90. On the Adiabatic Theorem for the Hamiltonian System of Differential Equations in the Classical Mechanics. III

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6. We consider the Hamiltonian system containing a parameter $\lambda(>0)$

(14) $$dp/dt=-\lambda\partial H/\partial q(p,q,t),\ dq/dt=\lambda\partial H/\partial p(p,q,t)$$

in D. If $(p^0,q^0,t^0)\in D$ and $\lambda>0$, there is a unique solution of (14) in D passing through $(p^0,q^0,t^0,)$ and prolonged as far as possible to the both directions of the time t, by the regularity of $H(p,q,s)$ in Assumption 1.[1] We denote it by

(15) $$p=\tilde{\mathfrak{p}}(t,p^0,q^0,t^0,\lambda),\ q=\tilde{\mathfrak{q}}(t,p^0,q^0,t^0,\lambda).$$

For a fixed $(p^0,q^0,t^0)\in D$ and a fix $\lambda(>0)$, $\tilde{\mathfrak{p}}(t,p^0,q^0,t^0,\lambda)$, $\tilde{\mathfrak{q}}(t,p^0,q^0,t^0,\lambda)$ are defined on a subinterval of the time interval $a\leqq t\leqq b$ which may be open, closed or half-open according to (p^0,q^0,t^0,λ).[1]

Since $\partial\tilde{\mathfrak{J}}/\partial s$ is continuous on $\bar{D}$ and $\bar{D}$ is compact, there is a number $M(>0)$ such that

(16) $$|\partial\tilde{\mathfrak{J}}/\partial s|\leqq M \qquad \text{on } D.$$

THEOREM 3. *Let a' and b' be two numbers such that $a\leqq a'<b'\leqq b$ and $(b'-a')<(J_2^*-J_1^*)/(2M)$ and let us put $J_2=J_2^*-M(b'-a')$, $J_1=J_1^*+M(b'-a')$. Then the solution of* (14) *passing through (p^0,q^0,a') where $(p^0,q^0)\in I(J_1,J_2,a')$ can be prolonged in D to the time interval $a'\leqq t\leqq b'$ for every $\lambda(>0)$.*

PROOF. Let β be the least upper bound of β' such that the solution in D of (14), $p=\tilde{\mathfrak{p}}(t,p^0,q^0,a',\lambda)$, $q=\tilde{\mathfrak{q}}(t,p^0,q^0,a',\lambda)$ for a fixed $(p^0,q^0)\in I(J_1,J_2,a')$ and a fixed $\lambda>0$, can be defined for the time interval $a'\leqq t<\beta'$ and such that $a'<\beta'\leqq b'$. Then $a'<\beta\leqq b'$ and this solution in D can be defined on the the time interval $a'\leqq t<\beta$. Since $\partial H/\partial p$, $\partial H/\partial q$ are bounded on D by their continuity on the compact set $\bar{D}$, the functions $\tilde{\mathfrak{p}}_i(t,p^0,q^0,a',\lambda)$ $\tilde{\mathfrak{q}}_i(t,p^0,q^0,a',\lambda)$ $(i=1,\cdots,n)$ of t representing a solution of (14) in D, are uniformly continuous on the interval $a'\leqq t<\beta$. Hence the limits

$$\tilde{\mathfrak{p}}(t,p^0,q^0,a',\lambda)\to p'(t\to\beta-0)$$
$$\tilde{\mathfrak{q}}(t,p^0,q^0,a',\lambda)\to q'(t\to\beta-0)$$

exist and $(p',q',\beta)\in\bar{D}$.

We shall sometimes abbreviate $\tilde{\mathfrak{p}}(t,p^0,q^0,a',\lambda)$ and $\tilde{\mathfrak{q}}(t,p^0,q^0,a',\lambda)$

1) Cf. E. Kamke [1, pp. 135–136 and pp. 137–142].

as $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{q}}$ in this proof of Theorem 3.

Now for $a' \leqq t < \beta$, we have[2)]

$$\frac{d}{dt}\tilde{\mathfrak{J}}\{\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda), t\}$$

$$(17)\qquad \begin{aligned} &= \frac{d}{dt}\mathfrak{J}\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\} \\ &= \frac{\partial \mathfrak{J}}{\partial E}\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\}\frac{d}{dt}H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) + \frac{\partial \mathfrak{J}}{\partial s}\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\} \\ &= \frac{\partial \mathfrak{J}}{\partial E}\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\}\frac{\partial H}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) + \frac{\partial \mathfrak{J}}{\partial s}\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\} \\ &= \frac{\partial \tilde{\mathfrak{J}}}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) \end{aligned}$$

since we can easily verify that

$$\frac{d}{dt}H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) = \frac{\partial H}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$$

for the solution $(p = \tilde{\mathfrak{p}}, q = \tilde{\mathfrak{q}})$ of (14). Hence by (16) we have

$$|\tilde{\mathfrak{J}}(p', q', \beta) - \tilde{\mathfrak{J}}(p^0, q^0, a')| = \left|\int_{a'}^{\beta}\frac{d}{dt}\tilde{\mathfrak{J}}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right|$$

$$= \left|\int_{a'}^{\beta}\frac{\partial \tilde{\mathfrak{J}}}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right| \leqq M(\beta - a') \leqq M(b' - a').$$

Hence we have

$$J_2^* = J_2 + M(b' - a') > \tilde{\mathfrak{J}}(p', q', \beta) > J_1 - M(b' - a') = J_1^*$$

since $J_2 > \tilde{\mathfrak{J}}(p^0, q^0, a') > J_1$ by $(p^0, q^0) \in I(J_1, J_2, a')$. Thus we have proved that $(p', q', \beta) \in D$ and that the solution $p = \tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda)$, $q = \tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda)$ of (14) in D can be defined on the interval $a' \leqq t \leqq \beta$.[1)] If $a' \leqq \beta < b'$, then $(p', q', \beta) \in D^0$[3)] and the solution can be continued beyond $t = \beta$.[1)] This contradicts the definition of β. Hence $\beta = b'$. This completes the proof of Theorem 2.

7. When $a \leqq a' < b' \leqq b$, $(p^0, q^0) \in I(a')$ and $\lambda > 0$, we define $\Delta(a', b', p^0, q^0, \lambda)$ as follows. We put

$$\Delta(a', b', p^0, q^0, \lambda) = \max_{a' \leq t \leq b'}|\tilde{\mathfrak{J}}\{\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t, p^0, q^0\, a', \lambda), t\} - \tilde{\mathfrak{J}}(p^0, q^0, a')|$$

if the solution $p = \tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda)$, $q = \tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda)$ of (14) can be continued to $t = b'$ in D and we put

$$\Delta(a', b', p^0, q^0, \lambda) = +\infty,$$

if the above solution of (14) can not be continued to $t = b'$ in D. When $a \leqq a' < b' \leqq b$, $\lambda > 0$ and $\delta > 0$, we denote the subset of $I(a')$, $\{(p^0, q^0) \mid (p^0, q^0) \in I(a'), \Delta(a', b', p^0, q^0, \lambda) < \delta\}$ by $L(a', b', \lambda, \delta)$. We can easily

2) $\partial\mathfrak{J}/\partial E\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\}$, $\partial\mathfrak{J}/\partial s\{H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t), t\}$ are the values of $\partial\mathfrak{J}/\partial E(E, s)$, $\partial\mathfrak{J}/\partial s(E, s)$ for $E = H(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$, $s = t$ and $\partial H/\partial s(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$, $\partial\tilde{\mathfrak{J}}/\partial s(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$ are the values of $\partial H/\partial s(p, q, s)$, $\partial\tilde{\mathfrak{J}}/\partial s(p, q, s)$ for $p = \tilde{\mathfrak{p}}$, $q = \tilde{\mathfrak{q}}$, $s = t$.

3) Cf. footnote 11) of Part II.

prove that $L(a', b', \lambda, \delta)$ is an open set in R^{2n} by the continuity of $\tilde{\mathfrak{J}}$ on D and by the theorems[4)] on the dependence of the solutions of (14) in D on the initial conditions.

In the following, we denote the m-dimensional Lebesgue measure of a measurable set in R^m by $\mu_m[\ \ ]$.

LEMMA 8. *Let a', b', J_1 and J_2 be the same with those in Theorem 3. Then for any fixed $\delta(>0)$, $\mu_{2n}[I(J_1, J_2, a') - L(a', b', \lambda, \delta)] \to 0$ $(\lambda \to +\infty)$.*

PROOF. By Theorem 3, the solution $p = \tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda)$, $q = \tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda)$ of (14) can be continued in D to $t = b'$ if $(p^0, q^0) \in I(J_1, J_2, a')$.

Now we take any positive number ε. Then by Lemma 7, we can take a function $f_0(p, q, s) \in C_0^1(D^0)$ such that

$$\left(\int_D \left|(H, f_0) - \frac{\partial \tilde{\mathfrak{J}}}{\partial s}\right|^2 dp dq ds\right)^{\frac{1}{2}} < \frac{\varepsilon}{2}[\mu_{2n+1}(D)]^{-\frac{1}{2}}.$$

Hence if we put $g_0(p, q, s) = \partial \tilde{\mathfrak{J}}/\partial s - (H, f_0)$ on D, then we have by Schwartz inequality

$$\int_D |g_0(p, q, s)| \, dp dq ds < \frac{\varepsilon}{2}. \tag{18}$$

We have in the same way as in (17)

$$\begin{aligned} &\frac{d}{dt}\tilde{\mathfrak{J}}\{\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda), t\} \\ &= \frac{\partial \tilde{\mathfrak{J}}}{\partial s}\{\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda), t\} \end{aligned} \tag{19}$$

for $a' \leqq t \leqq b'$ and $(p^0, q^0) \in I(J_1, J_2, a')$. We shall sometimes abbreviate $\tilde{\mathfrak{p}}(t, p^0, q^0, a', \lambda)$ and $\tilde{\mathfrak{q}}(t, p^0, q^0, a', \lambda)$ as $\tilde{\mathfrak{p}}$ and $\tilde{\mathfrak{q}}$ in this proof of Lemma 8. By (19) and the definition of $g_0(p, q, s)$, we get

$$\begin{aligned} &|\tilde{\mathfrak{J}}\{\tilde{\mathfrak{p}}(t', p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t', p^0, q^0, a', \lambda), t'\} - \tilde{\mathfrak{J}}(p^0, q^0, a')| \\ &= \left|\int_{a'}^{t'} \frac{d}{dt}\tilde{\mathfrak{J}}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) dt\right| = \left|\int_{a'}^{t'} \frac{\partial \tilde{\mathfrak{J}}}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) dt\right| \leqq \left|\int_{a'}^{t'} g_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) dt\right| \\ &+ \left|\int_{a'}^{t'} (H, f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) dt\right|^{5)}. \end{aligned}$$

Hence by the definition of $\Delta(a', b', p^0, q^0, \lambda)$, we have

$$\Delta(a', b', p^0, q^0, \lambda) \leqq \int_{a'}^{b'} |g_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)| \, dt + \max_{a' \leqq t' \leqq b'} \left|\int_{a'}^{t'} (H, f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t) dt\right|. \tag{20}$$

Now for any fixed $s(a' \leqq s \leqq b')$ and for any fixed $\lambda(>0)$, the one-to-one mapping of $I(J_1, J_2, a')$ onto an open set $V(s, \lambda)$ in $I(s)$

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4) Cf. E. Kamke [1, pp. 149–153].

5) $(H, f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$ is the value of the Poisson bracket $(H, f_0)(p, q, s)$ for $p = \tilde{\mathfrak{p}}$, $q = \tilde{\mathfrak{q}}$, $s = t$.

$$(p^0, q^0)\to\{\tilde{\mathfrak{p}}(s, p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(s, p^0, q^0, a', \lambda)\}$$

is measure-preserving, since (14) is a Hamiltonian system (Theorem of Liouville).[6] Hence

$$\begin{aligned}(21)\qquad &\int_{I(J_1,J_2,a')}\left(\int_{a'}^{b'}|g_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)|\,dt\right)dp^0dq^0\\ &=\int_{a'}^{b'}\left(\int_{I(J_1,J_2,a')}|g_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)|\,dp^0dq^0\right)dt\\ &=\int_{a'}^{b'}\left(\int_{V(s,\lambda)}|g_0(p, q, s)|\,dpdq\right)ds\\ &\leqq\int_D|g_0(p, q, s)|\,dpdqds,\end{aligned}$$

since $V(s, \lambda)\subset I(s)$ and $D=\{(p, q, s)\,|\,(p, q)\in I(s), a\leqq s\leqq b\}$.

On the other hand[7]

$$\begin{aligned}(H,f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)&=\frac{1}{\lambda}\frac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)-\frac{1}{\lambda}\left\{\frac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)-\lambda(H,f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)\right\}\\ &=\frac{1}{\lambda}\frac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)-\frac{1}{\lambda}\frac{d}{dt}f_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)\end{aligned}$$

since $\dfrac{d}{dt}f_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)=-\lambda(H,f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)+\dfrac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)$ for the solution $p=\tilde{\mathfrak{p}}$, $q=\tilde{\mathfrak{q}}$ of (14) in D as can be easily verified. Hence for $a'\leqq t'\leqq b'$

$$\begin{aligned}\left|\int_{a'}^{t'}(H,f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right|&\leqq\frac{1}{\lambda}\left|\int_{a'}^{t'}\frac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right|\\ &+\frac{1}{\lambda}\left|\int_{a'}^{t'}\frac{d}{dt}f_0(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)\,dt\right|\leqq\frac{1}{\lambda}\int_{a'}^{b'}\left|\frac{\partial f_0}{\partial s}(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)\right|dt\\ &+\frac{1}{\lambda}\left|f_0\{\tilde{\mathfrak{p}}(t', p^0, q^0, a', \lambda), \tilde{\mathfrak{q}}(t', p^0, q^0, a', \lambda), t'\}-f_0(p^0, q^0, a')\right|.\end{aligned}$$

Now there is an M' such that $|f_0|, |\partial f_0/\partial s|\leqq M'$ on D since $f_0\in C_0^1(D^0)$. Therefore we have

$$(22)\qquad \max_{a'\leqq t'\leqq b'}\left|\int_{a'}^{t'}(H,f_0)(\tilde{\mathfrak{p}}, \tilde{\mathfrak{q}}, t)dt\right|\leqq\frac{1}{\lambda}((b'-a')+2)M'.$$

By (18), (20), (21) and (22), we get

$$\int_{I(J_1,J_2,a')}\Delta(a', b', p^0, q^0, \lambda)dp^0dq^0$$

$\leqq\dfrac{\varepsilon}{2}+\dfrac{1}{\lambda}\{(b'-a')+2\}M'\mu_{2n}[I(J_1, J_2, a')]\leqq\dfrac{\varepsilon}{2}+\dfrac{\varepsilon}{2}\leqq\varepsilon$ if $\lambda\geqq 2\{(b'-a')+2\}\times M'\mu_{2n}[I(J_1, J_2, a')]/\varepsilon$. Thus we have proved that

$$\int_{I(J_1,J_2,a')}\Delta(a', b', p^0, q^0, \lambda)dp^0dq^0\to 0\quad(\lambda\to+\infty).$$

From this we can easily deduce the desired results. Q. E. D.

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6) Cf. E. Kamke [1, pp. 155-161].

7) Cf. footnote 2) and 5).

8. Now we state and prove a form of the adiabatic theorem.

THEOREM 4. *Under Assumptions* 1, 2 *and* 3,

$$\mu_{2n}[I(J_1, J_2, a)-L(a, b, \lambda, \delta)]\to 0 \ (\lambda\to+\infty)$$

for any fixed J_1, J_2, δ *such that* $J_2^*>J_2>J_1>J_1^*$ *and* $\delta>0$.

PROOF. We fix any J_1, J_2 such that $J_2^*>J_2>J_1>J_1^*$, then by Lemma 8, if $b\geqq\beta'>a$ and β' is sufficiently close to a,

$$(23)\qquad \mu_{2n}[I(J_1, J_2, a)-L(a, \beta', \lambda, \delta)]\to 0 \ (\lambda\to+\infty)$$

for all $\delta>0$. We denote by β the least upper bound of β' such that $b\geqq\beta'>a$ and (23) holds for all $\delta>0$. Also we fix a $\delta_0>0$ such that $J_2^*>J_2+\delta_0>J_1-\delta_0>J_1^*$. Then if $b\geqq\beta''\geqq\beta>\alpha''>a$ and $\beta''-\alpha''$ is sufficiently small, we have by Lemma 8

$$(24)\qquad \mu_{2n}[I(J_1-\delta_0, J_2+\delta_0, \alpha'')-L(\alpha'', \beta'', \lambda, \delta/2)]\to 0 \ (\lambda\to+\infty)$$

for all $\delta>0$. Also by the definition of β, there is an α'' arbitrarily close to β such that $\beta>\alpha''>a$ and

$$(25)\qquad \mu_{2n}[I(J_1, J_2, a)-L(a, \alpha'', \lambda, \delta/2)]\to 0 \ (\lambda\to+\infty)$$

for all $\delta>0$. Hence we can take for any β'' such that $b\geqq\beta''\geqq\beta$ and $\beta''-\beta$ is sufficiently small or zero, an α'' such that $\beta>\alpha''>a$ and (24), (25) are satisfied for all $\delta>0$. We take such α'' and β'' in the following.

Now if $(p^0, q^0)\in L(a, \alpha'', \lambda, \delta/2)$, then the solution $p=\tilde{\mathfrak{p}}(t, p^0, q^0, a, \lambda)$, $q=\tilde{\mathfrak{q}}(t, p^0, q^0, a, \lambda)$ of (14) can be continued in D to $t=\alpha''$, by the definition of $L(a, \alpha'', \lambda, \delta/2)$. We denote by $\mathfrak{A}_{\lambda,\delta}$ the one-to-one mapping of $L(a, \alpha'', \lambda, \delta/2)$ into $I(\alpha'')$

$$(p^0, q^0)\to\{\tilde{\mathfrak{p}}(\alpha'', p^0, q^0, a, \lambda),\ \tilde{\mathfrak{q}}(\alpha'', p^0, q^0, a, \lambda)\}$$

and also by $R(\lambda, \delta)$ the set $\mathfrak{A}_{\lambda,\delta}[I(J_1, J_2, a)\cap L(a, \alpha'', \lambda, \delta/2)]$. Then by the definition of $L(a, \alpha'', \lambda, \delta/2)$ we have for $0<\delta<2\delta_0$, $\lambda>0$

$$(26)\qquad R(\lambda, \delta)\subset I(J_1-\delta_0, J_2+\delta_0, \alpha'').$$

Also the mapping $\mathfrak{A}_{\lambda,\delta}$ is measure-preserving since (14) is a Hamiltonian system (Theorem of Liouville).[6] Hence for $\delta>0, \lambda>0$

$$(27)\qquad \mu_{2n}[R(\lambda, \delta)]=\mu_{2n}[I(J_1, J_2, a)\cap L(a, \alpha'', \lambda, \delta/2)].$$

By (25) and (27), we have for all $\delta>0$

$$(28)\qquad \mu_{2n}[R(\lambda, \delta)]\to\mu_{2n}[I(J_1, J_2, a)] \ (\lambda\to+\infty).$$

From (28), (24) and (26), we have for $0<\delta<2\delta_0$

$$\mu_{2n}[R(\lambda, \delta)\cap L(\alpha'', \beta'', \lambda, \delta/2)]\to\mu_{2n}[I(J_1, J_2, a)] \ (\lambda\to+\infty).$$

Therefore we have for $0<\delta<2\delta_0$,

$$\mu_{2n}\{\mathfrak{A}_{\lambda,\delta}^{-1}[R(\lambda, \delta)\cap L(\alpha'', \beta'', \lambda, \delta/2)]\}\to\mu_{2n}[I(J_1, J_2, a)] \ (\lambda\to+\infty)$$

since $\mathfrak{A}_{\lambda,\delta}$ and so $\mathfrak{A}_{\lambda,\delta}^{-1}$ is measure-preserving. Hence we have for $0<\delta<2\delta_0$,

$$\mu_{2n}[I(J_1, J_2, a)\cap L(a, \beta'', \lambda, \delta)]\to\mu_{2n}[I(J_1, J_2, a)] \ (\lambda\to+\infty)$$

since we can easily see that $I(J_1, J_2, a)\cap L(a, \beta'',\lambda, \delta)\supset\mathfrak{A}_{\lambda,\delta}^{-1}[R(\lambda, \delta)\cap L(\alpha'', \beta'', \lambda, \delta/2)]$ by the definitions of $L(a', b', \lambda, \delta)$ and $R(\lambda, \delta)$. Thus we get

$$(29)\qquad \mu_{2n}[I(J_1, J_2, a)-L(a, \beta'', \lambda, \delta)]\to 0 \ (\lambda\to+\infty)$$

for $0<\delta<2\delta_0$ and so for all $\delta>0$, since $L(a, \beta'', \lambda, \delta_2)\supset L(a, \beta'', \lambda, \delta_1)$ if $\delta_2>\delta_1$.

If $\beta<b$, then we can take the above β'' in such a manner that $b\geqq\beta''>\beta$. But then (29) contradicts the definition of β. Hence $\beta=b$. Then if we take $\beta''=\beta=b$ in the above argument, we have from (29) the desired results. Q. E. D.

Reference

[1] Kamke, E.,: Differentialgleichungen reeller Funktionen, Leipzig (1930).