* matrix Lie rings $\mathfrak{R}_1, \mathfrak{R}_2, \ldots, \mathfrak{R}_l$ with r. p. homomorphic to $(x_1, x_2, \ldots, x_s)$ with r. p. such that i) the *sum Lie ring*

$$\mathfrak{R} = \left\| \begin{matrix} \mathfrak{R}_1 & & & \\ & \mathfrak{R}_2 & & \\ & & \ddots & \\ & & & \mathfrak{R}_l \end{matrix} \right\|$$

with r. p. is isomorphic to $(x_1, x_2, \ldots, x_s)$ with s r. p. and ii) the Lie group generated by $\mathfrak{R}$ is simply connected. Such $\mathfrak{R}$ is precisely the Lie ring with r. p. of the desired group $\mathfrak{M}$.

IV. *The fundamental theorem of the tensor calculus.* The Lie ring $\{\mathfrak{R}_{\mathfrak{S}}\}$ with c. p. is isomorphic to the matrix Lie ring $\{\mathfrak{R}_{\mathfrak{A}}\}$ with c. p. Hence any matrix Lie ring with c. p. homomorphic to $\{\mathfrak{R}_{\mathfrak{S}}\}$ with c. p. is *completely reducible* and can be obtained by decomposing the *infinitesimal Kronecker's product* from the matrix Lie ring with c. p.

$$\left\| \begin{matrix} \{\mathfrak{R}_1\} & & & \\ & \{\mathfrak{R}_2\} & & \\ & & \ddots & \\ & & & \{\mathfrak{R}_l\} \end{matrix} \right\|.$$

Here $\mathfrak{R}_i$ are the Lie rings with r. p. obtained in III. Hence $\{\mathfrak{R}_i\}$ with c. p. are *irreducible* matrix Lie rings homomorphic to $\{\mathfrak{R}_{\mathfrak{S}}\}$. This constitutes a generalisation of the fundamental theorem of the tensor calculus.[4]

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1) J. von Neumann: Ann. of Math. **34** (1933). L. Pontrjagin: C. R. **198** (1934). H. Freudenthal: Ann. of Math. **37** (1936).

2) E. R. van Kampen: Ann. of Math. **37** (1936).

3) J. von Neumann: Trans. American Math. Soc. **36**, 3 (1934).

4) Cf. H. Weyl: loc. cit.