73. *Abstract Integral Equations and the Homogeneous Stochastic Process.*

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§ 1. *Introduction.* Let each point x of a complex Banach space $\mathfrak{B}$ represent a state (x) of a physical or a mathematical system. Consider a *temporally homogeneous stochastic process* by which the state (x) is transferred to the state (y) after the elapse of a unit time. We assume that this transition is realised by a *linear mapping* T in $\mathfrak{B}$: $y=T\cdot x$. Under their respective restrictions on T and on $\mathfrak{B}$, A. Markov, B. Hostinsky, M. Fréchet, N. Kryloff-N. Bogoliouboff and other authors investigated the asymptotic behaviour of the n-th iterate T^n of T for large n. In the present note I intend to treat the problem by the *abstract integral equations* due to F. Riesz[1] and the theory of *resolvents* due to M. Nagumo.[2] The theorem below is a generalisation of Fréchet-Kryloff-Bogoliouboff's theorem.[3] The lemma 1 and the lemma 3 respectively generalise the theorem of Riesz and that of Nagumo. I express my hearty thanks to S. Kakutani who kindly collaborated with me in the discussion of the present note.[4] In the next paper[5] the *mean ergodic theorem* of J. von Neumann is extended to $\mathfrak{B}$, in a way as to be applied to the problem of the homogeneous stochastic process.

§ 2. *The theorem.* A linear mapping T of a complex Banach space $\mathfrak{B}$ in $\mathfrak{B}$ is called a (linear) *operator* in $\mathfrak{B}$. T is called *continuous* if its *norm* (absolute value) $\|T\| = \underset{\|x\|\leq 1}{\text{l.u.b.}} \|T\cdot x\|$ is finite. A continuous operator T is called *completely continuous* if it maps the unit sphere $\|x\|\leq 1$ of $\mathfrak{B}$ on a compact point set in $\mathfrak{B}$.

Let T satisfy the following two conditions:

(1) there exists a completely continuous operator V such that $\|T-V\|<1$,

(2) there exists a constant α such that $\|T^n\|\leq\alpha$ for $n=1, 2, \ldots$.

Then we obtain the

Theorem. The proper values of T with modulus 1 are isolated proper values of finite multiplicities. Let these proper values be $\lambda_1, \lambda_2, \ldots\ldots, \lambda_k$. Then there exist completely continuous operators $T_1, T_2 \ldots\ldots, T_k$, a continuous operator S and positive constants β, ε such that

1) Acta Math. **41** (1918), 71–98.

2) Jap. J. of Math. **13** (1936), 75–80.

3) M. Fréchet: Quart. J. of Math. **5** (1934), 106–144. N. Kryloff and N. Bogoliouboff: C. R. Paris, **204** (1937), 1386–1388.

4) He also obtained another proof of our theorem, by virtue of the mean ergodic theorem in $\mathfrak{B}$. See the following paper of Kakutani.

5) Proc. **14** (1938), 292.

$$(3)\quad \begin{cases} T=\sum_{i=1}^{k}\lambda_i T_i+S,\ T_i^2=T_i,\ T_iT_j=0(i\neq j),\ T_iS=ST_i=0 \\ \qquad\qquad\qquad\qquad\qquad\qquad (i,j=1,2,\dots,k), \\ \|S^n\|\leqq \beta/(1+\varepsilon)^n \qquad (n=1,2,\dots\dots). \end{cases}$$

Corollary 1. There exist positive constants γ_λ *such that, if* $|\lambda|=1$,

$$(4)\quad \begin{cases} \left\| \dfrac{(T/\lambda)+(T/\lambda)^2+\cdots+(T/\lambda)^n}{n} - T_\infty(\lambda)\right\| \leq \gamma_\lambda/n \qquad (n=1,2,\dots), \\ T_\infty(\lambda)=T_i \ \textit{if}\ \lambda=\lambda_i,\ T_\infty(\lambda)=0\ \textit{if}\ \lambda\neq\lambda_1,\lambda_2,\dots,\lambda_k. \end{cases}$$

Corollary 2. $(T/\lambda)^n$ *converges (necessarily to* $T_\infty(\lambda)$*) if and only if there are no proper values of* T *with modulus 1 other than* λ.

Corollary 3. We replace the condition (1) by

$$(5)\quad \begin{cases} \textit{there exist positive integer } m \textit{ and a completely continuous} \\ \textit{operator } V \textit{ such that } \|T^m-V\|<1. \end{cases}$$

Then there exist positive constants γ'_λ *such that, if* $|\lambda|=1$,

$$\left\| \frac{(T/\lambda)+(T/\lambda)^2+\cdots+(T/\lambda)^n}{n} - T'_\infty(\lambda)\right\| \leq \gamma'_\lambda/n \qquad (n=1,2,\dots),$$

$$T'_\infty(\lambda)=\frac{(T/\lambda)+(T/\lambda)^2+\cdots+(T/\lambda)^{m-1}}{m-1}\lim_{n\to\infty}\frac{(T/\lambda)^m+(T/\lambda)^{2m}+\cdots+(T/\lambda)^{nm}}{n}.$$

Remark.[1] Put $T_0=E-\sum_{i=1}^{k}T_i$, where E denotes the identical mapping of $\mathfrak{B}$. Then, by (3), $T_0^2=T_0$, $T_0T_i=T_iT_0=0$ $(i\geq 1)$. Hence, if $\mathfrak{B}_j$ denotes the image of $\mathfrak{B}$ by T_j, we have the direct decomposition $\mathfrak{B}=\mathfrak{B}_0+\mathfrak{B}_1+\cdots+\mathfrak{B}_k$. Each point of $\mathfrak{B}_j$ is invariant by T_j, as $T_j^2=T_j$. $\mathfrak{B}_i$ $(i\geq 1)$ is of finite dimension by Riesz's theorem since $T_i^2=T_i$ and T_i $(i\geq 1)$ is completely continuous. Let $x\in\mathfrak{B}_0$, then $T\cdot x=TT_0\cdot x=S\cdot x$, $\dots$, $T^n\cdot x=S^nx$. Let $x\in\mathfrak{B}_i$ $(i\geq 1)$, then $T\cdot x=TT_i\cdot x=\lambda_iT_i\cdot x=\lambda_ix,\dots$, $T^n\cdot x=\lambda_i^n\cdot x$. Hence $\lim_{n\to\infty}T^n\cdot x=0$ uniformly for $x\in\mathfrak{B}_0$, and $T^n\cdot x$ $(x\in\mathfrak{B}_i, i\geq 1)$ moves in $\mathfrak{B}_i$ almost periodically with respect to n. $\mathfrak{B}_0$ and $\mathfrak{B}_i$ $(i\geq 1)$ may respectively be called the *dissipative* part and the *ergodic* part of $\mathfrak{B}$.

§ 3. *Three lemmas for the proof of the theorem.*

Lemma 1.[2] *Let* T *satisfy the condition (1). Then the proper values of* T *do not accumulate to the point not interior of the unit circle in the complex plane.*

Proof. Put $T=V+U$, then $\|U\|=\delta<1$. We have to derive a contradiction from

$$(6)\quad T\cdot x_i=\lambda_i\cdot x_i,\ x_i\in\mathfrak{B},\ x_i\neq 0,\ \lambda_i\neq\lambda_j\ (i\neq j,)\ \lim_{i\to\infty}\lambda_i=\lambda,\ |\lambda|\geq 1.$$

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1) Cf. N. Kryloff and N. Bogoliouboff: Bult. Soc. Math. France, **64** (1936), 49–56.

2) If T is completely continuous this lemma reduces to the Satz 12 in Riesz, loc. cit. p. 90: the only accumulation point of the proper values of T is the point zero. For, in this case, λT satisfies (1) for any λ.

We have $T^n = T^n - (T-V)^n + (T-V)^n$. $T^n - (T-V)^n$ is completely continuous with V, and $\|(T-V)^n\| \leq \delta^n$, $T^n \cdot x_i = \lambda_i^n \cdot x_i$. Therefore it suffices to derive a contradiction from (6) when $\delta < (1/4)$. This may be carried out as follows.

$x_1, x_2, \dots, x_n$ are linearly independent for any n. The proof is obtained by induction with respect to n. Let $x_1, x_2, \dots, x_{n-1}$ be linearly independent and let x_n be linearly dependent with $x_1, x_2, \dots, x_{n-1}$: $x_n = \sum_{i=1}^{n-1} \alpha_i x_i$. Then we obtain $\sum_{i=1}^{n-1} \alpha_i(\lambda_n - \lambda_i) x_i = 0$ from $T \cdot x_n = \lambda_n x_n$, $T \cdot x_n = \sum_{i=1}^{n-1} \alpha_i T \cdot x_i$, contrary to the hypothesis of the induction.

Thus the linear space $\mathfrak{N}_{n-1}$ spanned by $x_1, x_2, \dots, x_{n-1}$ is a proper subspace of the linear space $\mathfrak{N}_n$ spanned by $x_1, x_2, \dots, x_n$. By Riesz's theorem there exists a sequence $\{y_i\}$ such that $y_i \in \mathfrak{N}_i$, $\|y_i\| = 1$, $\|y_i - x\| > (1/2)$ for all $x \in \mathfrak{N}_{i-1}$. We have $T(y_i/\lambda_i) - T(y_j/\lambda_j) = y_i - \{y_i - T(y_i/\lambda_i) + T(y_j/\lambda_j)\}$. $y_i - T(y_i/\lambda_i) \in \mathfrak{N}_{i-1}$ as $y_i \in \mathfrak{N}_i$. Hence

$$(7) \quad \|V(y_i/\lambda_i) - V(y_j/\lambda_j)\| + \delta\|(y_i/\lambda_i) - (y_j/\lambda_j)\| > (1/2) \quad \text{for } j < i.$$

V being completely continuous and $\|y_i\| = 1$, $\lim_{i\to\infty} \lambda_i = \lambda$, $|\lambda| \geq 1$, there exists a partial sequence $\{i'\}$ of $\{i\}$ such that $\lim \|V(y_{i'}/\lambda_{i'}) - V(y_{j'}/\lambda_{i'})\| = 0$. Thus, by (7), $\delta < (1/4)$, $\|y_{i'}\| = 1$, $\lim \lambda_{i'} = \lambda$ and $|\lambda| \geq 1$, we obtain a contradiction.

Lemma 2. Let $\mathfrak{D}$ be a domain in the complex λ-plane. A family $V(\lambda)$ of completely continuous operators in $\mathfrak{B}$ be regular in $\lambda \in \mathfrak{D}$. Let $\mathfrak{J}$ denotes the set of points (in $\mathfrak{D}$) at each point of which the equation $(E + V(\lambda))x_\lambda = 0$ admits non-trivial solution $x_\lambda \neq 0$. Then for each $\lambda \in \mathfrak{D} - \mathfrak{J}$ $E + V(\lambda)$ has a unique (continuous) inverse $E + K(\lambda)$: $(E + V(\lambda))(E + K(\lambda)) = (E + K(\lambda))(E + V(\lambda)) = E$. $K(\lambda)$ is regular in $\lambda \in \mathfrak{D} - \mathfrak{J}$ and is completely continuous for each $\lambda \in \mathfrak{D} - \mathfrak{J}$.

Proof. By Riesz's theorem $E + V(\lambda)$ has a unique (continuous) inverse $E + K(\lambda)$ for each $\lambda \in \mathfrak{D} - \mathfrak{J}$. By $K(\lambda) = -V(\lambda) - K(\lambda)V(\lambda)$, we see that $K(\lambda)$ is completely continuous.

Let $\lambda_0 \in \mathfrak{D} - \mathfrak{J}$, then the series $\left[E + \sum_{n=1}^{\infty} \{E - (E + K(\lambda_0))(E + V(\lambda))\}^n\right] \cdot (E + K(\lambda_0))$ are absolutely and uniformly convergent for sufficiently small $|\lambda - \lambda_0|$. It is easy to see that this series are the demanded inverse $E + K(\lambda)$.

Lemma 3.[1] *Let T satisfy the condition (1). By the lemma 1, the proper values of T with modulus 1 are isolated proper values. Let these proper values be $\lambda_1, \lambda_2, \dots, \lambda_k$. Then there exists a positive ε such that $E + \lambda T$ admits a unique (continuous) inverse $E + \lambda R_\lambda$ for each λ, $1 - 2\varepsilon < |\lambda| < 1 + 2\varepsilon$, except for $\lambda = -\lambda_1^{-1}, -\lambda_2^{-1}, \dots, -\lambda_k^{-1}$. R_λ is regular in λ, $1 - 2\varepsilon < |\lambda| < 1 + 2\varepsilon$, except for poles $-\lambda_i^{-1}$ $(i = 1, 2, \dots, k)$.*

Proof. As $\lambda_1, \lambda_2, \dots, \lambda_k$ are isolated proper values of T, there exists

1) If T is completely continuous this lemma reduces to the Satz 12 in Nagumo, loc. cit. p. 79: the resolvent R_λ of T defined by $(E + \lambda T)(E + \lambda R_\lambda) = (E + \lambda R_\lambda)(E + \lambda T) = E$ is meromorphic in $|\lambda| < \infty$. For, in this case, λT satisfies (1) for any λ.

a positive η such that $(E+\lambda T)x_\lambda=0$ does not admit non-trivial solution $x_\lambda\neq 0$ for any λ, $1-\eta<|\lambda|<1+\eta$, except for $\lambda=-\lambda_1^{-1}, -\lambda_2^{-1}, \dots, -\lambda_k^{-1}$.

Put $T=V+U$. By $\|U\|=\delta<1$, $E+\lambda U$ admits a unique (continuous) inverse $E+\lambda U_\lambda=E+\sum_{n=1}^{\infty}(-\lambda U)^n$ which is regular in $|\lambda|<(1/\delta)$.

We have $(E+\lambda U_\lambda)(E+\lambda T)=E+\lambda V+\lambda^2 U_\lambda V$. Put $V(\lambda)=\lambda V+\lambda^2 U_\lambda V$. It is regular in $\lambda<(1/\delta)$ and is completely continuous with V for each λ, $\lambda<(1/\delta)$.

Let $2\varepsilon=\text{Min.}\big((1/\delta)-1, \eta\big)$. We denote by $\mathfrak{D}$ the domain $1-2\varepsilon<|\lambda|<1+2\varepsilon$, and let $\mathfrak{J}$ be the point set $(-\lambda_1^{-1}, -\lambda_2^{-1}, \dots, -\lambda_k^{-1})$. Then the equation $\big(E+V(\lambda)\big)x_\lambda=0$ does not admit non-trivial solution $x_\lambda\neq 0$ for any $\lambda\in\mathfrak{D}-\mathfrak{J}$. Assume that there exists a $x_{\lambda_0}\neq 0$, $\lambda_0\in\mathfrak{D}$, which satisfies $\big(E+V(\lambda_0)\big)x_{\lambda_0}=0$. Then we would have $(E+\lambda_0 T)x_{\lambda_0}=(E+\lambda_0 U)\big(E+V(\lambda_0)\big)x_{\lambda_0}=0$. This shows that $\lambda_0\in\mathfrak{J}$.

Thus, by the lemma 2, $E+V(\lambda)$ admits a unique (continuous) inverse $E+K(\lambda)$ for each $\lambda\in\mathfrak{D}-\mathfrak{J}$, and $K(\lambda)$ is regular in $\lambda\in\mathfrak{D}-\mathfrak{J}$. We easily verify that $E+\lambda R_\lambda=\big(E+K(\lambda)\big)(E+\lambda U_\lambda)$ is the inverse of $E+\lambda T$ for each $\lambda\in\mathfrak{D}-\mathfrak{J}$.

Let the Laurent expansion of $R_\lambda=\big(K(\lambda)+\lambda U_\lambda+\lambda K(\lambda)U_\lambda\big)/\lambda$ at the isolated singular point $\lambda=-\lambda_j^{-1}$ be

$$(8)\qquad \sum_{n=-\infty}^{\infty}(\lambda+\lambda_j^{-1})^n C_n(j)\,.$$

By Cauchy's theorem $C_{-1}(j)=\frac{1}{2\pi i}\int R_\lambda d\lambda$. U_λ being regular at $\lambda=-\lambda_j^{-1}$, we have $C_{-1}(j)=\frac{1}{2\pi i}\int\Big\{\big(K(\lambda)+\lambda K(\lambda)U_\lambda\big)/\lambda\Big\}d\lambda$. As $K(\lambda)$ is completely continuous we see that $C_{-1}(j)$ is also completely continuous.

By substituting (8) in the resolvent equation $R_\lambda-R_\mu=(\lambda-\mu)R_\lambda R_\mu$ we obtain

$$(9)\qquad C_{-n}(j)C_m(j)=C_m(j)C_{-n}(j)=0 \qquad (n>0, m\geq 0)\,,$$

$$(10)\qquad C_{-1}(j)^2=C_{-1}(j),\ C_{-n}(j)=C_{-n}(j)C_{-1}(j)=C_{-1}(j)C_{-n}(j),$$
$$C_{-(n+1)}(j)=C_{-2}^n(j) \qquad (n>0)\,.$$

$\mathfrak{B}$ is mapped on its linear subspace $\mathfrak{B}_j$ by $C_{-1}(j)$. By $C_{-1}^2(j)=C_{-1}(j)$ all the points of $\mathfrak{B}_j$ is invariant by $C_{-1}(j)$. The unit sphere in $\mathfrak{B}_j$ is compact since $C_{-1}(j)$ is completely continuous. Thus $\mathfrak{B}_j$ is of finite dimension by Riesz's theorem. By (10) $C_{-n}(j)$ maps $\mathfrak{B}_j$ in $\mathfrak{B}_j$ and hence $C_{-(n+1)}(j)$ is of the form $D_j^n C_{-1}(j)$, where D_j is a linear mapping of $\mathfrak{B}_j$ in $\mathfrak{B}_j$. Thus $\sum_{n=1}^{\infty}(\lambda+\lambda_j^{-1})^{-n}C_{-n}(j)=\sum_{n=0}^{\infty}(\lambda+\lambda_j^{-1})^{-(n+1)}D_j^n C_{-1}(j)$. As it converges for $|\lambda+\lambda_j^{-1}|>0$, the matrix D_j must be nilpotent: $D_i^n=0$ for large n.

Hence $\lambda=-\lambda_j^{-1}$ is a pole of R_λ.

§ 4. *The proof of the theorem.* By (1) and the lemma 3, the (continuous) inverse $E+\lambda R_\lambda$ of $E+\lambda T$ is regular in $1-2\varepsilon<|\lambda|<1+2\varepsilon$ except for poles $\lambda=-\lambda_1^{-1}, -\lambda_2^{-1}, \ldots, \lambda_k^{-1}$. By (2) we see that, for $|\lambda|<1$, R_λ is given by the absolutely and uniformly convergent series $\sum_{n=1}^{\infty}\lambda^{n-1}(-T)^n$: $\left\|\sum_{n=1}^{\infty}\lambda^{n-1}(-T)^n\right\|\leq a/(1-|\lambda|)$. Hence R_λ is regular in $|\lambda|<1+2\varepsilon$, except for simple poles $\lambda=-\lambda_1^{-1}, -\lambda_2^{-1}, \ldots, -\lambda_k^{-1}$. Let the Laurent expansion of R_λ at $\lambda=-\lambda_j^{-1}$ be

$$(11)\qquad (\lambda+\lambda_j^{-1})^{-1}T_j+\sum_{n=0}^{\infty}(\lambda+\lambda_j^{-1})^n T_{j,n}\,.$$

Then, by (8), (9), (10) and $R_0=-T$ we see that

$$T_j^2=T_j,\ T_iT_j=0 \quad (i\neq j)\,,\quad (T-\sum_{i=1}^{k}\lambda_iT_i)T_j=T_j(T-\sum_{i=1}^{k}\lambda_iT_i)=0 \qquad (i, j=1, 2, \ldots, k)\,.$$

Put $T=\sum_{i=1}^{k}\lambda_iT_i+S$. Then, by the above relations, we obtain for $|\lambda|<1$

$$R_\lambda=\sum_{n=1}^{\infty}\lambda^{n-1}(-T)^n=\sum_{j=1}^{k}\sum_{n=1}^{\infty}(-\lambda_j)^n\lambda^{n-1}T_j+\sum_{n=1}^{\infty}\lambda^{n-1}(-S)^n$$
$$=\sum_{j=1}^{k}(\lambda+\lambda_j^{-1})^{-1}T_j+\sum_{n=1}^{\infty}\lambda^{n-1}(-S)^n\,.$$

Therefore, by (11), we see that $\sum_{n=1}^{\infty}\lambda^{n-1}(-S)^n$ is regular in $|\lambda|<1+2\varepsilon$. Hence, by Cauchy's theorem, $\|S^n\|\leq\beta/(1+\varepsilon)^n$, $\beta=\text{l. u. b.}_{|\lambda|\leq 1+\frac{3}{2}\varepsilon}\left\|\sum_{n=1}^{\infty}(-\lambda S)^n\right\|$ for $n=1, 2, \ldots$.

§ 5. *Smoluchousky's equation.*[1] Let a family $T(t)$ of continuous operators in $\mathfrak{B}$ satisfy the equation of Smoluchousky: $T(t+s)=T(t)T(s)$ $(0<t, s<\infty)$. We assume that $T(t)$ is continuous in t: $\lim_{t\to t_0}\|T(t)-T(t_0)\|=0$, and that there exists a positive t_1 such that $T=T(t_1)$ satisfies (1) and (2).

By the theorem we have the representation (3). We put

$$T(t)=\sum_{j=1}^{k}T_j(t)+S(t),\ \ T_j(t)=T_jT(t)T_j\,.$$

$T_j(t)$ and $S(t)$ is continuous in t. T_j is commutative with every $T(t)$ by (4), and hence we obtain, for $0<t, s<\infty$, $T_j(t+s)=T_j(t)T_j(s)$, $T_j(t)S(s)=S(s)T_j(t)=0$ and $S(t+s)=S(t)S(s)$.

As $S=S(t_1)$ satisfies (4) we obtain, by positive a and b, $\|S(t)\|\leq a\cdot\exp(-bt)$ for $t_1\leq t<\infty$.

By $T_j^2=T_j$, $T_j(t)=T_jT_j(t)=T_j(t)T_j$ and the complete continuity of T_j we see, as in the proof of the lemma 3, that $T_j(t)=M_j(t)T_j$, where the finite dimentional matrix $M_j(t)$ is continuous in t and satisfies the equation of Smoluchousky. As $M_j(t_1)=$the unit matrix we see that

1) An analogus result is obtained by Kakutani also, by applying the theorem to the sequence $\{T(t/2^n)\}$.

$M_j(0)=\lim_{t\to 0} M_j(t)=$ the unit matrix. Hence,[1] if $\|M_j(t)-M_j(0)\|<1$ for $t\leq t_0$, $t_0>0$, we have $M_j(t)=\exp(C_j t/t_0)$, where $C_j=\log\big(M_j(t_0)\big)$. Thus, by $M_j(t_1)=$ the unit matrix, we see that $M_j(t)$ is similar to the matrix of the form

$$\begin{pmatrix} \exp(2\pi i a_{j1} t/t_0) & \cdots\cdots & 0 \\ \vdots & \ddots & \vdots \\ \times & \cdots\cdots & \exp(2\pi i a_{jl_j} t/t_0) \end{pmatrix} \quad \left(a\frac{t_1}{t_0} \text{ all real} \equiv 0 \text{ mod. } 1\right).$$

Therefore the theorem is extended to the continuous stochastic process.

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1) K. Yosida: Jap. J. of Math. **13** (1936), 25.