PAPERS COMMUNICATED

113. *Concircular Geometry IV. Theory of Subspaces.*

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In three previous papers,[1] we have considered the Concircular Geometry, that is to say, the geometry in which one seeks for the properties of Riemannian spaces invariant under the conformal transformations of the metric

$$\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu} \qquad (\lambda, \mu, \nu, \ldots=1, 2, 3, \ldots, n),$$

with functions ρ satisfying the following partial differential equations

$$\rho_{\mu\nu}\equiv\frac{\partial\rho_\mu}{\partial u^\nu}-\rho_\lambda\left\{{}^{\lambda}_{\mu\nu}\right\}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta\rho_{\mu\nu}=\phi g_{\mu\nu} \quad \left(\rho_\mu=\frac{\partial \log \rho}{\partial u^\mu}\right).$$

In the present paper, we shall deal with the theory of subspaces in the concircular geometry.

§ 1. Let us consider a subspace V_m immersed in a Riemannian space V_n whose parametric representation is

$$u^\lambda=u^\lambda(u^{\dot{1}}, u^{\dot{2}}, \ldots, u^{\dot{m}}) \tag{1.1}$$

where (u^λ) and (u^i) $(i, j, k, \ldots=\dot{1}, \dot{2}, \ldots, \dot{m})$ denote the coordinate systems of V_n and V_m respectively. A conformal transformation

$$\bar{g}_{\mu\nu}=\rho^2 g_{\mu\nu} \tag{1.2}$$

of the fundamental tensor of V_n, being a concircular one with the function ρ satisfying the equations

$$\rho_{\mu\nu}\equiv\rho_{\mu;\nu}-\rho_\mu\rho_\nu+\frac{1}{2}g^{\alpha\beta}\rho_\alpha\rho_\beta g_{\mu\nu}=\phi g_{\mu\nu}\,, \tag{1.3}$$

where the semi-colon denotes the covariant differentiation with respect to the Christoffel symbols $\left\{{}^{\lambda}_{\mu\nu}\right\}$ formed with $g_{\mu\nu}$, the induced conformal transformation

$$g_{jk}=\rho^2 g_{jk} \tag{1.4}$$

of the fundamental tensor

$$g_{jk}=g_{\mu\nu}B_j^{\cdot\mu}B_k^{\cdot\nu} \qquad \left(B_j^{\cdot\mu}=\frac{\partial u^\mu}{\partial u^j}\right) \tag{1.5}$$

of the subspace is not in general a concircular one.

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1) K. Yano, Concircular Geometry I, Proc. **16** (1940), 195–200, II, Proc. **16** (1940), 359–360 and III, **16** (1940), 442–458.

We shall, first, seek for the subspace V_m for which the induced conformal transformation is also a concircular one.

Putting

$$\rho_{jk} \equiv \rho_{j;k} - \rho_j \rho_k + \frac{1}{2} g^{ab} \rho_a \rho_b g_{jk}, \tag{1.6}$$

where

$$\rho_j = \frac{\partial \log \rho}{\partial u^j} = \rho_\mu B_j^{\cdot \mu}, \tag{1.7}$$

and $\rho_{j;k}$ denotes the covariant derivative of ρ_j with respect to the three-index symbols of Christoffel $\{{}^{i}_{jk}\}$ formed with g_{jk}, we obtain

$$\rho_{jk} = \rho_{\mu;\nu} B_j^{\cdot\mu} B_k^{\cdot\nu} + \rho_\mu H_{jk}^{\cdot\cdot\mu} - \rho_\mu \rho_\nu B_j^{\cdot\mu} B_k^{\cdot\nu} + \frac{1}{2} \rho_\alpha \rho_\beta B_a^{\cdot\alpha} B_b^{\cdot\beta} g^{ab} g_{\mu\nu} B_j^{\cdot\mu} B_k^{\cdot\nu} \tag{1.8}$$

or

$$\rho_{jk} = \rho_{\mu\nu} B_j^{\cdot\mu} B_k^{\cdot\nu} + \rho_\mu H_{jk}^{\cdot\cdot\mu} - \frac{1}{2} \rho_\alpha \rho_\beta B_A^{\cdot\alpha} B_A^{\cdot\beta} g_{jk} \tag{1.9}$$

where $B_A^{\cdot\alpha}$ $(A, B, \dots = \dot{m}+\dot{1}, \dots\dots, \dot{n})$ are $n-m$ mutually orthogonal unit vectors normal to V_m and

$$H_{jk}^{\cdot\cdot\mu} = \frac{\partial B_j^{\cdot\mu}}{\partial u^k} + B_j^{\cdot\alpha} B_k^{\cdot\beta} \{{}^{\mu}_{\alpha\beta}\} - B_a^{\cdot\mu} \{{}^{a}_{jk}\}. \tag{1.10}$$

The conformal transformation (1.2) being a concircular one, we have

$$\rho_{\mu\nu} = \phi g_{\mu\nu}.$$

Substituting these equations in (1.9), we have

$$\rho_{jk} = \rho_\mu H_{jk}^{\cdot\cdot\mu} + \left(\phi - \frac{1}{2} \rho_\alpha \rho_\beta B_A^{\cdot\alpha} B_A^{\cdot\beta}\right) g_{jk}. \tag{1.11}$$

If we suppose that the induced conformal transformation (1.4) is also concircular, we must have the equations of the form

$$\rho_\mu M_{jk}^{\cdot\cdot\mu} = 0 \tag{1.12}$$

where

$$M_{jk}^{\cdot\cdot\mu} = H_{jk}^{\cdot\cdot\mu} - \frac{1}{m} g^{ab} H_{ab}^{\cdot\cdot\mu} g_{jk}. \tag{1.13}$$

Conversely, if the equation (1.12) is satisfied, it is easily seen that the conformal transformation (1.4) is a concircular one.

Thus we have the following theorems:

Theorem I. The necessary and sufficient condition that a concircular transformation of the fundamental tensor of a Riemannian space induce a concircular transformation on a subspace is that the function ρ satisfy the equations $\rho_\mu M_{jk}^{\cdot\cdot\mu} = 0$ as well as (1.3).

Theorem II. The conformal transformation induced on a totally umbilical subspace by a concircular transformation is always a concircular one.

§ 2. We have seen, in a previous paper,[1] that the curvature tensor of V_n defined by

$$(2.1) \qquad Z^{\lambda}_{\mu\nu\omega}=R^{\lambda}_{\mu\nu\omega}-\frac{R}{n(n-1)}(g_{\mu\nu}\delta^{\lambda}_{\omega}-g_{\mu\omega}\delta^{\lambda}_{\nu})$$

is a concircular invariant. When the subspace V_m is not a totally umbilical one, the curvature tensor of V_m

$$(2.2) \qquad Z^{i}_{jkh}=R^{i}_{jkh}-\frac{g^{ab}R_{ab}}{m(m-1)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})$$

where

$$(2.3) \qquad R^{i}_{jkh}=\frac{\partial\{^{\,i}_{jk}\}}{\partial u^h}-\frac{\partial\{^{\,i}_{jh}\}}{\partial u^k}+\{^{\,a}_{jk}\}\{^{\,i}_{ah}\}-\{^{\,a}_{jh}\}\{^{\,i}_{ak}\}$$

is not in general a concircular invariant.

But the Weyl conformal curvature tensor

$$(2.4) \qquad C^{i}_{jkh}=R^{i}_{jkh}-\frac{1}{m-2}(R_{jk}\delta^{i}_{h}-R_{jh}\delta^{i}_{k}+g_{jk}R^{i}_{\cdot h}-g_{jh}R^{i}_{\cdot k}) +\frac{g^{ab}R_{ab}}{(m-1)(m-2)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})$$

is, of course, a concircular invariant. This conformal curvature tensor C^{i}_{jkh} may be expressed by means of Z^{i}_{jkh} and $Z_{jk}=Z^{i}_{jki}$ as follows:

$$(2.5) \qquad C^{i}_{jkh}=Z^{i}_{jkh}-\frac{1}{m-2}(Z_{jk}\delta^{i}_{h}-Z_{jh}\delta^{i}_{k}+g_{jk}Z^{i}_{\cdot h}-g_{jh}Z^{i}_{\cdot k})$$

where

$$Z^{i}_{\cdot h}=g^{ik}Z_{kh}\,.$$

We shall, in the following, establish the relations between the concircular curvature tensor $Z^{\lambda}_{\mu\nu\omega}$ and the conformal curvature tensor C^{i}_{jkh}. The equations of Gauss of V_m in V_n are

$$(2.6) \qquad R^{i}_{jkh}=B^{i\mu\nu\omega}_{\lambda jkh}R^{\lambda}_{\mu\nu\omega}+H^{\cdot\cdot\lambda}_{jk}H^{i}_{\cdot h\lambda}-H^{\cdot\cdot\lambda}_{jh}H^{i}_{\cdot k\lambda}$$

where

$$B^{i\mu\nu\omega}_{\lambda jkh}=B^{i}_{\cdot\lambda}B^{\cdot\mu}_{j}B^{\cdot\nu}_{k}B^{\cdot\omega}_{h}\,,\quad B^{i}_{\cdot\lambda}=g^{ij}g_{\lambda\mu}B^{\cdot\mu}_{j}\quad\text{and}\quad H^{i}_{\cdot h\lambda}=g^{ik}g_{\lambda\mu}H^{\cdot\cdot\mu}_{kh}\,.$$

Contracting (2.1) with $B^{i\mu\nu\omega}_{\lambda jkh}$, we have

$$B^{i\mu\nu\omega}_{\lambda jkh}Z^{\lambda}_{\mu\nu\omega}=B^{i\mu\nu\omega}_{\lambda jkh}R^{\lambda}_{\mu\nu\omega}-\frac{R}{n(n-1)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})\,.$$

Then substituting these equations in (2.6), we obtain

$$(2.7) \qquad R^{i}_{jkh}=B^{i\mu\nu\omega}_{\lambda jkh}Z^{\lambda}_{\mu\nu\omega}+H^{\cdot\cdot\lambda}_{jk}H^{i}_{\cdot h\lambda}-H^{\cdot\cdot\lambda}_{jh}H^{i}_{\cdot k\lambda}+\frac{R}{n(n-1)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})\,.$$

From (2.7), we find by contration

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1) K. Yano: Concircular geometry I. loc. cit.

$$(2.8)\qquad R_{jk}=B^{\mu\nu}_{jk}B^{\omega}_{\lambda}Z^{\lambda}_{\mu\nu\omega}+H^{\cdot\cdot\lambda}_{jk}H^{b}_{\cdot b\lambda}-H^{\cdot\cdot\lambda}_{jb}H^{b}_{\cdot k\lambda}+\frac{(m-1)}{n(n-1)}Rg_{jk}\,,$$

and

$$(2.9)\qquad g^{jk}R_{jk}=B^{\mu\nu}B^{\omega}_{\lambda}Z^{\lambda}_{\mu\nu\omega}+H^{a\cdot\lambda}_{\cdot a}H^{b}_{\cdot b\lambda}-H^{a\cdot\lambda}_{\cdot b}H^{b}_{\cdot a\lambda}+\frac{m(m-1)}{n(n-1)}R\,,$$

where

$$B^{\omega}_{\lambda}=B^{i}_{\cdot\lambda}B^{\cdot\omega}_{i}\,,\quad B^{\mu\nu}_{jk}=B^{\cdot\mu}_{j}B^{\cdot\nu}_{k}\,,\quad B^{\mu\nu}=B^{\mu\nu}_{ab}g^{ab}\,,\quad \text{and}\quad H^{i\cdot\lambda}_{\cdot k}=g^{ij}H^{\cdot\cdot\lambda}_{jk}\,.$$

The equations (2.7), (2.8) and (2.9) give us

$$(2.10)\qquad Z^{i}_{jkh}=B^{i\mu\nu\omega}_{\lambda jkh}Z^{\lambda}_{\mu\nu\omega}-\frac{B^{\mu\nu}B^{\omega}_{\lambda}Z^{\lambda}_{\mu\nu\omega}}{m(m-1)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})$$
$$+H^{\cdot\cdot\lambda}_{jk}H^{i}_{\cdot h\lambda}-H^{\cdot\cdot\lambda}_{jh}H^{i}_{\cdot k\lambda}-\frac{H^{a\cdot\lambda}_{\cdot a}H^{b}_{\cdot b\lambda}}{m(m-1)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})$$
$$+\frac{H^{a\cdot\lambda}_{\cdot b}H^{b}_{\cdot a\lambda}}{m(m-1)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})$$

and

$$(2.11)\qquad Z_{jk}=B^{\mu\nu}_{jk}B^{\omega}_{\lambda}Z^{\lambda}_{\mu\nu\omega}-\frac{1}{m}B^{\mu\nu}B^{\omega}_{\lambda}Z^{\lambda}_{\mu\nu\omega}g_{jk}+H^{\cdot\cdot\lambda}_{jk}H^{b}_{\cdot b\lambda}$$
$$-H^{\cdot\cdot\lambda}_{jb}H^{b}_{\cdot k\lambda}-\frac{1}{m}H^{a\cdot\lambda}_{\cdot a}H^{b}_{\cdot b\lambda}g_{jk}+\frac{1}{m}H^{a\cdot\lambda}_{\cdot b}H^{b}_{\cdot a\lambda}g_{jk}\,.$$

Substituting the equations (2.10) and (2.11) in (2.4), we obtain

$$(2.12)\qquad C^{i}_{jkh}=Z^{i}_{jkh}-\frac{1}{m-2}(Z_{jk}\delta^{i}_{h}-Z_{jh}\delta^{i}_{k}+g_{jk}Z^{i}_{\cdot h}-g_{jh}Z^{i}_{\cdot k})$$
$$=B^{i\mu\nu\omega}_{\lambda jkh}Z^{\lambda}_{\mu\nu\omega}-\frac{1}{m-2}(B^{\mu\nu}_{jk}B^{\omega}_{\lambda}Z^{\lambda}_{\mu\nu\omega}\delta^{i}_{h}-B^{\mu\nu}_{jh}B^{\omega}_{\lambda}Z^{\lambda}_{\mu\nu\omega}\delta^{i}_{k}$$
$$+g_{jk}B^{\mu\nu}_{ah}B^{\omega}_{\lambda}g^{ai}Z^{\lambda}_{\mu\nu\omega}-g_{jh}B^{\mu\nu}_{ak}B^{\omega}_{\lambda}g^{ai}Z^{\lambda}_{\mu\nu\omega})$$
$$+\frac{B^{\mu\nu}B^{\omega}_{\lambda}Z^{\lambda}_{\mu\nu\omega}}{(m-1)(m-2)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})+H^{\cdot\cdot\lambda}_{jk}H^{i}_{\cdot h\lambda}-H^{\cdot\cdot\lambda}_{jh}H^{i}_{\cdot k\lambda}$$
$$-\frac{H^{a\cdot\lambda}_{\cdot a}H^{b}_{\cdot b\lambda}}{m(m-1)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})+\frac{H^{a\cdot\lambda}_{\cdot b}H^{b}_{\cdot a\lambda}}{m(m-1)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})$$
$$-\frac{1}{m-2}\Big[H^{\cdot\cdot\lambda}_{jk}H^{b}_{\cdot b\lambda}\delta^{i}_{h}-H^{\cdot\cdot\lambda}_{jh}H^{b}_{\cdot b\lambda}\delta^{i}_{k}+g_{jk}H^{i\cdot\lambda}_{\cdot h}H^{b}_{\cdot b\lambda}$$
$$-g_{jh}H^{i\cdot\lambda}_{\cdot k}H^{b}_{\cdot b\lambda}-H^{\cdot\cdot\lambda}_{jb}H^{b}_{\cdot k\lambda}\delta^{i}_{h}+H^{\cdot\cdot\lambda}_{jb}H^{b}_{\cdot h\lambda}\delta^{i}_{k}-g_{jk}H^{i\cdot\lambda}_{\cdot b}H^{b}_{\cdot h\lambda}$$
$$+g_{jh}H^{i\cdot\lambda}_{\cdot b}H^{b}_{\cdot k\lambda}-\frac{2H^{a\cdot\lambda}_{\cdot a}H^{b}_{\cdot b\lambda}}{m}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})$$
$$+\frac{2H^{a\cdot\lambda}_{\cdot b}H^{b}_{\cdot a\lambda}}{m}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k})\Big]\,.$$

We have, on the other hand,

$$H_{jk}^{\cdot\cdot\lambda}=M_{jk}^{\cdot\cdot\lambda}+\frac{1}{m}H_{\cdot a}^{a\cdot\lambda}g_{jk}\,,$$

$$H_{\cdot h\lambda}^{i}=M_{\cdot h\lambda}^{i}+\frac{1}{m}H_{\cdot a\lambda}^{a}\delta_{h}^{i}\,,\qquad (M_{\cdot h\lambda}^{i}=g^{ij}g_{\lambda\mu}M_{jh}^{\cdot\cdot\mu})\,.$$

Substituting these equations in (2.12), we obtain finally

$$\begin{aligned}(2.13)\qquad & Z_{jkh}^{i}-\frac{1}{m-2}(Z_{jk}\delta_{h}^{i}-Z_{jh}\delta_{k}^{i}+g_{jk}Z_{\cdot h}^{i}-g_{jh}Z_{\cdot k}^{i})\\ &=B_{\lambda jkh}^{i\mu\nu\omega}Z_{\mu\nu\omega}^{\lambda}-\frac{1}{m-2}(B_{jk}^{\mu\nu}B_{\lambda}^{\omega}Z_{\mu\nu\omega}^{\lambda}\delta_{h}^{i}-B_{jh}^{\mu\nu}B_{\lambda}^{\omega}Z_{\mu\nu\omega}^{\lambda}\delta_{k}^{i}\\ &+g_{jk}B_{ah}^{\mu\nu}B_{\lambda}^{\omega}g^{ai}Z_{\mu\nu\omega}^{\lambda}-g_{jh}B_{ak}^{\mu\nu}B_{\lambda}^{\omega}g^{ai}Z_{\mu\nu\omega}^{\lambda})\\ &+\frac{B^{\mu\nu}B_{\lambda}^{\omega}Z_{\mu\nu\omega}^{\lambda}}{(m-1)(m-2)}(g_{jk}\delta_{h}^{i}-g_{jh}\delta_{k}^{i})+M_{jk}^{\cdot\cdot\lambda}M_{\cdot h\lambda}^{i}-M_{jh}^{\cdot\cdot\lambda}M_{\cdot k\lambda}^{i}\\ &+\frac{1}{m-2}\Big[(g_{jk}M_{\cdot a}^{i\cdot\lambda}M_{\cdot h\lambda}^{a}+M_{ja}^{\cdot\cdot\lambda}M_{\cdot k\lambda}^{a}\delta_{h}^{i})\\ &-(g_{jh}M_{\cdot a}^{i\cdot\lambda}M_{\cdot k\lambda}^{a}+M_{ja}^{\cdot\cdot\lambda}M_{\cdot h\lambda}^{a}\delta_{k}^{i})\Big]\\ &-\frac{M_{\cdot b}^{a\cdot\lambda}M_{\cdot a\lambda}^{b}}{(m-1)(m-2)}(g_{jk}\delta_{h}^{i}-g_{jh}\delta_{k}^{i})\,.\end{aligned}$$

The left member of (2.13) representing the Weyl conformal curvature tensor, the equations (2.13) are the equations of Gauss of V_m in V_n in our concircular geometry.

§ 3. Let $B_{A}^{\cdot\lambda}$ $(A, B, C, \ldots=\dot{m}+\dot{1}, \dot{m}+\dot{2}, \ldots, \dot{n})$ be $n-m$ mutually orthogonal unit vectors normal to the subspace V_m, then the equations of Weingarten may be written in the form

$$(3.1)\qquad B_{A;j}^{\cdot\lambda}=-B_{a}^{\cdot\lambda}H_{\cdot jA}^{a}+L_{ABj}B_{B}^{\cdot\lambda}$$

where we have put

$$H_{\cdot jA}^{a}=g_{\lambda\mu}H_{\cdot j}^{a\cdot\lambda}B_{A}^{\cdot\mu}\quad\text{and}\quad L_{ABj}=g_{\lambda\mu}B_{A;j}^{\cdot\lambda}B_{B}^{\cdot\mu}\,.$$

From (3.1) we can derive the equations of Codazzi

$$(3.2)\qquad B_{\lambda Ajk}^{i\mu\nu\omega}R_{\mu\nu\omega}^{\lambda}=-H_{\cdot jA;k}^{i}+H_{\cdot kA;j}^{i}+H_{\cdot jB}^{i}L_{ABk}-H_{\cdot kB}^{i}L_{ABj}\,,$$

where

$$B_{\lambda Ajk}^{i\mu\nu\omega}=B_{\cdot\lambda}^{i}B_{A}^{\cdot\mu}B_{j}^{\cdot\nu}B_{k}^{\cdot\omega}\,.$$

Multiplying (2.1) by $B_{\lambda Ajk}^{i\mu\nu\omega}$ and contracting with respect to the indices λ, μ, ν and ω, we find $B_{\lambda Ajk}^{i\mu\nu\omega}Z_{\mu\nu\omega}^{\lambda}=B_{\lambda Ajk}^{i\mu\nu\omega}R_{\mu\nu\omega}^{\lambda}$, consequently

$$(3.3)\qquad B_{\lambda Ajk}^{i\mu\nu\omega}Z_{\mu\nu\omega}^{\lambda}=-H_{\cdot jA;k}^{i}+H_{\cdot kA;j}^{i}+H_{\cdot jB}^{i}L_{ABk}-H_{\cdot kB}^{i}L_{ABj}\,.$$

These are the equations of Codazzi in our concircular geometry.

From (3.3) we can conclude that the tensor whose compoments are

$$-H^{i}_{\cdot jA;k}+H^{i}_{\cdot kA;j}+H^{i}_{\cdot jB}L_{ABk}-H^{i}_{\cdot kB}L_{ABj} \tag{3.4}$$

is a semi-concircular tensor, that is to say, it will be multiplied by a power of ρ by the concircular transformation.

If we consider a hypersurface V_{n-1} in V_n and denote by B^λ the unit vector normal to V_{n-1}, we have

$$H^{\cdot\cdot\lambda}_{ij}=H_{ij}B^\lambda\,, \qquad L_{ABj}=0 \tag{3.5}$$

and the equations (3.3) reduce to

$$B^{i}_{\cdot\lambda}B^{\mu}B^{\nu\omega}_{jk}Z^{\lambda}_{\mu\nu\omega}=-H^{i}_{\cdot j;k}+H^{i}_{\cdot k;j}$$

or

$$B^{i}_{\cdot\lambda}B^{\mu}B^{\nu\omega}_{jk}Z^{\lambda}_{\mu\nu\omega}=-M^{i}_{\cdot j;k}+M^{i}_{\cdot k;j}-\frac{1}{n-1}H^{a}_{\cdot a;k}\delta^{i}_{j}+\frac{1}{n-1}H^{a}_{\cdot a;j}\delta^{i}_{k} \tag{3.6}$$

where

$$H^{i}_{\cdot k}=g^{ij}H_{jk}\,,\quad M^{\cdot\cdot\lambda}_{ij}=M_{ij}B^\lambda \quad\text{and}\quad M^{i}_{\cdot k}=g^{ij}M_{jk}\,.$$

§ 4. In this paragraph, we state some of theorems which may be easily deduced from the formulae proved in three preceding paragraphs. They are all well known theorems, but it may not be of no use to emphasize here that they are theorems which may be considered in the concircular geometry.

Theorem III. A totally umbilical subspace in a concircularly flat space is also concircularly flat.

Proof. For a totally umbilical subspace, we have $H^{\cdot\cdot\lambda}_{jk}=\frac{1}{m}H^{a\cdot\lambda}_{\cdot a}g_{jk}$. In such a case equations (2.10) become

$$Z^{i}_{jkh}=B^{i\mu\nu\omega}_{\lambda jkh}Z^{\lambda}_{\mu\nu\omega}-\frac{B^{\mu\nu}B^{\omega i}_{\lambda}Z^{\lambda}_{\mu\nu\omega}}{m(m-1)}(g_{jk}\delta^{i}_{h}-g_{jh}\delta^{i}_{k}). \tag{4.1}$$

Thus we can see that if the enveloping space V_n is a concircularly flat one ($Z^{\lambda}_{\mu\nu\omega}=0$), the subspace is also concircularly flat one ($Z^{i}_{jkh}=0$).

Theorem IV. The mean curvature of totally umbilical hypersurface in a concircularly flat space is constant.

Proof. The conditions $Z^{\lambda}_{\mu\nu\omega}=0$, $M_{ij}=0$ and equations (3.6) give us

$$H^{a}_{\cdot a;k}\delta^{i}_{j}-H^{a}_{\cdot a;j}\delta^{i}_{k}=0$$

from which we have

$$H^{a}_{\cdot a;k}=0\,. \tag{4.2}$$

Thus the theorem is proved.

Theorem V. If there exists always a totally umbilical hypersurface of constant mean curvature touching an arbitrary hyperplane passing through any point of the enveloping space, then the enveloping space is concircularly flat.

Proof. $M^{i}_{\cdot j}$ and $H^{a}_{\cdot a;j}$ being zero, we have from (3.6)

$$B^{i}_{\cdot\lambda}B^{\mu}B^{\nu\omega}_{jk}Z^{\lambda}_{\mu\nu\omega}=0\,, \tag{4.3}$$

which must be satisfied for any $B^{\cdot\lambda}_{i}$ and B^{λ} satisfying

$$g_{\lambda\mu}B^{\cdot\lambda}_{i}B^{\mu}=0\,.$$

Consequently we have from (4.3)

$$Z^{\lambda}_{\mu\nu\omega}=0\,.$$

This proves the theorem.